Hierarchy of coupled Burgers–like equations induced by conditional symmetries

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Lie groups, geometry, integrability, hydrodynamics

Sophus Lie Conference Center, Nordfjordeid, July 1–5, 2024

Classical Lie symmetries

Considering an r–th order DE

$$\Delta\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}\right) = 0,$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ and $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ denote the independent and dependent variables, whereas $\mathbf{u}^{(r)}$ are the derivatives of $\mathbf{u}$ w.r.t. $\mathbf{x}$ up to the order r.
A Lie point symmetry is characterized by the infinitesimal operator

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}$$

such that

$$\Xi^{(r)}(\Delta)\Big|_{\Delta=0} = 0,$$

where $\Xi^{(r)}$ is the *rth* order prolongation.

Nonclassical Symmetries

Q–conditional symmetries [Bluman & Cole, J.M.M. 1969; Cherniha, J.M.A.A. 2007]

Consider the invariant surface conditions

$$Q_\alpha \equiv \sum_{i=1}^{n} \xi_i(\mathbf{x},\mathbf{u})\frac{\partial u_\alpha}{\partial x_i} - \eta_\alpha(\mathbf{x},\mathbf{u}) = 0, \qquad \alpha = 1,\dots,m.$$

The Q–conditional symmetries are expressed by vector fields Ξ such that

$$\Xi^{(r)}(\Delta)\Big|_{\mathcal{M}} = 0,$$

where $\mathcal{M}$ is the manifold of the jet space defined by

$$\Delta = 0, \qquad Q_\alpha = 0, \qquad \frac{D}{Dx_{j_1}}\cdots\frac{D}{Dx_{j_k}}Q_\alpha = 0,$$

with $1 \leq j_1, j_2, \dots, j_k \leq n$, $1 \leq k \leq r-1$, and $\alpha = 1,\dots,m$.

Nonclassical Symmetries

Properties

1. Ξ is a classical symmetry $\Rightarrow$ Ξ is a Q–conditional symmetry;
2. Ξ is a Q–conditional symmetry $\Rightarrow \lambda\Xi$, with $\lambda \equiv \lambda(\mathbf{x}, \mathbf{u})$ arbitrary smooth function, is a Q–conditional symmetry;
3. in general, Q–conditional symmetries admitted by a DE form a set which is neither a Lie algebra nor a linear space.

We can look for Q–conditional symmetries in n different situations along with the following constraints:

- $\xi_1 = 1$;
- $\xi_i = 1$ and $\xi_j = 0$ for $1 \leq j < i \leq n$.

Nonclassical Symmetries

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In this talk...

We will be concerned with second order PDEs ruling unknown functions depending on t and x, and consider Q–conditional symmetries corresponding to the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi(t, x, u_\beta)\frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \eta_\alpha(t, x, u_\beta)\frac{\partial}{\partial u_\alpha}.$$

Q–Conditional symmetries of the Burgers' equation

Let us consider the Burgers' equation

$$\Delta_1 \equiv u^{(1)}_{,t} + u^{(1)} u^{(1)}_{,x} - u^{(1)}_{,xx} = 0$$

(the subscripts denote partial derivatives) for the unknown $u^{(1)}(t,x)$, and let us consider the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi(t,x,u^{(1)})\frac{\partial}{\partial x} + \eta(t,x,u^{(1)})\frac{\partial}{\partial u^{(1)}}.$$

The Q–conditional symmetries of Δ_1 are the symmetries of the manifold $\mathcal{M}_1$ defined by

$$\begin{cases} \Delta_1 = 0, \\ Q_1 \equiv \Xi\left(u^{(1)} - u^{(1)}(t,x)\right) = 0, \\ \dfrac{DQ_1}{Dt} = \dfrac{DQ_1}{Dx} = 0. \end{cases}$$

