Perturbed PDEs: approximate symmetries and conservation laws

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Applied Mathematics Seminars

Messina, February 22–23, 2024

Classical Lie symmetries

Considering an r–th order system of DEs

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}\right) = \mathbf{0},$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ and $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ denote the independent and dependent variables, whereas $\mathbf{u}^{(r)}$ are the derivatives of $\mathbf{u}$ w.r.t. $\mathbf{x}$ up to the order r.

A Lie point symmetry is characterized by the Lie generator

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}$$

such that

$$\Xi^{(r)}(\boldsymbol{\Delta})\Big|_{\boldsymbol{\Delta}=\mathbf{0}} = \mathbf{0},$$

where $\Xi^{(r)}$ is the *rth* order prolongation.

Question

What happens with perturbed PDEs?

Perturbed PDEs and Lie symmetries

Consider a system of differential equations of order r involving a small parameter $\varepsilon \ll 1$,

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}; \varepsilon\right) = \mathbf{0}.$$

By looking for classical Lie point symmetries, in general, it is not guaranteed that the infinitesimal generators depend on the parameter ε. Nevertheless, it is not uncommon that this system possesses few symmetries compared with the unperturbed system

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}; 0\right) = 0.$$

The applicability of Lie group methods is limited!

Example

The Korteweg–deVries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits a four–dimensional Lie algebra of exact point symmetries spanned by the vector fields:

$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad \Xi_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}.$$

On the contrary, by considering the Korteweg–deVries–Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0,$$

we lose the scaling group and have only three symmetries:

$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Remark

Differential equations containing small terms are commonly and successfully investigated by means of perturbative techniques!

Approximate symmetry theories

Baikov, Gazizov, Ibragimov, Mat. Sb., 1988

Considering a system of differential equations involving a small parameter

$$\boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\widetilde{\boldsymbol{\Delta}}_{(k)}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=O(\varepsilon^{p+1}),$$

the Lie generator is expanded in a perturbation series:

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial u_\alpha}\equiv\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}\widetilde{\xi}_{(k)i}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\widetilde{\eta}_{(k)\alpha}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_\alpha}\right).$$

Then, the approximate invariance is defined:

$$\Xi^{(r)}(\boldsymbol{\Delta})\Big|_{\boldsymbol{\Delta}=O(\varepsilon^{p+1})}=O(\varepsilon^{p+1}).$$

- Pros: quite elegant theory, since all the useful properties of exact Lie symmetries are moved to the approximate world;
- Cons: the expanded generator is not consistent with principles of perturbation analysis since the dependent variables are not expanded!

Approximate symmetry theories

Fushchich and Shtelen, J. Phys. A., 1989

The dependent variables are expanded in a perturbation series as done in usual perturbation analysis:

$$\mathbf{u}(\mathbf{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathbf{u}_{(k)}(\mathbf{x}) + O(\varepsilon^{p+1});$$

by separating at each order of approximation, a coupled system to be solved in hierarchy is obtained:

$$\widetilde{\boldsymbol{\Delta}}_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}_{(0)}^{(r)}, \dots, \mathbf{u}_{(k)}, \mathbf{u}_{(k)}^{(r)}\right) = 0, \qquad k = 0, \dots, p.$$

Approximate symmetries of the original DE defined as the *exact symmetries* of the DE obtained from perturbations!

- Pros: approach with a simple and coherent basis.
- Cons: a lot of algebra (especially for higher-order perturbations) is required; the basic assumption of a fully coupled system is too strong, since the equations at a level should not be influenced by those at higher levels. No possibility to work in a hierarchy!

A consistent approach[1]

Consider DEs containing a small term ε,

$$\mathbf{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)=\mathbf{0},$$

and take a Lie generator with infinitesimals depending on ε,

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial u_\alpha}.$$

Expand the dependent variables in power series of ε

$$\mathbf{u}(\mathbf{x};\varepsilon)=\sum_{k=0}^{p}\varepsilon^k\mathbf{u}_{(k)}(\mathbf{x})+O(\varepsilon^{p+1}),$$

whereupon DEs write as

$$\mathbf{\Delta}\equiv\sum_{k=0}^{p}\varepsilon^k\widetilde{\mathbf{\Delta}}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}_{(0)}^{(r)},\dots,\mathbf{u}_{(k)},\mathbf{u}_{(k)}^{(r)}\right)=O(\varepsilon^{p+1}).$$

[1] Di Salvo, Gorgone, Oliveri, Nonlinear Dyn., 2018

Expansions of infinitesimals

$$\xi_i \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\xi}_{(k)i}, \qquad \eta_\alpha \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\eta}_{(k)\alpha},$$

where $\widetilde{\xi}_{(k)i}$ and $\widetilde{\eta}_{(k)\alpha}$ $(k > 0)$ are suitable polynomials in $\mathbf{u}_{(1)}, \dots, \mathbf{u}_{(k)}$ with coefficients given by $\xi_{(j)i}\left(\mathbf{x}, \mathbf{u}_{(0)}\right)$ and $\eta_{(j)\alpha}\left(\mathbf{x}, \mathbf{u}_{(0)}\right)$ $(j = 0, \dots, p)$ and their derivatives with respect to $\mathbf{u}_{(0)}$.
More precisely:

$$\begin{aligned} \widetilde{\xi}_{(0)i} &= \xi_{(0)i} = \xi_i(\mathbf{x}, \mathbf{u}_{(0)}; 0), & \widetilde{\eta}_{(0)\alpha} &= \eta_{(0)\alpha} = \eta_\alpha(\mathbf{x}, \mathbf{u}_{(0)}; 0), \\ \widetilde{\xi}_{(k+1)i} &= \frac{1}{k+1}\mathcal{R}[\widetilde{\xi}_{(k)i}], & \widetilde{\eta}_{(k+1)\alpha} &= \frac{1}{k+1}\mathcal{R}[\widetilde{\eta}_{(k)\alpha}], \end{aligned}$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives and such that

$$\begin{aligned} &\mathcal{R}\left[\frac{\partial^{|\tau|} f_{(k)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right] = \frac{\partial^{|\tau|} f_{(k+1)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}} + \sum_{i=1}^{m} \frac{\partial}{\partial u_{(0)i}} \left(\frac{\partial^{|\tau|} f_{(k)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right) u_{(1)i}, \\ &\mathcal{R}[u_{(k)j}] = (k+1) u_{(k+1)j}, \end{aligned}$$

where $k \geq 0$, $j = 1, \dots, m$, $|\tau| = \tau_1 + \dots + \tau_m$.

