Reduction of nonlinear PDEs to homogeneous and autonomous form

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Symmetry and Perturbation Theory

Otranto, June 4–10, 2023

Problem of this talk

If we have a general first order nonlinear system of PDEs,

$$\Delta\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0}, \qquad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^m,$$

we can be interested to map it to an equivalent special form by means of an invertible transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}).$$

Lie symmetries can help us

The admitted Lie symmetries can give information about the final form, and allow to algorithmically construct a mapping from a (SOURCE) DE to a (TARGET) suitable DE.
Of course, considering 1–1 point mappings, a 1–1 correspondence between symmetries of source and target DEs arises, *i.e.*, the corresponding Lie algebras of infinitesimal generators are isomorphic.

Necessary and sufficient conditions can be determined if the source nonlinear first order system is polynomial in the derivatives and it admits a suitable $(n+1)$–dimensional solvable Lie algebra as subalgebra of the algebra of its Lie point symmetries.

Systems polynomial in the derivatives

Let us consider a nonlinear system of first order PDEs which is polynomial in the derivatives, with coefficients depending at most on $\mathbf{x}$ and $\mathbf{u}$, say a system made by equations like

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = \mathbf{0}, \qquad s = 1,\dots,m,$$

where $\boldsymbol{\alpha}$ is the multi–index $(\alpha_1,\dots,\alpha_r)$, $\mathbf{j}$ the multi–index $(j_1,\dots,j_r)$, $\alpha_k = 1,\dots,m$, $j_k = 1,\dots,n$, N_s are integers, and $A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u})$, $B^s(\mathbf{x},\mathbf{u})$ smooth functions.

The aim is to determine necessary and sufficient conditions for the construction of an invertible point transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}),$$

mapping the source system into an equivalent one that is autonomous and polynomially homogeneous in the derivatives, *i.e.*, made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = \mathbf{0}, \qquad s = 1,\dots,m,$$

for some integers $\overline{N}_s$.

Remark

A special case is that of a nonhomogeneous and nonautonomous quasilinear first order system.
We require the target system to be autonomous and polynomially homogeneous in the derivatives!
Of course, it may occur that the target system turns out to be linear in the derivatives, *i.e.*, $\overline{N}_s = 1$ $(s = 1, \dots, m)$, whereupon we have an autonomous and homogeneous quasilinear system.

Lemma

Given a first order system of PDEs polynomial in the derivatives, say

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = \mathbf{0}, \quad s = 1, \dots, m,$$

with coefficients depending on the independent and the dependent variables, then an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}\left(\mathbf{x},\mathbf{u}\right), \qquad \mathbf{w} = \mathbf{W}\left(\mathbf{x},\mathbf{u}\right),$$

produces a first order system still polynomial in the derivatives.

Theorem

The nonlinear first order system of PDEs polynomial in the derivatives

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = \mathbf{0}, \qquad s = 1,\dots,m,$$

is mapped by an invertible point transformation, say

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}),$$

to the equivalent nonlinear first order autonomous system polynomially homogeneous in the derivatives, say

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = \mathbf{0}, \qquad s = 1,\dots,m,$$

for some integers $\overline{N}_s$,

. . . Theorem

. . . iff there exists an $(n+1)$–dimensional subalgebra of the Lie algebra of point symmetries, admitted by the source system, spanned by the vector fields

$$\Xi_i = \sum_{j=1}^{n} \xi_i^j(\mathbf{x},\mathbf{u}) \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{m} \eta_i^\alpha(\mathbf{x},\mathbf{u}) \frac{\partial}{\partial u_\alpha}, \qquad i = 1,\dots,n+1,$$

such that

$$\begin{aligned} &[\Xi_i, \Xi_j] = 0, && i = 1,\dots,n-1, \qquad i < j \leq n, \\ &[\Xi_i, \Xi_{n+1}] = \Xi_i, && i = 1,\dots,n. \end{aligned}$$

Furthermore, the $(n+1)\times n$ matrix with entries ξ_i^j $(i = 1,\dots,n+1,\ j = 1,\dots,n)$ must have rank n, and the variables $\mathbf{w}$, which by constructions are invariants of $\Xi_1,\dots,\Xi_n$, must result invariant with respect to Ξ_{n+1} too.