Q–Conditional symmetries of the Burgers' equation

By imposing

$$\Xi^{(2)}(\Delta)\Big|_{\mathcal{M}_1} = 0,$$

it is obtained the following polynomial of third degree in the derivative $u^{(1)}_{,x}$:

$$\begin{aligned}
&\frac{\partial^2 \xi}{\partial u^{(1)2}}\left(u^{(1)}_{,x}\right)^3 + \left(2\frac{\partial^2 \xi}{\partial x \partial u^{(1)}} - \frac{\partial^2 \eta}{\partial u^{(1)2}} + 2\frac{\partial \xi}{\partial u^{(1)}} u^{(1)} - 2\xi\frac{\partial \xi}{\partial u^{(1)}}\right)\left(u^{(1)}_{,x}\right)^2 \\
&+ \left(\frac{\partial^2 \xi}{\partial x^2} - 2\frac{\partial^2 \eta}{\partial x \partial u^{(1)}} - \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} u^{(1)} - 2\xi\frac{\partial \xi}{\partial x} + 2\frac{\partial \xi}{\partial u^{(1)}}\eta + \eta\right) u^{(1)}_{,x} \\
&- \frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial \xi}{\partial x}\eta + \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} u^{(1)} = 0.
\end{aligned}$$

Q–Conditional symmetries of the Burgers' equation

Annihilating the coefficients of this polynomial, after simple algebra, we get

$$\begin{aligned} \xi &= \kappa u^{(1)} + \frac{1}{2} u_1^{(2)}, \\ \eta &= \frac{\kappa(1-\kappa)}{3}\left(u^{(1)}\right)^3 - \frac{\kappa}{2} u_1^{(2)}\left(u^{(1)}\right)^2 + \frac{1}{4} u^{(1)} u_2^{(2)} + \frac{1}{4} u_3^{(2)}, \end{aligned}$$

where κ is a constant such that

$$\kappa(\kappa-1)(2\kappa+1) = 0,$$

whereas $u_1^{(2)}(t,x)$, $u_2^{(2)}(t,x)$ and $u_3^{(2)}(t,x)$ are functions depending on the indicated arguments.

Q–Conditional symmetries of the Burgers' equation

For $\kappa = -1/2$, the functions $u_1^{(2)}(t,x)$, $u_2^{(2)}(t,x)$ and $u_3^{(2)}(t,x)$ satisfy the system

$$\mathbf{\Delta}_3 \equiv \begin{cases} u_{1,t}^{(2)} + u_1^{(2)} u_{1,x}^{(2)} - u_{1,xx}^{(2)} + u_{2,x}^{(2)} = 0, \\ u_{2,t}^{(2)} + u_2^{(2)} u_{1,x}^{(2)} - u_{2,xx}^{(2)} + u_{3,x}^{(2)} = 0, \\ u_{3,t}^{(2)} + u_3^{(2)} u_{1,x}^{(2)} - u_{3,xx}^{(2)} = 0. \end{cases}$$

P. J. Olver, E. M. Vorob'ev (1996)
Nonclassical and conditional symmetries
CRC handbook of Lie group analysis of differential equations 3, 291–328.

Q–Conditional symmetries of three coupled Burgers-like equations

Problem

If we look for conditional symmetries of system

$$\mathbf{\Delta}_3 \equiv \begin{cases} u^{(2)}_{1,t} + u^{(2)}_1 u^{(2)}_{1,x} - u^{(2)}_{1,xx} + u^{(2)}_{2,x} = 0, \\ u^{(2)}_{2,t} + u^{(2)}_2 u^{(2)}_{1,x} - u^{(2)}_{2,xx} + u^{(2)}_{3,x} = 0, \\ u^{(2)}_{3,t} + u^{(2)}_3 u^{(2)}_{1,x} - u^{(2)}_{3,xx} = 0. \end{cases}$$

what happens?

Q–Conditional symmetries of three coupled Burgers-like equations

Problem

If we look for conditional symmetries of system

$$\boldsymbol{\Delta}_3 \equiv \begin{cases} u^{(2)}_{1,t} + u^{(2)}_1 u^{(2)}_{1,x} - u^{(2)}_{1,xx} + u^{(2)}_{2,x} = 0, \\ u^{(2)}_{2,t} + u^{(2)}_2 u^{(2)}_{1,x} - u^{(2)}_{2,xx} + u^{(2)}_{3,x} = 0, \\ u^{(2)}_{3,t} + u^{(2)}_3 u^{(2)}_{1,x} - u^{(2)}_{3,xx} = 0. \end{cases}$$

what happens?

Answer

The system $\boldsymbol{\Delta}_3$ of coupled Burgers–like equations admits Q-conditional symmetries expressed in terms of five functions satisfying a new system of coupled Burgers–like equations!