...

We get the approximate Lie generator

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right).$$

Then define prolongations in the usual way (*i.e.*, preserving contact conditions) and impose the approximate invariance conditions:

$$\sum_{k=0}^{p} \varepsilon^k \sum_{\ell=0}^{k} \widetilde{\Xi}^{(r)}_{(\ell)} \widetilde{\mathbf{\Delta}}_{(k-\ell)} \Bigg|_{\mathbf{\Delta}=O(\varepsilon^{p+1})} = O(\varepsilon^{p+1}).$$

Computational cost

The consistent approach requires more computations than that required for determining exact Lie symmetries; nevertheless, there is the general and freely available package ReLie[a] able to do automatically all the needed work.

[a] Oliveri, Symmetry, 2021

Example

For $p = 1$, the approximate Lie generator reads

$$
\begin{aligned}
\Xi \approx & \sum_{i=1}^{n} \left(\widetilde{\xi}_{(0)i}(\mathbf{x}, \mathbf{u}_{(0)}) + \varepsilon \widetilde{\xi}_{(1)i}(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}_{(1)})\right) \frac{\partial}{\partial x_i} \\
& + \sum_{\alpha=1}^{m} \left(\widetilde{\eta}_{(0)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}) + \varepsilon \widetilde{\eta}_{(1)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}_{(1)})\right) \frac{\partial}{\partial u_\alpha} \\
& = \sum_{i=1}^{n} \left(\xi_{(0)i} + \varepsilon \left(\xi_{(1)i} + \sum_{\beta=1}^{m} \frac{\partial \xi_{(0)i}}{\partial u_{(0)\beta}} u_{(1)\beta}\right)\right) \frac{\partial}{\partial x_i} \\
& + \sum_{\alpha=1}^{m} \left(\eta_{(0)\alpha} + \varepsilon \left(\eta_{(1)\alpha} + \sum_{\beta=1}^{m} \frac{\partial \eta_{(0)\alpha}}{\partial u_{(0)\beta}} u_{(1)\beta}\right)\right) \frac{\partial}{\partial u_\alpha},
\end{aligned}
$$

where $\xi_{(0)i}$, $\xi_{(1)i}$, $\eta_{(0)\alpha}$ and $\eta_{(1)\alpha}$ depend on $(\mathbf{x}, \mathbf{u}_{(0)})$.

Remarks

- The Lie generator $\widetilde{\Xi}_{(0)}$ is always a symmetry of the unperturbed equations ($\varepsilon = 0$); the correction terms $\displaystyle\sum_{k=1}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$ give the deformation of the symmetry due to the terms involving ε.
- Not all the symmetries of the unperturbed equations are admitted as the zeroth terms of the approximate symmetries; the symmetries of the unperturbed equations that are the zeroth terms of the approximate symmetries are called stable symmetries.
- If Ξ is the generator of an approximate Lie point symmetry of a differential equation, $\varepsilon\Xi$ is a generator of an approximate Lie point symmetry too, but the converse is not true in general.
- The approximate Lie point symmetries of a DE are the elements of an approximate Lie algebra.

Approximate symmetries of Korteweg–deVries–Burgers equation

Consider the Korteweg–deVries equation perturbed with the addition of a small dissipative term,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0.$$

The first order approximate symmetries are spanned by the following vector fields:

$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$
$$\Xi_4 = \varepsilon\frac{\partial}{\partial t}, \qquad \Xi_5 = \varepsilon\frac{\partial}{\partial x}, \qquad \Xi_6 = \varepsilon\left(t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$
$$\Xi_7 = \varepsilon\left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u_0\frac{\partial}{\partial u}\right).$$

Approximate symmetries of Korteweg–deVries–Burgers equation

Consider the Korteweg–deVries equation perturbed with the addition of a small dissipative term,

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The first order approximate symmetries are spanned by the following vector fields:

$$\begin{aligned}
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&\Xi_4 = \varepsilon\frac{\partial}{\partial t}, \qquad \Xi_5 = \varepsilon\frac{\partial}{\partial x}, \qquad \Xi_6 = \varepsilon\left(t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),\\
&\Xi_7 = \varepsilon\left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u_0\frac{\partial}{\partial u}\right).
\end{aligned}$$

Approximate invariant solutions

Approximate invariant solutions can be found requiring that

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}\widetilde{\xi}_{(k)i}(\mathbf{x},\mathbf{u}_{(0)},\dots,\mathbf{u}_{(k)})\frac{\partial \mathbf{u}}{\partial x_i} - \widetilde{\boldsymbol{\eta}}_{(k)}(\mathbf{x},\mathbf{u}_{(0)},\dots,\mathbf{u}_{(k)})\right) = O(\varepsilon^{p+1}).$$

Application: approximate invariant solutions

Consider the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(u^2 \frac{\partial u}{\partial x}\right) + \varepsilon \frac{\partial u}{\partial t} = 0.$$

