Decoupling of First Order Quasilinear Systems

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Problem [Courant]

When can an autonomous and homogeneous quasilinear first order system of PDEs

$$\frac{\partial u_\ell}{\partial x_1} = \sum_{m=1}^{n} A_m^\ell(u_1, \ldots, u_n) \frac{\partial u_m}{\partial x_2} \qquad (\ell = 1, \ldots, n)$$

be decoupled — in some new coordinates $v_1(\mathbf{u}), \ldots, v_n(\mathbf{u})$ — into k non-interacting subsystems

$$\frac{\partial v_{m_j+i}}{\partial x_1} = \sum_{\ell=1}^{n_j} \tilde{A}_{m_j+\ell}^{m_j+i}(v_{m_j+1}, \ldots, v_{m_j+n_j}) \frac{\partial v_{m_j+\ell}}{\partial x_2}$$

$$(j = 1, \ldots, k, \quad i = 1, \ldots, n_j, \quad m_j = n_1 + \ldots + n_{j-1})$$

of some orders $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$?

Necessary and Sufficient Conditions

Theorem [Nijenhuis]

The necessary and sufficient condition for the complete decoupling of the system

$$\frac{\partial u_\ell}{\partial x_1} = \sum_{m=1}^{n} A^\ell_m(u_1, \dots, u_n) \frac{\partial u_m}{\partial x_2} \qquad (\ell = 1, \dots, n)$$

into n non-interacting one-dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor

$$N^j_{ik} = A^\alpha_i \frac{\partial A^j_k}{\partial u_\alpha} - A^\alpha_k \frac{\partial A^j_i}{\partial u_\alpha} + A^j_\alpha \left(\frac{\partial A^\alpha_i}{\partial u_k} - \frac{\partial A^\alpha_k}{\partial u_i} \right),$$

provided that all eigenvalues of $A^i_j(\mathbf{u})$ are real and distinct.

General Decoupling Problem

Let us consider an hyperbolic system of PDEs

$$\frac{\partial \mathbf{u}}{\partial x_1} + p(\mathbf{x}, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \qquad (\textit{source system})$$

where $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^m$, $p \in M_{m,m}(\mathbb{R})$, $\mathbf{g} \in \mathbb{R}^m$, $p_{ij}(\mathbf{x}, \mathbf{u})$ and $g_i(\mathbf{x}, \mathbf{u})$ functions of the indicated arguments.

Through an invertible transformation like

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{u} = \mathbf{h}(\mathbf{x}, \mathbf{U}),$$

or, equivalently,

$$\mathbf{x} = \mathbf{z}(\mathbf{X}), \qquad \mathbf{U} = \mathbf{H}(\mathbf{x}, \mathbf{u}),$$

the quasilinear form is preserved, and we obtain

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U})\frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}) \qquad (\textit{target system})$$

Transformation

By defining

$$J = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} \end{bmatrix},$$

we get

$$P = (\nabla_{\mathbf{U}} \mathbf{h})^{-1} (J_{11} I + J_{12} p)^{-1} (J_{21} I + J_{22} p) \nabla_{\mathbf{U}} \mathbf{h},$$

$$\mathbf{G} = (\nabla_{\mathbf{U}} \mathbf{h})^{-1} (J_{11} I + J_{12} p)^{-1} \left(\mathbf{g} - \frac{\partial \mathbf{h}}{\partial x_1} - p \frac{\partial \mathbf{h}}{\partial x_2} \right) =$$

$$= \frac{1}{\det J} \left((J_{22} I - J_{12} P) \left((\nabla_{\mathbf{u}} \mathbf{H}) \cdot \mathbf{g} + \frac{\partial \mathbf{H}}{\partial x_1} \right) - (J_{21} I - J_{11} P) \frac{\partial \mathbf{H}}{\partial x_2} \right),$$

$$\mathbf{L}^{(i)} = \mathbf{l}^{(i)} \cdot (\nabla_{\mathbf{u}} \mathbf{H})^{-1}, \qquad \mathbf{R}^{(i)} = \nabla_{\mathbf{u}} \mathbf{H} \cdot \mathbf{r}^{(i)}, \qquad \Lambda_i = \frac{J_{21} + J_{22} \lambda_i}{J_{11} + J_{12} \lambda_i},$$