Q–Conditional symmetries of three coupled Burgers-like equations

Proposition

The vector field

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{3} \eta_\alpha \frac{\partial}{\partial u_\alpha^{(2)}},$$

gives a Q–conditional symmetry of the system $\boldsymbol{\Delta}_3$ provided that

$$\xi = \frac{1}{2}\left(-u_1^{(2)} + u_1^{(3)}\right),$$

$$\eta_1 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_1^{(2)} - u_1^{(2)} u_2^{(2)} - u_2^{(2)} u_1^{(2)} + u_1^{(3)}\left(u_1^{(2)}\right)^2 + u_2^{(3)} u_1^{(2)} + u_1^{(3)} u_2^{(2)} - u_3^{(2)} + u_3^{(3)}\right),$$

$$\eta_2 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_2^{(2)} - u_1^{(2)} u_3^{(2)} - \left(u_2^{(2)}\right)^2 + u_1^{(3)} u_1^{(2)} u_2^{(2)} + u_2^{(3)} u_2^{(2)} + u_1^{(3)} u_3^{(2)} + u_4^{(3)}\right),$$

$$\eta_3 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_3^{(2)} - u_2^{(2)} u_3^{(2)} + u_1^{(3)} u_1^{(2)} u_3^{(2)} + u_2^{(3)} u_3^{(2)} + u_5^{(3)}\right),$$

Q–Conditional symmetries of three coupled Burgers-like equations

where the functions $u_\alpha^{(3)}(t,x)$ $(\alpha = 1,\dots,5)$ satisfy the constraints

$$\mathbf{\Delta}_5 \equiv \begin{cases} u_{1,t}^{(3)} + u_1^{(3)} u_{1,x}^{(3)} - u_{1,xx}^{(3)} + u_{2,x}^{(3)} = 0, \\ u_{2,t}^{(3)} + u_2^{(3)} u_{1,x}^{(3)} - u_{2,xx}^{(3)} + u_{3,x}^{(3)} = 0, \\ u_{3,t}^{(3)} + u_3^{(3)} u_{1,x}^{(3)} - u_{3,xx}^{(3)} + u_{4,x}^{(3)} = 0, \\ u_{4,t}^{(3)} + u_4^{(3)} u_{1,x}^{(3)} - u_{4,xx}^{(3)} + u_{5,x}^{(3)} = 0, \\ u_{5,t}^{(3)} + u_5^{(3)} u_{1,x}^{(3)} - u_{5,xx}^{(3)} = 0. \end{cases}$$

Remarks

- We note that the system $\mathbf{\Delta}_5$ has the same structure as the system $\mathbf{\Delta}_3$, even if it involves two more unknowns.
- The lengthy computations can be done by using the program `ReLie`[a] written in the CAS Reduce.

[a] F. Oliveri, ReLie: a Reduce program for Lie group analysis of differential equations, Symmetry, **13**, 1–39, 2021.

Q–Conditional symmetries of five coupled Burgers–like equations

Proposition

The system $\boldsymbol{\Delta}_5$ admits the vector field Ξ of the Q-conditional symmetries, say

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{5} \eta_\alpha \frac{\partial}{\partial u_\alpha^{(3)}},$$

with

$$\xi = \frac{1}{2}\left(-u_1^{(3)} + u_1^{(4)}\right),$$

$$\eta_1 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^3 - 2u_1^{(3)}u_2^{(3)} + u_1^{(4)}\left(u_1^{(3)}\right)^2 + u_2^{(4)}u_1^{(3)} + u_1^{(4)}u_2^{(3)} - u_3^{(3)} + u_3^{(4)}\right),$$

$$\eta_2 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_2^{(3)} - u_1^{(3)}u_3^{(3)} - \left(u_2^{(3)}\right)^2 + u_1^{(4)}u_1^{(3)}u_2^{(3)} + u_2^{(4)}u_2^{(3)} + u_1^{(4)}u_3^{(3)} - u_4^{(3)} + u_4^{(4)}\right),$$

$$\eta_3 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_3^{(3)} - u_1^{(3)}u_4^{(3)} - u_2^{(3)}u_3^{(3)} + u_1^{(4)}u_1^{(3)}u_3^{(3)} + u_2^{(4)}u_3^{(3)} + u_1^{(4)}u_4^{(3)} - u_5^{(3)} + u_5^{(4)}\right),$$

$$\eta_4 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_4^{(3)} - u_1^{(3)}u_5^{(3)} - u_2^{(3)}u_4^{(3)} + u_1^{(4)}u_1^{(3)}u_4^{(3)} + u_2^{(4)}u_4^{(3)} + u_1^{(4)}u_5^{(3)} + u_6^{(4)}\right),$$

$$\eta_5 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_5^{(3)} - u_2^{(3)}u_5^{(3)} + u_1^{(4)}u_1^{(3)}u_5^{(3)} + u_2^{(4)}u_5^{(3)} + u_7^{(4)}\right),$$