Baikov-Gazizov-Ibragimov

The first order approximate symmetries are generated by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x},\\
&\Xi_3 = \left(t + \varepsilon \frac{t^2}{6}\right)\frac{\partial}{\partial t} - \left(u + \varepsilon \frac{tu}{3}\right)\frac{\partial}{\partial u},\\
&\Xi_4 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\\
&\Xi_5 = \varepsilon \Xi_1, \quad \Xi_6 = \varepsilon \Xi_2, \quad \Xi_7 = \varepsilon \Xi_3, \quad \Xi_8 = \varepsilon \Xi_4.
\end{aligned}$$

The approximate invariant solutions w.r.t. Ξ_3 are

$$u(t,x) = \pm\left(\frac{x}{t} - \varepsilon \frac{x}{6}\right).$$

Consistent approach

The first order approximate symmetries are generated by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x},\\
&\Xi_3 = \left(t + \varepsilon \frac{t^2}{6}\right)\frac{\partial}{\partial t} - \left(u_0 + \varepsilon \left(u_1 + \frac{tu_0}{3}\right)\right)\frac{\partial}{\partial u},\\
&\Xi_4 = x\frac{\partial}{\partial x} + (u_0 + \varepsilon u_1)\frac{\partial}{\partial u},\\
&\Xi_5 = \varepsilon \Xi_1, \quad \Xi_6 = \varepsilon \Xi_2, \quad \Xi_7 = \varepsilon \Xi_3, \quad \Xi_8 = \varepsilon \Xi_4.
\end{aligned}$$

The approximate invariant solutions w.r.t. Ξ_3 are

$$u(t,x) = \pm\frac{x}{t} + \varepsilon\left(\frac{k_1}{tx^3} + \frac{k_2}{t} \mp \frac{x}{6}\right).$$

Applications of the new consistent approach

The consistent approach has been applied to:

- lower the order or solve by quadrature, for ODEs, and determine approximate invariant solutions, for ODEs and PDEs:
  - Di Salvo, Gorgone, Oliveri, Nonlinear Dynamics, 2018;
  - Gorgone, International Journal of Non-Linear Mechanics, 2018;
- find approximate conditional symmetries (and determine the corresponding approximate invariant solutions):
  - Gorgone, Oliveri, Electronic Journal of Differential Equations, 2018;
  - Gorgone, Oliveri, Zeitschrift für Angewandte Mathematik und Physik, 2021;
- find approximate Noether symmetries and derive approximate conservation laws:
  - Gorgone, Oliveri, Mathematics, 2021;
- derive approximate conservation laws by means of a direct approach:
  - Gorgone, Inferrera, European Physical Journal Plus, 2023.

Approximate conservation laws

Definition

Given a system of DEs of order r involving a small parameter $\varepsilon \ll 1$

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}; \varepsilon\right) = \mathbf{0},$$

an approximate conservation law of order r, compatible with the system, is a divergence expression

$$\sum_{i=1}^{n} D_i\left(\Phi^i\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r-1)}; \varepsilon\right)\right) = O(\varepsilon^{p+1}),$$

holding for all solutions of the system, where $\Phi^i\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r-1)}; \varepsilon\right)$ are the fluxes of the approximate conservation law, and D_i is the Lie derivative.

Unperturbed variational problems

The determination of CLaws is ruled by Noether's theorem, establishing a correspondence between symmetries of the action integral and conservation laws through an explicit formula involving the infinitesimals and the Lagrangian itself. The same can be done in the approximate framework!

Approximate conservation law: consistent definition

Given a system

$$\boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\widetilde{\boldsymbol{\Delta}}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}_{(0)}^{(r)},\dots,\mathbf{u}_{(k)},\mathbf{u}_{(k)}^{(r)}\right)=O(\varepsilon^{p+1}),$$

an approximate conservation law of order r, compatible with the system, is an approximate divergence expression

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}D_i\left(\widetilde{\Phi}^i_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}_{(0)}^{(r-1)},\dots,\mathbf{u}_{(k)},\mathbf{u}_{(k)}^{(r-1)}\right)\right)\right)=O(\varepsilon^{p+1}),$$

holding for all solutions of the system, where

$$\sum_{k=0}^{p}\varepsilon^k\widetilde{\Phi}^i_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}_{(0)}^{(r-1)},\dots,\mathbf{u}_{(k)},\mathbf{u}_{(k)}^{(r-1)}\right),\qquad i=1,\dots,n$$

are the expansions at order p of the fluxes $\Phi^i\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r-1)};\varepsilon\right)$ of the conservation law, and

$$D_i=\frac{D}{Dx_i}=\frac{\partial}{\partial x_i}+\sum_{k=0}^{p}\sum_{\alpha=1}^{m}\left(u_{(k)\alpha,i}\frac{\partial}{\partial u_{(k)\alpha}}+\sum_{j=1}^{n}u_{(k)\alpha,ij}\frac{\partial}{\partial u_{(k)\alpha,j}}+\dots\right)$$

is the approximate Lie derivative, with $u_{(k)\alpha,i}=\dfrac{\partial u_{(k)\alpha}}{\partial x_i}$, $u_{(k)\alpha,ij}=\dfrac{\partial^2 u_{(k)\alpha}}{\partial x_i\partial x_j}$, ...