I identity matrix, $\mathbf{l}^{(i)}$ and $\mathbf{r}^{(i)}$ left and right eigenvector of p corresponding to the eigenvalue λ_i, $\mathbf{L}^{(i)}$ and $\mathbf{R}^{(i)}$ left and right eigenvector of P corresponding to the eigenvalue Λ_i

Case $m = 2$ (two dependent variables)

By considering the target system

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}),$$

we have two main cases:

1. **PARTIAL DECOUPLING**

$$P_{12} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial G_1}{\partial U_2} = 0$$

2. **FULL DECOUPLING**

$$P_{12} = P_{21} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial P_{22}}{\partial U_1} = \frac{\partial G_1}{\partial U_2} = \frac{\partial G_2}{\partial U_1} = 0$$

Case $m=2$

1. PARTIAL DECOUPLING

- $P_{12}=0 \Longrightarrow R_1^{(2)}=\nabla_{\mathbf{u}} H_1 \cdot \mathbf{r}^{(2)}=0;$
- Since $R_1^{(2)}=0$ means $\frac{\partial \mathbf{h}}{\partial U_2} \parallel \mathbf{r}^{(2)}$ and $P_{11}=\Lambda_1=\dfrac{J_{21}+J_{22}\lambda_1}{J_{11}+J_{12}\lambda_1}$,

$$\frac{\partial P_{11}}{\partial U_2}=\frac{\partial \Lambda_1}{\partial U_2}=0 \Longrightarrow \nabla_{\mathbf{u}}\lambda_1 \cdot \mathbf{r}^{(2)}=0;$$

- Since

$$G_1=\frac{(J_{11}P_{11}-J_{21})\frac{\partial H_1}{\partial x_2}+(J_{22}-J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1}+g_2\frac{\partial H_1}{\partial u_2}+\frac{\partial H_1}{\partial x_1})}{\det J},$$

$$\frac{\partial G_1}{\partial U_2}=0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}} H_1 \cdot \mathbf{g}+\frac{\partial H_1}{\partial x_1}+\lambda_1\frac{\partial H_1}{\partial x_2}\right)\cdot \mathbf{r}^{(2)}=0.$$

Case $m = 2$

1. PARTIAL DECOUPLING CONDITIONS

$$\begin{cases} \nabla_{\mathbf{u}}\lambda_1 \cdot \mathbf{r}^{(2)} = 0, \\ \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_1 \cdot \mathbf{g} + \dfrac{\partial H_1}{\partial x_1} + \lambda_1 \dfrac{\partial H_1}{\partial x_2}\right) \cdot \mathbf{r}^{(2)} = 0, \\ \nabla_{\mathbf{u}}H_1 \cdot \mathbf{r}^{(2)} = 0. \end{cases}$$

$$U_1 = H_1(x_1, x_2, u_1, u_2)$$

Case $m = 2$

2. FULL DECOUPLING

- $P_{12} = 0 \Longrightarrow R_1^{(2)} = \nabla_{\mathbf{u}} H_1 \cdot \mathbf{r}^{(2)} = 0;$
- $P_{21} = 0 \Longrightarrow R_2^{(1)} = \nabla_{\mathbf{u}} H_2 \cdot \mathbf{r}^{(1)} = 0;$
- Since $R_1^{(2)} = R_2^{(1)} = 0$ mean $\frac{\partial \mathbf{h}}{\partial U_1} \parallel \mathbf{r}^{(1)}$ and $\frac{\partial \mathbf{h}}{\partial U_2} \parallel \mathbf{r}^{(2)}$,

$$P_{11} = \Lambda_1 = \frac{J_{21} + J_{22}\lambda_1}{J_{11} + J_{12}\lambda_1} \text{ and } P_{22} = \Lambda_2 = \frac{J_{21} + J_{22}\lambda_2}{J_{11} + J_{12}\lambda_2},$$

$$\frac{\partial P_{11}}{\partial U_2} = \frac{\partial \Lambda_1}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_1 \cdot \mathbf{r}^{(2)} = 0;$$

$$\frac{\partial P_{22}}{\partial U_1} = \frac{\partial \Lambda_2}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_2 \cdot \mathbf{r}^{(1)} = 0;$$