Q–Conditional symmetries of five coupled Burgers–like equations

and the functions $u_\alpha^{(4)}(t,x)$ $(\alpha = 1,\dots,7)$ satisfy the system

$$\mathbf{\Delta}_7 \equiv \begin{cases} u^{(4)}_{1,t} + u^{(4)}_1 u^{(4)}_{1,x} - u^{(4)}_{1,xx} + u^{(4)}_{2,x} = 0, \\ u^{(4)}_{2,t} + u^{(4)}_2 u^{(4)}_{1,x} - u^{(4)}_{2,xx} + u^{(4)}_{3,x} = 0, \\ u^{(4)}_{3,t} + u^{(4)}_3 u^{(4)}_{1,x} - u^{(4)}_{3,xx} + u^{(4)}_{4,x} = 0, \\ u^{(4)}_{4,t} + u^{(4)}_4 u^{(4)}_{1,x} - u^{(4)}_{4,xx} + u^{(4)}_{5,x} = 0, \\ u^{(4)}_{5,t} + u^{(4)}_5 u^{(4)}_{1,x} - u^{(4)}_{5,xx} + u^{(4)}_{6,x} = 0, \\ u^{(4)}_{6,t} + u^{(4)}_6 u^{(4)}_{1,x} - u^{(4)}_{6,xx} + u^{(4)}_{7,x} = 0, \\ u^{(4)}_{7,t} + u^{(4)}_7 u^{(4)}_{1,x} - u^{(4)}_{7,xx} = 0. \end{cases}$$

About the proof

The proof requires only straightforward though lengthy computations!

Hierarchy originating from Burgers' equation

Summary results

- The Q–conditional symmetries of the Burgers' equation are expressed in terms of three functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_3$ made of three coupled Burgers–like equations;
- the Q–conditional symmetries of $\boldsymbol{\Delta}_3$ are expressed in terms of five functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_5$ made of five coupled Burgers–like equations;
- the Q–conditional symmetries of $\boldsymbol{\Delta}_5$ are expressed in terms of seven functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_7$ made of seven coupled Burgers–like equations.

First Conjecture

It seems natural to conjecture that repeatedly searching for Q–conditional symmetries, and starting from the classical Burgers' equation, a hierarchy of systems made of an odd number of Burgers–like equations may arise.

Problem

What happens if we consider coupled systems made of an even number of Burgers–like equations?

Q–Conditional symmetries of two coupled Burgers–like equations

Consider as starting point the following system made by two coupled Burgers–like equations

$$\boldsymbol{\Delta}_2 \equiv \begin{cases} u^{(1)}_{1,t} + u^{(1)}_1 u^{(1)}_{1,x} - u^{(1)}_{1,xx} + u^{(1)}_{2,x} = 0, \\ u^{(1)}_{2,t} + u^{(1)}_2 u^{(1)}_{1,x} - u^{(1)}_{2,xx} = 0. \end{cases}$$

By looking for Q–conditional symmetries of $\boldsymbol{\Delta}_2$ in correspondence to the vector field

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u^{(1)}_1} + \eta_2 \frac{\partial}{\partial u^{(1)}_2},$$

and solving the determining equations arising from

$$\Xi^{(2)}(\boldsymbol{\Delta}_2)\Big|_{\mathcal{M}_2} = \mathbf{0},$$

where $\mathcal{M}_2$ is the manifold of the jet space defined by the system $\boldsymbol{\Delta}_2$ together with the invariant surface conditions and their differential consequences. . .