Perturbed variational problems

Perturbed first order Lagrangian function and Lagrangian action

$$\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\varepsilon\right)\equiv\mathcal{L}_0\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(1)}_{(0)}\right)+\sum_{k=1}^{p}\varepsilon^k\mathcal{L}_k\left(\mathbf{x},\mathbf{u}_{(0)},\ldots,\mathbf{u}_{(k)},\mathbf{u}^{(1)}_{(0)},\ldots,\mathbf{u}^{(1)}_{(k)}\right)+O(\varepsilon^{p+1})$$

$$\mathcal{J}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\varepsilon\right)=\int_\Omega\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\varepsilon\right)d\mathbf{x}\equiv\int_\Omega\left(\sum_{k=0}^{p}\varepsilon^k\mathcal{L}_k\left(\mathbf{x},\mathbf{u}_{(0)},\ldots,\mathbf{u}_{(k)},\mathbf{u}^{(1)}_{(0)},\ldots,\mathbf{u}^{(1)}_{(k)}\right)\right)d\mathbf{x}+O(\varepsilon^{p+1})$$

Approximate Euler–Lagrange equations

By requiring $\delta\mathcal{J}=O(\varepsilon^{p+1})$ under variations of order $O(\varepsilon^{p+1})$ at the boundary of Ω, we obtain

$$\sum_{k=0}^{p}\varepsilon^k\left(\frac{\partial\mathcal{L}_k}{\partial u_{(0)\alpha}}-\sum_{i=1}^{n}D_i\left(\frac{\partial\mathcal{L}_k}{\partial u_{(0)\alpha,i}}\right)\right)=O(\varepsilon^{p+1}),\qquad\alpha=1,\ldots,m.$$

Approximate Noether theorem[2]

Let us consider a variational system of DEs arising from a first order perturbed Lagrangian function.
The generator

$$\Xi = \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right).$$

of an approximate Lie symmetry leaves the Lagrangian action approximately invariant if

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{j=0}^{k} \left(\widetilde{\Xi}^{(1)}_{(j)} \mathcal{L}_{k-j} + \mathcal{L}_{k-j} \sum_{i=1}^{n} D_i \widetilde{\xi}_{(j)i} \right) - \sum_{i=1}^{n} D_i \phi^i_{(k)} \right) = O(\varepsilon^{p+1}),$$

with $\phi^i_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}\right)$ $(i = 1, \dots, n)$ functions to be suitably determined.
Then, we obtain the approximate conservation law

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} D_i \widetilde{\Phi}^i_{(k)} \right) = O(\varepsilon^{p+1}),$$

where

$$\widetilde{\Phi}^i_{(k)} = \sum_{\ell=0}^{k} \left(\sum_{\alpha=1}^{m} \left(\left(\widetilde{\eta}_{(\ell)\alpha} - \sum_{j=1}^{n} \widetilde{\xi}_{(\ell)j} u_{(\ell)\alpha,j} \right) \sum_{q=0}^{k-\ell} \frac{\partial \mathcal{L}_{k-\ell}}{\partial u_{(q)\alpha,i}} \right) + \widetilde{\xi}_{(\ell)i} \mathcal{L}_{k-\ell} \right) - \phi^i_{(k)}.$$

[2] Gorgone, Oliveri, Mathematics, 2021

The planar three–body problem

Motion equations:

$$\ddot{r}_1 + Gm_2\frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathbf{r}_{13}}{|\mathbf{r}_{13}|^3} = \mathbf{0},$$
$$\ddot{r}_2 - Gm_1\frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathbf{r}_{23}}{|\mathbf{r}_{23}|^3} = \mathbf{0},$$
$$\ddot{r}_3 - Gm_1\frac{\mathbf{r}_{13}}{|\mathbf{r}_{13}|^3} - Gm_2\frac{\mathbf{r}_{23}}{|\mathbf{r}_{23}|^3} = \mathbf{0},$$

with $\mathbf{r}_i \equiv (x_i(t), y_i(t), 0)$ $(i = 1, 2, 3)$ position vectors of the three masses m_α in a fixed frame reference, and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r_j}$ $(1 \leq i < j \leq 3)$. The system arises from the Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\right) + \frac{Gm_1m_2}{|\mathbf{r}_{12}|} + \varepsilon\left(m_3\dot{\mathbf{r}}_3^2 + \frac{Gm_1m_3}{|\mathbf{r}_{13}|} + \frac{Gm_2m_3}{|\mathbf{r}_{23}|}\right).$$

By expanding the dependent variables at first order in ε, i.e.,

$$\mathbf{r}_i = \mathbf{r}_{(0)i} + \varepsilon\mathbf{r}_{(1)i} + O(\varepsilon^2) \equiv \left(x_{(0)i}(t) + \varepsilon x_{(1)i}(t) + O(\varepsilon^2), y_{(0)i}(t) + \varepsilon y_{(1)i}(t) + O(\varepsilon^2), 0\right),$$

along with

$$\mathbf{r}_{ij} = \mathbf{r}_{(0)ij} + \varepsilon\mathbf{r}_{(1)ij} + O(\varepsilon^2) = \mathbf{r}_{(0)i} - \mathbf{r}_{(0)j} + \varepsilon\left(\mathbf{r}_{(1)i} - \mathbf{r}_{(1)j}\right) + O(\varepsilon^2),$$

we are able to determine the approximate variational Lie symmetries, together with the corresponding approximate conserved quantities.

The planar three–body problem – Results

- From
$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \phi_{(0)} = \phi_{(1)} = 0,$$
we have the approximate conservation of total energy:
$$\Phi_1 = \frac{1}{2}\left(m_1 \dot{\mathbf{r}}^2_{(0)1} + m_2 \dot{\mathbf{r}}^2_{(0)2}\right) - \frac{G m_1 m_2}{|\mathbf{r}_{(0)12}|}$$
$$+\varepsilon\left(\frac{1}{2} m_3 \dot{\mathbf{r}}^2_{(0)3} + m_1 \dot{\mathbf{r}}_{(0)1}\cdot\dot{\mathbf{r}}_{(1)1} + m_2 \dot{\mathbf{r}}_{(0)2}\cdot\dot{\mathbf{r}}_{(1)2} - \frac{G m_1 m_3}{|\mathbf{r}_{(0)13}|} - \frac{G m_2 m_3}{|\mathbf{r}_{(0)23}|} + \frac{G m_1 m_2}{|\mathbf{r}_{(0)23}|^3}\mathbf{r}_{(0)12}\cdot\mathbf{r}_{(1)12}\right);$$
- From
$$\Xi_{2a} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}, \qquad \Xi_{2b} = \sum_{i=1}^{3} \frac{\partial}{\partial y_i}, \qquad \phi_{(0)} = \phi_{(1)} = 0,$$
we have the approximate conservation of total linear momentum:
$$\Phi_2 = m_1 \dot{\mathbf{r}}_{(0)1} + m_2 \dot{\mathbf{r}}_{(0)2} + \varepsilon\left(m_1 \dot{\mathbf{r}}_{(1)1} + m_2 \dot{\mathbf{r}}_{(1)2} + m_3 \dot{\mathbf{r}}_{(0)3}\right);$$