Case $m = 2$

2. FULL DECOUPLING

- Since

$$G_1 = \frac{(J_{11}P_{11} - J_{21})\frac{\partial H_1}{\partial x_2} + (J_{22} - J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1} + g_2\frac{\partial H_1}{\partial u_2} + \frac{\partial H_1}{\partial x_1})}{\det J},$$

$$G_2 = \frac{(J_{11}P_{22} - J_{21})\frac{\partial H_2}{\partial x_2} + (J_{22} - J_{12}P_{22})(g_1\frac{\partial H_2}{\partial u_1} + g_2\frac{\partial H_2}{\partial u_2} + \frac{\partial H_2}{\partial x_1})}{\det J},$$

$$\frac{\partial G_1}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_1 \cdot \mathbf{g} + \frac{\partial H_1}{\partial x_1} + \lambda_1\frac{\partial H_1}{\partial x_2}\right) \cdot \mathbf{r}^{(2)} = 0,$$

$$\frac{\partial G_2}{\partial U_1} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_2 \cdot \mathbf{g} + \frac{\partial H_2}{\partial x_1} + \lambda_2\frac{\partial H_2}{\partial x_2}\right) \cdot \mathbf{r}^{(1)} = 0.$$

Case $m = 2$

2. FULL DECOUPLING CONDITIONS

$$\begin{cases} \nabla_{\mathbf{u}} \lambda_i \cdot \mathbf{r}^{(j)} = 0, \\ \nabla_{\mathbf{u}} \left(\nabla_{\mathbf{u}} H_i \cdot \mathbf{g} + \dfrac{\partial H_i}{\partial x_1} + \lambda_i \dfrac{\partial H_i}{\partial x_2} \right) \cdot \mathbf{r}^{(j)} = 0, \\ \nabla_{\mathbf{u}} H_i \cdot \mathbf{r}^{(j)} = 0, \qquad i, j = 1, 2 \qquad i \neq j \end{cases}$$

$$U_1 = H_1(x_1, x_2, u_1, u_2)$$
$$U_2 = H_2(x_1, x_2, u_1, u_2)$$

Case $m = 2$: Example

Let us consider a Galilean system, where u_1 is a velocity, of the form:

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} + u_1 \dfrac{\partial u_1}{\partial x_2} + f_1(u_2)\dfrac{\partial u_2}{\partial x_2} = g_1(u_2), \\ \dfrac{\partial u_2}{\partial x_1} + f_2(u_2)\dfrac{\partial u_1}{\partial x_2} + u_1 \dfrac{\partial u_2}{\partial x_2} = g_2(u_2), \end{cases}$$

f_i, g_i, $(i = 1, 2)$ functions of their arguments.
Integrating partial decoupling conditions, we get

$$\int \sqrt{\frac{f_1}{f_2}} du_2 = \sqrt{f_1 f_2} - k_1, \qquad g_1 = k_2 - \sqrt{\frac{f_1}{f_2}} g_2,$$

$$H_1 = H_1\left(u_1 + \int \sqrt{\frac{f_1}{f_2}} du_2\right),$$

k_1 and k_2 being arbitrary constants.