Q–Conditional symmetries of two coupled Burgers–like equations

...we obtain:

$$\xi = \frac{1}{2}\left(-u_1^{(1)} + u_1^{(2)}\right),$$
$$\eta_1 = \frac{1}{4}\left(-\left(u_1^{(1)}\right)^3 - 2u_1^{(1)}u_2^{(1)} + u_1^{(2)}\left(u_1^{(1)}\right)^2 + u_2^{(2)}u_1^{(1)} + u_1^{(2)}u_2^{(1)} + u_3^{(2)}\right),$$
$$\eta_2 = \frac{1}{4}\left(-\left(u_1^{(1)}\right)^2 u_2^{(1)} - \left(u_2^{(1)}\right)^2 + u_1^{(2)}u_1^{(1)}u_2^{(1)} + u_2^{(2)}u_2^{(1)} + u_4^{(2)}\right),$$

where the functions $u_\alpha^{(2)}(t,x)$ $(\alpha = 1,\dots,4)$ satisfy the constraints

$$\boldsymbol{\Delta}_4 \equiv \begin{cases} u_{1,t}^{(2)} + u_1^{(2)}u_{1,x}^{(2)} - u_{1,xx}^{(2)} + u_{2,x}^{(2)} = 0, \\ u_{2,t}^{(2)} + u_2^{(2)}u_{1,x}^{(2)} - u_{2,xx}^{(2)} + u_{3,x}^{(2)} = 0, \\ u_{3,t}^{(2)} + u_3^{(2)}u_{1,x}^{(2)} - u_{3,xx}^{(2)} + u_{4,x}^{(2)} = 0, \\ u_{4,t}^{(2)} + u_4^{(2)}u_{1,x}^{(2)} - u_{4,xx}^{(2)} = 0. \end{cases}$$

From 2 to 4?

Repeating the same algorithm for a system of four coupled Burgers–like equations, the admitted Q–conditional symmetries are expressed in terms of six arbitrary functions depending on t and x.

Q–Conditional symmetries of four coupled Burgers–like equations

Proposition

The system $\boldsymbol{\Delta}_4$ admits the vector field Ξ of the Q–conditional symmetries, say

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{4} \eta_\alpha \frac{\partial}{\partial u_\alpha^{(2)}},$$

where

$$\xi = \frac{1}{2}\left(-u_1^{(2)} + u_1^{(3)}\right),$$

$$\eta_1 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^3 - 2u_1^{(2)}u_2^{(2)} + u_1^{(3)}\left(u_1^{(2)}\right)^2 + u_2^{(3)}u_1^{(2)} + u_1^{(3)}u_2^{(2)} - u_3^{(2)} + u_3^{(3)}\right),$$

$$\eta_2 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_2^{(2)} - u_1^{(2)}u_3^{(2)} - \left(u_2^{(2)}\right)^2 + u_1^{(3)}u_1^{(2)}u_2^{(2)} + u_2^{(3)}u_2^{(2)} + u_1^{(3)}u_3^{(2)} - u_4^{(2)} + u_4^{(3)}\right),$$

$$\eta_3 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_3^{(2)} - u_1^{(2)}u_4^{(2)} - u_2^{(2)}u_3^{(2)} + u_1^{(3)}u_1^{(2)}u_3^{(2)} + u_2^{(3)}u_3^{(2)} + u_1^{(3)}u_4^{(2)} + u_5^{(3)}\right),$$

$$\eta_4 = \frac{1}{4}\left(-\left(u_1^{(2)}\right)^2 u_4^{(2)} - u_2^{(2)}u_4^{(2)} + u_1^{(3)}u_1^{(2)}u_4^{(2)} + u_2^{(3)}u_4^{(2)} + u_6^{(3)}\right),$$

Q–Conditional symmetries of four coupled Burgers–like equations

and the functions $u_\alpha^{(3)}(t,x)$ $(\alpha = 1,\dots,6)$ satisfy the system

$$\boldsymbol{\Delta}_6 \equiv \begin{cases} u^{(3)}_{1,t} + u^{(3)}_1 u^{(3)}_{1,x} - u^{(3)}_{1,xx} + u^{(3)}_{2,x} = 0, \\ u^{(3)}_{2,t} + u^{(3)}_2 u^{(3)}_{1,x} - u^{(3)}_{2,xx} + u^{(3)}_{3,x} = 0, \\ u^{(3)}_{3,t} + u^{(3)}_3 u^{(3)}_{1,x} - u^{(3)}_{3,xx} + u^{(3)}_{4,x} = 0, \\ u^{(3)}_{4,t} + u^{(3)}_4 u^{(3)}_{1,x} - u^{(3)}_{4,xx} + u^{(3)}_{5,x} = 0, \\ u^{(3)}_{5,t} + u^{(3)}_5 u^{(3)}_{1,x} - u^{(3)}_{5,xx} + u^{(3)}_{6,x} = 0, \\ u^{(3)}_{6,t} + u^{(3)}_6 u^{(3)}_{1,x} - u^{(3)}_{6,xx} = 0. \end{cases}$$

From 4 to 6?