The planar three–body problem – Results

- From

$$\Xi_{3a} = t\sum_{i=1}^{3}\frac{\partial}{\partial x_i}, \qquad \phi_{(0)} = -\sum_{i=1}^{2} m_i x_{(0)i}, \quad \phi_{(1)} = -\sum_{i=1}^{2} m_i x_{(1)i} - m_3 x_{(0)3},$$

$$\Xi_{3b} = t\sum_{i=1}^{3}\frac{\partial}{\partial y_i}, \qquad \phi_{(0)} = -\sum_{i=1}^{2} m_i y_{(0)i}, \quad \phi_{(1)} = -\sum_{i=1}^{2} m_i y_{(1)i} - m_3 y_{(0)3},$$

we have

$$\begin{aligned}\Phi_3 = {} & m_1(t\dot{\mathbf{r}}_{(0)1} - \mathbf{r}_{(0)1}) + m_2(t\dot{\mathbf{r}}_{(0)2} - \mathbf{r}_{(0)2}) \\ & + \varepsilon\left(m_1(t\dot{\mathbf{r}}_{(1)1} - \mathbf{r}_{(1)1}) + m_2(t\dot{\mathbf{r}}_{(1)2} - \mathbf{r}_{(1)2}) + m_3(t\dot{\mathbf{r}}_{(0)3} - \mathbf{r}_{(0)3})\right),\end{aligned}$$

i.e., the approximate barycenter of the system has a uniform and rectilinear motion;

- From

$$\Xi_4 = \sum_{i=1}^{3}\left(\left(y_{(0)i} + \varepsilon y_{(1)i}\right)\frac{\partial}{\partial x_i} - \left(x_{(0)i} + \varepsilon x_{(1)i}\right)\frac{\partial}{\partial y_i}\right), \qquad \phi_{(0)} = \phi_{(1)} = 0,$$

we have the approximate conservation of total angular momentum:

$$\begin{aligned}\Phi_4 = {} & m_1\mathbf{r}_{(0)1}\wedge\dot{\mathbf{r}}_{(0)1} + m_2\mathbf{r}_{(0)2}\wedge\dot{\mathbf{r}}_{(0)2} \\ & + \varepsilon\left(m_1\left(\mathbf{r}_{(0)1}\wedge\dot{\mathbf{r}}_{(1)1} + \mathbf{r}_{(1)1}\wedge\dot{\mathbf{r}}_{(0)1}\right) + m_2\left(\mathbf{r}_{(0)2}\wedge\dot{\mathbf{r}}_{(1)2} + \mathbf{r}_{(1)2}\wedge\dot{\mathbf{r}}_{(0)2}\right) + m_3\mathbf{r}_{(0)3}\wedge\dot{\mathbf{r}}_{(0)3}\right).\end{aligned}$$

Perturbed non–variational problems: direct method[3]

Given a system of DEs,

$$\boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)=\mathbf{0},$$

we want to determine sets of non–singular (when evaluated on the solutions of the system) multipliers $\Lambda^{\nu}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)$ $(\nu=1,\dots,q)$ provided that

$$\sum_{\nu=1}^{q}\left(\Lambda^{\nu}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\Delta^{\nu}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\right)\equiv\sum_{i=1}^{n}D_i\left(\Phi^{i}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r-1)};\varepsilon\right)\right)=O(\varepsilon^{p+1})$$

is an approximate divergence expression holding for all solutions of the system.

Key aspects of the direct approach

- Any divergence expression is annihilated by the Euler operators associated to all dependent variables;
- All the sets of multipliers can be found algorithmically by solving a linear system of determining equations.

[3] Bluman, Anco, Eur. J. Appl. Math., 2002

Direct approaches to approximate conservation laws

First method: without expansion of dependent variables

Given a sistem of differential equations involving a small parameter

$$\Delta^\nu\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\Delta^\nu_{(k)}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=O(\varepsilon^{p+1}),\qquad \nu=1,\dots,q,$$

an expansion of the Lagrange multipliers is considered,

$$\Lambda^\nu\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\Lambda^\nu_{(k)}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right),\qquad \nu=1,\dots,q,$$

and the following Euler operator in the algorithmic procedure is used:

$$E_{u_\alpha}=\frac{\partial}{\partial u_\alpha}-\sum_{i=1}^{n}D_i\left(\frac{\partial}{\partial u_{\alpha,i}}\right)+\dots+(-1)^s\sum_{i_1=1}^{n}\dots\sum_{i_s=i_{s-1}}^{n}D_{i_1}\dots D_{i_s}\left(\frac{\partial}{\partial u_{\alpha,i_1\dots i_s}}\right),\quad \alpha=1,\dots,m.$$

Remark

This approach moves within the same framework of BGI method for approximate symmetries.