Remark

The condition that the n–dimensional Abelian Lie subalgebra of the symmetries generate a distribution of rank n ensures that we may construct the complete set of the new independent variables $\mathbf{z}$.

Sketch of the Proof: Necessary Conditions

The target system

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = \mathbf{0}, \qquad s = 1, \dots, m.$$

admits the Lie point symmetries

$$\Xi_i = \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n, \qquad \Xi_{n+1} = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i},$$

spanning an $(n+1)$–dimensional solvable Lie algebra containing an n–dimensional Abelian Lie subalgebra, with the only non–zero commutators

$$[\Xi_i, \Xi_{n+1}] = \Xi_i, \quad i = 1, \dots, n.$$

The same algebraic structure has to be possessed by the Lie symmetries of source system!

Sketch of the Proof: Sufficient Conditions

Suppose the conditions of the theorem are satisfied. Let us introduce a set of canonical variables for all the vector fields Ξ_i $(i = 1, \ldots, n)$, say z_i $(i = 1, \ldots, n)$ and w_α $(\alpha = 1, \ldots, m)$ such that

$$\Xi_i z_j = \delta_{ij}, \qquad \Xi_i w_\alpha = 0, \qquad i = 1, \ldots n.$$

Consequently, in the variables **z** and **w**, we have an autonomous system.
Then, since $[\Xi_i, \Xi_{n+1}] = \Xi_i$ for $i = 1, \ldots, n$, it is

$$\begin{aligned} \Xi_i(\Xi_{n+1} z_j) &= \Xi_{n+1}(\Xi_i z_j) + \Xi_i z_j = \delta_{ij}, \\ \Xi_i(\Xi_{n+1} w_\alpha) &= \Xi_{n+1}(\Xi_i w_\alpha) + \Xi_i w_\alpha = 0; \end{aligned}$$

these relations allow the vector field Ξ_{n+1} to gain the representation

$$\Xi_{n+1} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.$$

As a consequence, the resulting system in the variables **z**, **w** must be necessarily polynomially homogeneous in the derivatives.

Problem

Consider a quasilinear nonautonomous and nonhomogeneous first order system of PDEs,

$$\sum_{i=1}^{n} A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^m.$$

that we want to reduce into an equivalent quasilinear autonomous and homogeneous form

$$\sum_{i=1}^{n} \widetilde{A}^i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0},$$

by means of an invertible mapping like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}).$$

Remark

The previous theorem ensures that a quasilinear first order system can be mapped to a system which is autonomous and polynomially homogeneous in the derivatives.
But the target system, in general, is not necessarily quasilinear!
Additional conditions to preserve the quasilinear structure are needed!

Example: $n = 3$ and $m = 2$

Consider a general quasilinear first order system

$$A^1(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_1} + A^2(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} + A^3(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_3} = \mathbf{B}(\mathbf{x},\mathbf{u}).$$

and the transformation $\mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u})$. The system in the new dependent variables remains quasilinear. Therefore, we can limit to consider what happens when we transform the independent variables, say

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}).$$

The conditions

$$\begin{aligned}
&A^1_{i1}\left(\frac{\partial Z_j}{\partial u_2}\frac{\partial Z_k}{\partial x_1} - \frac{\partial Z_j}{\partial x_1}\frac{\partial Z_k}{\partial u_2}\right) - A^1_{i2}\left(\frac{\partial Z_j}{\partial u_1}\frac{\partial Z_k}{\partial x_1} - \frac{\partial Z_j}{\partial x_1}\frac{\partial Z_k}{\partial u_1}\right)\\
&+ A^2_{i1}\left(\frac{\partial Z_j}{\partial u_2}\frac{\partial Z_k}{\partial x_2} - \frac{\partial Z_j}{\partial x_2}\frac{\partial Z_k}{\partial u_2}\right) - A^2_{i2}\left(\frac{\partial Z_j}{\partial u_1}\frac{\partial Z_k}{\partial x_2} - \frac{\partial Z_j}{\partial x_2}\frac{\partial Z_k}{\partial u_1}\right)\\
&+ A^3_{i1}\left(\frac{\partial Z_j}{\partial u_2}\frac{\partial Z_k}{\partial x_3} - \frac{\partial Z_j}{\partial x_3}\frac{\partial Z_k}{\partial u_2}\right) - A^3_{i2}\left(\frac{\partial Z_j}{\partial u_1}\frac{\partial Z_k}{\partial x_3} - \frac{\partial Z_j}{\partial x_3}\frac{\partial Z_k}{\partial u_1}\right) - B_i\left(\frac{\partial Z_j}{\partial u_1}\frac{\partial Z_k}{\partial u_2} - \frac{\partial Z_j}{\partial u_2}\frac{\partial Z_k}{\partial u_1}\right) = 0,
\end{aligned}$$

with $i = 1, 2$ and $1 \leq j < k \leq 3$ are required to preserve the quasilinear structure.