We can repeat the same procedure for the system $\boldsymbol{\Delta}_6$ made of six coupled Burgers–like equations!

Q–Conditional symmetries of six coupled Burgers–like equations

Proposition

The system $\boldsymbol{\Delta}_6$ admits the vector field Ξ of the Q–conditional symmetries, say

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{6} \eta_\alpha \frac{\partial}{\partial u_\alpha^{(3)}}, \qquad \text{where}$$

$$\xi = \frac{1}{2}\left(-u_1^{(3)} + u_1^{(4)}\right),$$

$$\eta_1 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^3 - 2u_1^{(3)}u_2^{(3)} + u_1^{(4)}\left(u_1^{(3)}\right)^2 + u_2^{(4)}u_1^{(3)} + u_1^{(4)}u_2^{(3)} - u_3^{(3)} + u_3^{(4)}\right),$$

$$\eta_2 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_2^{(3)} - u_1^{(3)}u_3^{(3)} - \left(u_2^{(3)}\right)^2 + u_1^{(4)}u_1^{(3)}u_2^{(3)} + u_2^{(4)}u_2^{(3)} + u_1^{(4)}u_3^{(3)} - u_4^{(3)} + u_4^{(4)}\right),$$

$$\eta_3 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_3^{(3)} - u_1^{(3)}u_4^{(3)} - u_2^{(3)}u_3^{(3)} + u_1^{(4)}u_1^{(3)}u_3^{(3)} + u_2^{(4)}u_3^{(3)} + u_1^{(4)}u_4^{(3)} - u_5^{(3)} + u_5^{(4)}\right),$$

$$\eta_4 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_4^{(3)} - u_1^{(3)}u_5^{(3)} - u_2^{(3)}u_4^{(3)} + u_1^{(4)}u_1^{(3)}u_4^{(3)} + u_2^{(4)}u_4^{(3)} + u_1^{(4)}u_5^{(3)} - u_6^{(3)} + u_6^{(4)}\right),$$

$$\eta_5 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_5^{(3)} - u_1^{(3)}u_6^{(3)} - u_2^{(3)}u_5^{(3)} + u_1^{(4)}u_1^{(3)}u_5^{(3)} + u_2^{(4)}u_5^{(3)} + u_1^{(4)}u_6^{(3)} + u_7^{(4)}\right),$$

$$\eta_6 = \frac{1}{4}\left(-\left(u_1^{(3)}\right)^2 u_6^{(3)} - u_2^{(3)}u_6^{(3)} + u_1^{(4)}u_1^{(3)}u_6^{(3)} + u_2^{(4)}u_6^{(3)} + u_8^{(4)}\right),$$

Q–Conditional symmetries of six coupled Burgers–like equations

... Proposition

and the functions $u^{(4)}_\alpha(t,x)$ $(\alpha = 1,\dots,8)$ satisfy the system

$$\mathbf{\Delta}_8 \equiv \begin{cases} u^{(4)}_{1,t} + u^{(4)}_1 u^{(4)}_{1,x} - u^{(4)}_{1,xx} + u^{(4)}_{2,x} = 0, \\ u^{(4)}_{2,t} + u^{(4)}_2 u^{(4)}_{1,x} - u^{(4)}_{2,xx} + u^{(4)}_{3,x} = 0, \\ u^{(4)}_{3,t} + u^{(4)}_3 u^{(4)}_{1,x} - u^{(4)}_{3,xx} + u^{(4)}_{4,x} = 0, \\ u^{(4)}_{4,t} + u^{(4)}_4 u^{(4)}_{1,x} - u^{(4)}_{4,xx} + u^{(4)}_{5,x} = 0, \\ u^{(4)}_{5,t} + u^{(4)}_5 u^{(4)}_{1,x} - u^{(4)}_{5,xx} + u^{(4)}_{6,x} = 0, \\ u^{(4)}_{6,t} + u^{(4)}_6 u^{(4)}_{1,x} - u^{(4)}_{6,xx} + u^{(4)}_{7,x} = 0, \\ u^{(4)}_{7,t} + u^{(4)}_7 u^{(4)}_{1,x} - u^{(4)}_{7,xx} + u^{(4)}_{8,x} = 0, \\ u^{(4)}_{8,t} + u^{(4)}_8 u^{(4)}_{1,x} - u^{(4)}_{8,xx} = 0. \end{cases}$$