Direct approaches to approximate conservation laws

Second method: with expansion of dependent variables

Given a sistem of differential equations involving a small parameter

$$\Delta^{\nu}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)=\mathbf{0},\qquad \nu=1,\dots,q,\qquad (*)$$

the dependent variables are expanded in a perturbation series as done in usual perturbation analysis:

$$\mathbf{u}^{(r)}(\mathbf{x};\varepsilon)=\sum_{k=0}^{p}\varepsilon^{k}\mathbf{u}^{(r)}_{(k)}(\mathbf{x})+O(\varepsilon^{p+1}),$$

with $\mathbf{u}^{(r)}_{(k)}\equiv\left(u^{(r)}_{(k)1},\dots,u^{(r)}_{(k)N}\right)$.

By separating at each order of approximation, we have a coupled system to be solved in a hierarchy:

$$\Delta^{\nu}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(r)}_{(0)},\dots,\mathbf{u}_{(k)},\mathbf{u}^{(r)}_{(k)}\right)=0,\qquad k=0,\dots,p,\qquad \nu=1,\dots,q.\qquad (**)$$

Approaches to approximate conservation laws

. . . Multipliers and Euler operators

The approximate multipliers of system $(*)$ are defined as the exact multipliers

$$\Lambda^{\nu}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(r)}_{(0)},\dots,\mathbf{u}_{(p)},\mathbf{u}^{(r)}_{(p)}\right),\qquad k=0,\dots,p,\qquad \nu=1,\dots,q,$$

of system $(**)$ obtained from perturbations.
The following Euler operator is considered:

$$E_{u_{(k)\alpha}}=\frac{\partial}{\partial u_{(k)\alpha}}-\sum_{i=1}^{n}D_i\left(\frac{\partial}{\partial u_{(k)\alpha,i}}\right)+\dots+(-1)^s\sum_{i_1=1}^{n}\dots\sum_{i_s=i_{s-1}}^{n}D_{i_1}\dots D_{i_s}\left(\frac{\partial}{\partial u_{(k)\alpha,i_1\dots i_s}}\right),$$

with $k=0,\dots,p$ and $\alpha=1,\dots,m$.

Remark

This approach moves within the same framework of FS method for approximate symmetries.

Aim

We use a method that, besides being coherent with perturbation analysis, does not require a huge computational cost. Essentially, we combine the direct procedure with the consistent approach to approximate Lie symmetries.

Main ingredients of the approximate direct method

- Expand the dependent variables in power series of ε.
- Assume the Lagrange multipliers to be dependent on the small parameter ε, *i.e.*,

$$\Lambda^{\nu} = \Lambda^{\nu}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}; \varepsilon\right), \qquad \nu = 1, \dots, q,$$

  and consider a consistent expansion.
- Define approximate multipliers.
- Use a consistent definition of approximate Euler operators.

Expansions of multipliers

$$\Lambda^{\nu}\left(\mathbf{x},\mathbf{u}^{(r)};\varepsilon\right)=\sum_{k=0}^{p}\varepsilon^{k}\widetilde{\Lambda}^{\nu}_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)},\dots,\mathbf{u}^{(r)}_{(k)}\right)+O(\varepsilon^{p+1}),\qquad \nu=1,\dots,q$$

where $\widetilde{\Lambda}^{\nu}_{(k)}$ $(k>0)$ are suitable polynomials in $\mathbf{u}^{(r)}_{(1)},\dots,\mathbf{u}^{(r)}_{(k)}$ with coefficients given by $\Lambda^{\nu}_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)$ $(k=0,\dots,p)$ and their derivatives with respect to $\mathbf{u}^{(r)}_{(0)}$.

Expansions of multipliers

$$\Lambda^\nu\left(\mathbf{x},\mathbf{u}^{(r)};\varepsilon\right)=\sum_{k=0}^{p}\varepsilon^k\widetilde{\Lambda}^\nu_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)},\dots,\mathbf{u}^{(r)}_{(k)}\right)+O(\varepsilon^{p+1}),\qquad \nu=1,\dots,q$$

where $\widetilde{\Lambda}^\nu_{(k)}$ $(k>0)$ are suitable polynomials in $\mathbf{u}^{(r)}_{(1)},\dots,\mathbf{u}^{(r)}_{(k)}$ with coefficients given by $\Lambda^\nu_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)$ $(k=0,\dots,p)$ and their derivatives with respect to $\mathbf{u}^{(r)}_{(0)}$.

In fact:

$$\widetilde{\Lambda}^\nu_{(0)}=\Lambda^\nu_{(0)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)=\Lambda^\nu\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)};0\right),\qquad \widetilde{\Lambda}^\nu_{(k+1)}=\frac{1}{k+1}\mathcal{R}[\widetilde{\Lambda}^\nu_{(k)}],$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives defined as

$$\mathcal{R}\left[\frac{\partial^{|\tau|}\Lambda^\nu_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}\right]=\frac{\partial^{|\tau|}\Lambda^\nu_{(k+1)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}+\sum_{i=1}^{N}\frac{\partial}{\partial u^{(r)}_{(0)i}}\left(\frac{\partial^{|\tau|}\Lambda^\nu_{(k)}\left(\mathbf{x},\mathbf{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}\right)u^{(r)}_{(1)i},$$

$$\mathcal{R}[u^{(r)}_{(k)j}]=(k+1)u^{(r)}_{(k+1)j},$$

where $k\geq 0$, $j=1,\dots,N$, $|\tau|=\tau_1+\dots+\tau_N$.