Lemma

Given the system

$$\sum_{i=1}^{n} A^{i}\left(\mathbf{x},\mathbf{u}\right)\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}\left(\mathbf{x},\mathbf{u}\right),$$

an invertible mapping of the form

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}),$$

produces a system still in quasilinear form.

Proof

Straightforward, by using the chain rule.

Theorem

$$\sum_{i=1}^{n} A^i(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x},\mathbf{u}) \quad \Leftrightarrow \quad \sum_{i=1}^{n} \widehat{A}^i(\mathbf{w})\frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}$$

through an invertible point transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}),$$

if and only if it admits as subalgebra of its Lie point symmetries an $(n+1)$–dimensional Lie algebra spanned by

$$\Xi_i = \sum_{j=1}^{n} \xi_i^j(\mathbf{x})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_i^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A}, \qquad i = 1,\dots,n+1,$$

such that

$$[\Xi_i,\Xi_j] = 0, \qquad [\Xi_i,\Xi_{n+1}] = \Xi_i, \qquad i,j = 1,\dots,n.$$

Furthermore, the $(n+1)\times n$ matrix with entries ξ_i^j $(i = 1,\dots,n+1,\ j = 1,\dots,n)$ must have rank n, and the variables $\mathbf{w}$, which by constructions are invariants of $\Xi_1,\dots,\Xi_n$, must result invariant with respect to Ξ_{n+1} too.

Remark

It is worth of noticing that, since the infinitesimals $\xi_i^j(\mathbf{x})$ $(i,j = 1,\dots n)$ depend on the independent variables only, it is $\mathbf{z} = \mathbf{Z}(\mathbf{x})$.

Rotating shallow water equations

$$
\begin{aligned}
&h_t + uh_x + vh_y + h(u_x + v_y) = 0,\\
&u_t + uu_x + vu_y + gh_x = 2\omega v,\\
&v_t + uv_x + vv_y + gh_y = -2\omega u,
\end{aligned}
$$

where h is the height of the fluid, (u, v) the components of its velocity, g the gravitational constant, and ω the constant angular velocity of the fluid around the z–axis responsible for the Coriolis force.

This system is autonomous, nonhomogeneous, and involves 3 independent variables: hence, a suitable 4–dimensional Lie algebra of point symmetries is needed.

9–dimensional Lie algebra of point symmetries

$$\Xi_1 = \partial_t, \qquad \Xi_2 = \partial_x, \qquad \Xi_3 = \partial_y,$$
$$\Xi_4 = y\partial_x - x\partial_y + v\partial_u - u\partial_v,$$
$$\Xi_5 = \cos(2\omega t)\partial_x - \sin(2\omega t)\partial_y - 2\omega\sin(2\omega t)\partial_u - 2\omega\cos(2\omega t)\partial_v,$$
$$\Xi_6 = \sin(2\omega t)\partial_x + \cos(2\omega t)\partial_y + 2\omega\cos(2\omega t)\partial_u - 2\omega\sin(2\omega t)\partial_v,$$
$$\Xi_7 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h,$$
$$\begin{aligned}\Xi_8 = {} & \sin(2\omega t)\partial_t + \omega(x\cos(2\omega t) + y\sin(2\omega t))\partial_x + \omega(y\cos(2\omega t) - x\sin(2\omega t))\partial_y \\ & + \omega((2\omega y - u)\cos(2\omega t) + (-2\omega x + v)\sin(2\omega t))\partial_u \\ & - \omega((2\omega x + v)\cos(2\omega t) + (2\omega y + u))\sin(2\omega t))\partial_v - 2\omega h\cos(2\omega t)\partial_h,\end{aligned}$$
$$\begin{aligned}\Xi_9 = {} & \cos(2\omega t)\partial_t + \omega(y\cos(2\omega t) - x\sin(2\omega t))\partial_x - \omega(x\cos(2\omega t) + y\sin(2\omega t))\partial_y \\ & - \omega((2\omega x - v)\cos(2\omega t) + (2\omega y - u)\sin(2\omega t))\partial_u \\ & - \omega((2\omega y + u)\cos(2\omega t) - (2\omega x + v))\sin(2\omega t))\partial_v - 2\omega h\sin(2\omega t)\partial_h.\end{aligned}$$