Hierarchy originating from a pair of coupled Burgers–like equations

Summary results

- The Q–conditional symmetries of the system $\boldsymbol{\Delta}_2$ made by two coupled Burgers–like equations are expressed in terms of four functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_4$ made of four coupled Burgers–like equations;
- the Q–conditional symmetries of $\boldsymbol{\Delta}_4$ are expressed in terms of six functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_6$ made of six coupled Burgers–like equations;
- the Q–conditional symmetries of $\boldsymbol{\Delta}_6$ are expressed in terms of eight functions representing arbitrary solutions of the system $\boldsymbol{\Delta}_8$ made of eight coupled Burgers–like equations.

Second Conjecture

These results suggest to conjecture that repeatedly searching for Q–conditional symmetries, and starting from a pair of coupled Burgers–like equations, a hierarchy of systems made of an even number of coupled Burgers–like equations arises.
Indeed, the two conjectures can be unified and proved to be true!

The general hierarchy of Burgers–like equations

Theorem

Let m be a positive integer, and let $k = \lceil m/2 \rceil$. The system of Burgers–like equations

$$\boldsymbol{\Delta}_m \equiv \begin{cases} u^{(k)}_{\alpha,t} + u^{(k)}_{\alpha} u^{(k)}_{1,x} - u^{(k)}_{\alpha,xx} + u^{(k)}_{\alpha+1,x} = 0, \\ u^{(k)}_{m,t} + u^{(k)}_{m} u^{(k)}_{1,x} - u^{(k)}_{m,xx} = 0, \end{cases}$$

with $\alpha = 1, \dots, m-1$, admits the Q–conditional symmetries generated by

$$\Xi = \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \eta_\alpha \frac{\partial}{\partial u^{(k)}_\alpha},$$

where

$$\xi = \frac{1}{2}\left(-u^{(k)}_1 + u^{(k+1)}_1\right),$$

$$\eta_\alpha = \frac{1}{4}\left(-\left(u^{(k)}_1\right)^2 u^{(k)}_\alpha - u^{(k)}_1 u^{(k)}_{\alpha+1} - u^{(k)}_2 u^{(k)}_\alpha + u^{(k+1)}_1 u^{(k)}_1 u^{(k)}_\alpha \right.$$
$$\left. + u^{(k+1)}_2 u^{(k)}_\alpha + u^{(k+1)}_1 u^{(k)}_{\alpha+1} - u^{(k)}_{\alpha+2} + u^{(k+1)}_{\alpha+2}\right), \qquad \alpha = 1, \dots, m-2,$$

The general hierarchy of Burgers–like equations

. . . Theorem

$$\eta_{m-1} = \frac{1}{4}\Bigg(-\left(u_1^{(k)}\right)^2 u_{m-1}^{(k)} - u_1^{(k)}u_m^{(k)} - u_2^{(k)}u_{m-1}^{(k)} + u_1^{(k+1)}u_1^{(k)}u_{m-1}^{(k)} + u_2^{(k+1)}u_{m-1}^{(k)} + u_1^{(k+1)}u_m^{(k)} + u_{m+1}^{(k+1)}\Bigg),$$

$$\eta_m = \frac{1}{4}\left(-\left(u_1^{(k)}\right)^2 u_m^{(k)} - (1-\delta_{1m})u_2^{(k)}u_m^{(k)} + u_1^{(k+1)}u_1^{(k)}u_m^{(k)} + u_2^{(k+1)}u_m^{(k)} + u_{m+2}^{(k+1)}\right),$$

provided that the functions $u_\alpha^{(k+1)}(t,x)$ satisfy the system

$$\mathbf{\Delta}_{m+2} \equiv \begin{cases} u_{\alpha,t}^{(k+1)} + u_\alpha^{(k+1)}u_{1,x}^{(k+1)} - u_{\alpha,xx}^{(k+1)} + u_{\alpha+1,x}^{(k+1)} = 0, \\ u_{m+2,t}^{(k+1)} + u_{m+2}^{(k+1)}u_{1,x}^{(k+1)} - u_{m+2,xx}^{(k+1)} = 0, \end{cases}$$

with $\alpha = 1, \dots, m+1$.