Approximate multipliers

Functions $\Lambda^\nu(\mathbf{x},\mathbf{u},\mathbf{u}^{(s)};\varepsilon)$ $(\nu=1,\dots,q)$ are approximate multipliers depending on s–th order derivatives if, after expanding in pertubation series of ε up to the order p, *i.e.*,

$$\Lambda^\nu\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(s)};\varepsilon\right)=\sum_{k=0}^{p}\varepsilon^k\widetilde{\Lambda}^\nu_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(s)}_{(0)},\dots,\mathbf{u}_{(k)},\mathbf{u}^{(s)}_{(k)}\right)+O(\varepsilon^{p+1}),\qquad \nu=1,\dots,q,$$

the relation

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right)-\sum_{i=1}^{n}D_i\widetilde{\Phi}^i_{(k)}\right)=O(\varepsilon^{p+1})$$

holds for arbitrary $\mathbf{u}^{(s)}_{(\ell)}(\mathbf{x})$ and some suitable functions $\widetilde{\Phi}^i_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(s-1)}_{(0)},\dots,\mathbf{u}_{(k)},\mathbf{u}^{(s-1)}_{(k)}\right)$.

Then, if $\Lambda^\nu\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(s)};\varepsilon\right)$ are non–singular, an approximate conservation law can be recovered:

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right)\right)\equiv\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}D_i\widetilde{\Phi}^i_{(k)}\right)=O(\varepsilon^{p+1}).$$

Theorem

The non–singular approximate multipliers

$$\Lambda^{\nu}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}; \varepsilon\right), \qquad \nu = 1, \dots, q$$

yield an approximate conservation law iff the set of relations

$$E_{u_{(0)\alpha}}\left(\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^{\nu}_{(\ell)}\widetilde{\Delta}^{\nu}_{(k-\ell)}\right)\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha = 1, \dots, m$$

holds for arbitrary $\mathbf{u}^{(r)}_{(k)}(\mathbf{x})$ $(k = 0, \dots, p)$, where

$$E_{u_{(0)\alpha}} = \frac{\partial}{\partial u_{(0)\alpha}} - \sum_{i=1}^{n} D_i\left(\frac{\partial}{\partial u_{(0)\alpha,i}}\right) + \dots + (-1)^r \sum_{i_1=1}^{n} \dots \sum_{i_r=i_{r-1}}^{n} D_{i_1}\dots D_{i_r}\left(\frac{\partial}{\partial u_{(0)\alpha,i_1\dots i_r}}\right)$$

are the approximate Euler operators.

Approximate direct method with the consistent approach: algorithm

- Expand the dependent variables in power series of ε: $\mathbf{u}(\mathbf{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathbf{u}_{(k)}(\mathbf{x}) + O(\varepsilon^{p+1})$;
- Expand in perturbation series of ε the multipliers, so obtaining:

$$\Lambda^\nu\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(s)};\varepsilon\right) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^\nu_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}^{(s)}_{(0)},\dots,\mathbf{u}_{(k)},\mathbf{u}^{(s)}_{(k)}\right) + O(\varepsilon^{p+1}), \qquad \nu = 1,\dots,q;$$

- Apply the approximate Euler operators, *i.e.*,

$$E_{u_{(0)\alpha}}\left(\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right)\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha = 1,\dots,m;$$

- Separate the resulting conditions at each order in ε, and split into an overdetermined system for the unknown approximate multipliers;
- Insert the recovered approximate multipliers in

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right) - \sum_{i=1}^{n} D_i\widetilde{\Phi}^i_{(k)}\right) = O(\varepsilon^{p+1})$$

and, if possible, find the approximate fluxes.

Approximate direct method – Applications

By means of the approximate direct method, approximate conservation laws have been determined for:

- Perturbed Van der Pool equation:

$$\ddot{u} + u - \varepsilon\left(1 - u^2\right)\dot{u} = 0;$$

- Perturbed KdV–Burgers equation:

$$u_{,t} + uu_{,x} + u_{,xxx} - \varepsilon u_{,xx} = 0;$$

- A perturbed nonlinear wave equation:

$$u_{,xx} - \frac{1}{c^2}u_{,tt} - \lambda u^3 - \varepsilon f(u) = 0;$$

- Two perturbed nonlinear Schrödinger equations:

$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon|p|^4 p = 0;$$

$$\mathrm{i}p_{,t} + \frac{1}{2}p_{,xx} + |p|^2 p + \mathrm{i}\varepsilon\left(\beta_1 p_{,xxx} + \beta_2|p|^2 p_{,x} + \beta_3 p(|p|^2)_{,x}\right) = 0;$$

- The generalized Kaup–Newell equation:

$$u_{,t} - \frac{1}{2}u_{,xx} + uvu_{,x} + \frac{1}{2}u^2 v_{,x} + 2\varepsilon uu_{,x} = 0,$$

$$v_{,t} + \frac{1}{2}v_{,xx} + uvv_{,x} + \frac{1}{2}v^2 u_{,x} + 2\varepsilon(vu_{,x} + uv_{,x}) = 0.$$

Perturbed nonlinear second order Schrödinger equation:

$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon |p|^4 p = 0,$$

with $p \equiv p(t, x; \varepsilon)$ the complex-valued envelope of the wave. By decomposing into real and imaginary parts:

$$\begin{aligned}\Delta^1 =& u_{,t} + v_{,xx} + 2v(u^2+v^2) - \varepsilon v \left(u^2+v^2\right)^2 = 0,\\ \Delta^2 =& v_{,t} - u_{,xx} - 2u(u^2+v^2) + \varepsilon u \left(u^2+v^2\right)^2 = 0.\end{aligned}$$