Case $m = 2$: Example

By choosing

$$U_1 = u_1 + \sqrt{f_1 f_2} - k_1, \qquad U_2 = u_2,$$

and replacing its inverse transformation in the source system, *i.e.*

$$u_1 = U_1 - \sqrt{f_1 f_2} + k_1, \qquad u_2 = U_2,$$

we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial U_1}{\partial X_1} + (U_1 + k_1)\dfrac{\partial U_1}{\partial X_2} = k_2, \\ \dfrac{\partial U_2}{\partial X_1} + f_2(U_2)\dfrac{\partial U_1}{\partial X_2} + (U_1 - 2\sqrt{f_1(U_2) f_2(U_2)} + k_1)\dfrac{\partial U_2}{\partial X_2} = g_2(U_2). \end{cases}$$

Case $m = 2$: Example

The system so considered, when

$$f_2(u_2) = u_2, \qquad g_1(u_2) = g_2(u_2) = 0,$$

represents the one–dimensional Euler equations of a barotropic fluid with density u_2 and pressure

$$p(u_2) = \frac{1}{u_2} \int f_1(u_2)\, du_2.$$

The constitutive law

$$p(u_2) = \frac{k}{3} u_2^3, \qquad k \text{ constant}$$

allows to satisfy the decoupling condition and, in the new variables

$$U_1 = u_1 + \sqrt{k} u_2, \qquad U_2 = u_2,$$

we have a partially decoupled system

$$\frac{\partial U_1}{\partial X_1} + U_1 \frac{\partial U_1}{\partial X_2} = 0, \qquad \frac{\partial U_2}{\partial X_1} + U_2 \frac{\partial U_1}{\partial X_2} + (U_1 - 2\sqrt{k} U_2) \frac{\partial U_2}{\partial X_2} = 0.$$

Case $m=3$ (three dependent variables)

PARTIAL DECOUPLING – 1a

$$P_{13}=P_{23}=\frac{\partial P_{11}}{\partial U_3}=\frac{\partial P_{12}}{\partial U_3}=\frac{\partial P_{21}}{\partial U_3}=\frac{\partial P_{22}}{\partial U_3}=\frac{\partial G_1}{\partial U_3}=\frac{\partial G_2}{\partial U_3}=0$$

PARTIAL DECOUPLING – 1b

$$P_{31}=P_{32}=\frac{\partial P_{33}}{\partial U_1}=\frac{\partial P_{33}}{\partial U_2}=\frac{\partial G_3}{\partial U_1}=\frac{\partial G_3}{\partial U_2}=0$$

FULL DECOUPLING IN TWO SUBSYSTEMS

$$P_{13}=P_{23}=\frac{\partial P_{11}}{\partial U_3}=\frac{\partial P_{12}}{\partial U_3}=\frac{\partial P_{21}}{\partial U_3}=\frac{\partial P_{22}}{\partial U_3}=\frac{\partial G_1}{\partial U_3}=\frac{\partial G_2}{\partial U_3}=0$$

$$P_{31}=P_{32}=\frac{\partial P_{33}}{\partial U_1}=\frac{\partial P_{33}}{\partial U_2}=\frac{\partial G_3}{\partial U_1}=\frac{\partial G_3}{\partial U_2}=0$$

Case $m = 3$

1. PARTIAL DECOUPLING – 1a

- $P_{13} = 0 \Longrightarrow R_1^{(3)} = \nabla_{\mathbf{u}} H_1 \cdot \mathbf{r}^{(3)} = 0;$
- $P_{23} = 0 \Longrightarrow R_2^{(3)} = \nabla_{\mathbf{u}} H_2 \cdot \mathbf{r}^{(3)} = 0;$
- $\dfrac{\partial P_{11}}{\partial U_3} = \dfrac{\partial P_{12}}{\partial U_3} = \dfrac{\partial P_{21}}{\partial U_3} = \dfrac{\partial P_{22}}{\partial U_3} = 0$ are equivalent to
$$\frac{\partial \Lambda_1}{\partial U_3} = \frac{\partial \Lambda_2}{\partial U_3} = \partial_{U_3}\left(\frac{R_2^{(1)}}{R_1^{(1)}}\right) = \partial_{U_3}\left(\frac{R_1^{(2)}}{R_2^{(2)}}\right) = 0;$$
Since $R_1^{(3)} = R_2^{(3)} = 0$ mean both $\dfrac{\partial \mathbf{h}}{\partial U_3} \parallel \mathbf{r}^{(3)}$ and
$$\nabla_{\mathbf{u}} H_k = \alpha_k(x_1, x_2, u_1, u_2)\mathbf{l}^{(1)} + \beta_k(x_1, x_2, u_1, u_2)\mathbf{l}^{(2)}, \quad k = 1, 2,$$
(α_k and β_k functions such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$)