Thee 4–dimensional subalgebra spanned by the vector fields

$$\widehat{\Xi}_1 = \Xi_1 + \omega\Xi_4 - \Xi_9, \qquad \widehat{\Xi}_2 = \Xi_3 - \Xi_6,$$
$$\widehat{\Xi}_3 = -\Xi_2 + \Xi_5, \qquad \widehat{\Xi}_4 = \frac{1}{2}\left(\Xi_7 - \frac{1}{\omega}\Xi_8\right),$$

with non–zero commutators

$$[\widehat{\Xi}_i, \widehat{\Xi}_4] = \widehat{\Xi}_i, \qquad i = 1, \dots, 3,$$

allows us to introduce

$$\tau = -\frac{1}{2\omega}\cot(\omega t), \quad \xi = \frac{1}{2}(y - x\cot(\omega t)), \quad \eta = -\frac{1}{2}(x + y\cot(\omega t)),$$
$$U = -\frac{1}{2}(u\sin(2\omega t) - v(1-\cos(2\omega t)) - 2\omega x),$$
$$V = -\frac{1}{2}(u(1-\cos(2\omega t)) + v\sin(2\omega t)) - 2\omega y), \qquad H = \frac{1 - cos(2\omega t)}{2}h,$$

whereupon we get

$$H_\tau + UH_\xi + VH_\eta + H(U_\xi + V_\eta) = 0,$$
$$U_\tau + UU_\xi + VU_\eta + gH_\xi = 0,$$
$$V_\tau + UV_\xi + VV_\eta + gH_\eta = 0.$$

Nonlinear hyperbolic heat equation

$$\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = -\frac{u_1}{\tau_0}, \qquad \frac{\partial u_2}{\partial t} - \frac{\chi_0^2}{u_1^2}\frac{\partial u_1}{\partial x} = 0,$$

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \exp(-t/\tau_0)\left(\tau_0\frac{\partial}{\partial t} - u_1\frac{\partial}{\partial u_1}\right), \qquad \Xi_3 = x\frac{\partial}{\partial x} + \tau_0\frac{\partial}{\partial t} - u_1\frac{\partial}{\partial u_1}.$$

$$\xi = x, \qquad \tau = \exp\left(\frac{t}{\tau_0}\right), \qquad w_1 = \exp\left(\frac{t}{\tau_0}\right)u_1, \qquad w_2 = u_2,$$

$$\frac{\partial w_1}{\partial \tau} - \tau_0\frac{\partial w_2}{\partial \xi} = 0, \qquad \frac{\partial w_2}{\partial \tau} - \frac{\chi_0^2\tau_0}{w_1^2}\frac{\partial w_1}{\partial \xi} = 0.$$

Rate–type materials

$$\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0,$$
$$\frac{\partial u_2}{\partial t} - \phi(t, u_2)\frac{\partial u_1}{\partial x} = \psi(t, u_2),$$
$$\phi(t, u_2) = \varphi(\exp(-t)u_2),$$
$$\psi(t, u_2) = u_2 - \varphi(\exp(-t)u_2),$$

$$\Xi_1 = \exp(-t)\left(\frac{\partial}{\partial t} + \exp(-t)u_2\frac{\partial}{\partial u_2}\right),$$
$$\Xi_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u_1},$$
$$\Xi_3 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + x\frac{\partial}{\partial u_1} + \exp(-t)u_2\frac{\partial}{\partial u_2}.$$

$$\xi = x, \qquad \tau = \exp(t),$$
$$w_1 = u_1 - x, \qquad w_2 = u_2\exp(-t),$$

$$\frac{\partial w_1}{\partial \tau} - \frac{\partial w_2}{\partial \xi} = 0,$$
$$\frac{\partial w_2}{\partial \tau} - \varphi(w_2)\frac{\partial w_1}{\partial \xi} = 0.$$