Expand $u(t, x; \varepsilon)$ and $v(t, x; \varepsilon)$ at first order in ε and look for the approximate multipliers of the form

$$\Lambda^\nu = \Lambda^\nu_{(0)} + \varepsilon \left(\Lambda^\nu_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0)}} u_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0)}} v_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0),x}} u_{(1),x} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0),x}} v_{(1),x} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0),xx}} u_{(1),xx} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0),xx}} v_{(1),xx}\right),$$

where $\Lambda^\nu_{(k)} \equiv \Lambda^\nu_{(k)}\left(t, x, u_{(0)}, v_{(0)}, u_{(0),x}, v_{(0),x}, u_{(0),xx}, v_{(0),xx}\right)$ $(k = 0, 1$ and $\nu = 1, 2)$.
By solving the approximate determining equations

$$E_{u_{(0)}}\left(\Lambda^1\Delta^1 + \Lambda^2\Delta^2\right) = 0, \qquad E_{v_{(0)}}\left(\Lambda^1\Delta^1 + \Lambda^2\Delta^2\right) = 0,$$

where $E_{w_{(0)}} = \frac{\partial}{\partial w_{(0)\alpha}} - D_t\left(\frac{\partial}{\partial w_{(0),t}}\right) - D_x\left(\frac{\partial}{\partial w_{(0),x}}\right) + D_t D_t\left(\frac{\partial}{\partial w_{(0),tt}}\right) + D_t D_x\left(\frac{\partial}{\partial w_{(0),tx}}\right) + D_x D_x\left(\frac{\partial}{\partial w_{(0),xx}}\right)$,

we obtain the sets of approximate multipliers with the corresponding approximate conservation laws.

- $$\Lambda_1^1 = v_{(0),xx} + 2v_{(0)}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon\left(v_{(1),xx} - v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(u_{(0)}^2 v_{(1)} + 2u_{(0)}u_{(1)}v_{(0)} + 3v_{(0)}^2 v_{(1)})\right),$$
$$\Lambda_1^2 = -u_{(0),xx} - 2u_{(0)}(u_{(0)}^2 + v_{(0)}^2) - \varepsilon\left(u_{(1),xx} - u_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(v_{(0)}^2 u_{(1)} + 2u_{(0)}v_{(0)}v_{(1)} + 3u_{(0)}^2 u_{(1)})\right),$$
with
$$\begin{aligned} & D_t\left(\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 - (u_{(0)}^2 + v_{(0)}^2)^2\right) + \varepsilon\left(\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),x}\right.\right. \\ & \left.\left.+\left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),x} - 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right) \\ & +D_x\left(-(u_{(0),t}u_{(0),x} + v_{(0),t}v_{(0),x}) - \varepsilon\left(u_{(0),x}u_{(1),t} + v_{(0),x}v_{(1),t}\right.\right. \\ & \left.\left.+\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),t} + \left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),t}\right)\right) = 0. \end{aligned}$$

- $$\Lambda_2^1 = u_{(0),x} + \varepsilon u_{(1),x}, \qquad \Lambda_2^2 = v_{(0),x} + \varepsilon v_{(1),x},$$
with
$$\begin{aligned} & D_t\left(v_{(0)}u_{(0),x} + \varepsilon\left((tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),x} + tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),x} + v_{(0)}u_{(1),x}\right)\right) \\ & +D_x\left(-\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 + (u_{(0)}^2 + v_{(0)}^2)^2\right) - v_{(0)}u_{(0),t}\right. \\ & -\varepsilon\left(u_{(0),x}u_{(1),x} + v_{(0),x}v_{(1),x} + (tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),t}\right. \\ & \left.\left.+ tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),t} + v_{(0)}u_{(1),t} + 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right) = 0. \end{aligned}$$

- $$\Lambda_3^1 = 2tu_{(0),x} + xv_{(0)} + \varepsilon\left(2tu_{(1),x} + xv_{(1)}\right), \qquad \Lambda_3^2 = 2tv_{(0),x} - xu_{(0)} + \varepsilon\left(2tv_{(1),x} - xu_{(1)}\right),$$

  with

$$\begin{aligned}
&D_t\Big(2tv_{(0)}u_{(0),x} + \frac{x}{2}(u_{(0)}^2 + v_{(0)}^2)\\
&+ \varepsilon\left(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),x} + t^2 v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),x} + 2tv_{(0)}u_{(1),x} + x(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\Big)\\
+&D_t\Big(-t\left(u_{(0),x}^2 + v_{(0),x}^2 + 2v_{(0)}u_{(0),t} + (u_{(0)}^2 + v_{(0)}^2)^2\right) - x(v_{(0)}u_{(0),x} - u_{(0)}v_{(0),x}) - u_{(0)}v_{(0)}\\
&- \varepsilon\Big(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),t} + (2tu_{(1),x} + xv_{(1)})u_{(0),x} + (2tv_{(1),x} - xu_{(1)})v_{(0),x}\\
&+ t^2 v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),t} + x(v_{(0)}u_{(1),x} - u_{(0)}v_{(1),x})\\
&+ 2tv_{(0)}u_{(1),t} + 4t(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)}) + u_{(0)}v_{(1)} + v_{(0)}u_{(1)}\Big)\Big) = 0.
\end{aligned}$$

- $$\Lambda_4^1 = v_{(0)} + \varepsilon v_{(1)}, \qquad \Lambda_4^2 = -u_{(0)} - \varepsilon u_{(1)},$$

  with

$$\begin{aligned}
&D_t\left(\frac{1}{2}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\\
+&D_x\left(u_{(0)}v_{(0),x} - v_{(0)}u_{(0),x} + \varepsilon(u_{(1)}v_{(0),x} - v_{(1)}u_{(0),x} + u_{(0)}v_{(1),x} - v_{(0)}u_{(1),x})\right) = 0.
\end{aligned}$$

Work in progress

Extensions of the consistent approach to:

- derive local transformations (suggested by the approximate symmetries) mapping differential equations to approximately equivalent ones;
- define approximate equivalence transformations for classes of differential equations involving small terms;
- include multiple scales in the independent variables in order to avoid the occurrence of secular-like terms in the solutions.


NOT SURE IF THEY'RE CLAPPING FOR MY PRESENTATION
OR BECAUSE ITS FINISHED