Remark

Note that if m is odd (even, respectively), a hierarchy of systems with an odd (even, respectively) number of equations is generated.

Hierarchy of of Burgers–like equations: Lie point symmetries

Proposition

Let m be a positive integer, and let $k = \lceil m/2 \rceil$. The system of Burgers–like equations

$$\boldsymbol{\Delta}_m \equiv \begin{cases} u^{(k)}_{\alpha,t} + u^{(k)}_{\alpha} u^{(k)}_{1,x} - u^{(k)}_{\alpha,xx} + u^{(k)}_{\alpha+1,x} = 0, \quad \alpha = 1, \dots, m-1, \\ u^{(k)}_{m,t} + u^{(k)}_{m} u^{(k)}_{1,x} - u^{(k)}_{m,xx} = 0, \end{cases}$$

for $m = 1$ (classical Burgers' equation) admits the Lie point symmetries generated by:

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \\ &\Xi_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u^{(1)}_1 \frac{\partial}{\partial u^{(1)}_1}, \\ &\Xi_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u^{(1)}_1}, \\ &\Xi_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tu^{(1)}_1)\frac{\partial}{\partial u^{(1)}_1}; \end{aligned}$$

Hierarchy of of Burgers–like equations: Lie point symmetries

for $m = 2$ the Lie point symmetries generated by:

$$
\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x},\\
\Xi_3 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u_1^{(1)}\frac{\partial}{\partial u_1^{(1)}} - 2u_2^{(1)}\frac{\partial}{\partial u_2^{(1)}},\\
\Xi_4 &= t\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u_1^{(1)}} - u_1^{(1)}\frac{\partial}{\partial u_2^{(1)}},\\
\Xi_5 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (2x - tu_1^{(1)})\frac{\partial}{\partial u_1^{(1)}} - (xu_1^{(1)} + 2tu_2^{(1)} + 2)\frac{\partial}{\partial u_2^{(1)}};
\end{aligned}
$$

Hierarchy of of Burgers–like equations: Lie point symmetries

for $m \geq 3$ the Lie point symmetries generated by:

$$
\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x},\\
\Xi_3 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \sum_{\alpha=1}^{m} \alpha u_\alpha^{(k)} \frac{\partial}{\partial u_\alpha^{(k)}},\\
\Xi_4 &= t\frac{\partial}{\partial x} + m\frac{\partial}{\partial u_1^{(k)}} + \sum_{\alpha=2}^{m} (\alpha - m - 1) u_{\alpha-1}^{(k)} \frac{\partial}{\partial u_\alpha^{(k)}},\\
\Xi_5 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(mx - tu_1^{(k)}\right)\frac{\partial}{\partial u_1^{(k)}} - \left((m-1)(xu_1^{(k)} + m) + 2tu_2^{(k)}\right)\frac{\partial}{\partial u_2^{(k)}}\\
&\quad - \sum_{\alpha=3}^{m} \left(\alpha t u_\alpha^{(k)} + (m-\alpha+1)(xu_{\alpha-1}^{(k)} - (m-\alpha+2)u_{\alpha-2}^{(k)})\right)\frac{\partial}{\partial u_\alpha^{(k)}}.
\end{aligned}
$$

Whatever the number m of coupled equations is, we have always a five–dimensional Lie algebra (time and space translation, scaling, Galilean and projective transformation, respectively): these Lie algebras, although realized in terms of vector fields on manifolds with different dimensionality, share the same structure constants and so they are all isomorphic.

Further investigation

Question 1

Are the elements of this hierarchy integrable systems?

Question 2

Are there other evolution equations able to produce hierarchies by repeatedly looking for conditional symmetries?

References

M. Gorgone, F. Oliveri and E. Sgroi (2024)
Hierarchy of coupled Burgers–like equations induced by conditional symmetries.
Submitted.

THANKS