Case $m=3$

1. PARTIAL DECOUPLING – 1a

we get

$$\frac{\partial \Lambda_1}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_1 \cdot \mathbf{r}^{(3)} = 0,$$
$$\frac{\partial \Lambda_2}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_2 \cdot \mathbf{r}^{(3)} = 0,$$
$$\partial_{U_3}\left(\frac{R_2^{(1)}}{R_1^{(1)}}\right) = 0 \Longrightarrow \mathbf{l}^{(2)} \cdot (\nabla_{\mathbf{u}} \mathbf{r}^{(3)} \cdot \mathbf{r}^{(1)} - \nabla_{\mathbf{u}} \mathbf{r}^{(1)} \cdot \mathbf{r}^{(3)}) = 0,$$
$$\partial_{U_3}\left(\frac{R_1^{(2)}}{R_2^{(2)}}\right) = 0 \Longrightarrow \mathbf{l}^{(1)} \cdot (\nabla_{\mathbf{u}} \mathbf{r}^{(3)} \cdot \mathbf{r}^{(2)} - \nabla_{\mathbf{u}} \mathbf{r}^{(2)} \cdot \mathbf{r}^{(3)}) = 0;$$

Case $m=3$

1. PARTIAL DECOUPLING – 1a

- Since

$$\mathbf{G} = \frac{1}{\det J}\Big((J_{22}I - J_{12}P)(\nabla_{\mathbf{u}}\mathbf{H})\cdot\mathbf{g} + (J_{22}I - J_{12}P)\frac{\partial \mathbf{H}}{\partial x_1} - (J_{21}I - J_{11}P)\frac{\partial \mathbf{H}}{\partial x_2}\Big),$$

$$\frac{\partial G_1}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_1\cdot\mathbf{g} + \frac{\partial H_1}{\partial x_1} + \lambda_1\frac{\partial H_1}{\partial x_2}\right)\cdot\mathbf{r}^{(3)} = 0,$$

$$\frac{\partial G_2}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_2\cdot\mathbf{g} + \frac{\partial H_2}{\partial x_1} + \lambda_2\frac{\partial H_2}{\partial x_2}\right)\cdot\mathbf{r}^{(3)} = 0.$$

Case $m = 3$

1. PARTIAL DECOUPLING CONDITIONS – 1a

$$
\begin{cases}
\nabla_{\mathbf{u}}\lambda_i \cdot \mathbf{r}^{(3)} = 0, \\
\mathbf{l}^{(i)} \cdot (\nabla_{\mathbf{u}}\mathbf{r}^{(3)} \cdot \mathbf{r}^{(j)} - \nabla_{\mathbf{u}}\mathbf{r}^{(j)} \cdot \mathbf{r}^{(3)}) = 0, \\
\nabla_{\mathbf{u}} \left(\nabla_{\mathbf{u}} H_i \cdot \mathbf{g} + \dfrac{\partial H_i}{\partial x_1} + \lambda_i \dfrac{\partial H_i}{\partial x_2} \right) \cdot \mathbf{r}^{(3)} = 0, \\
\nabla_{\mathbf{u}} H_i \cdot \mathbf{r}^{(3)} = 0, \qquad i, j = 1, 2 \qquad i \neq j
\end{cases}
$$

$$
\begin{aligned}
U_1 &= H_1(x_1, x_2, u_1, u_2, u_3) \\
U_2 &= H_2(x_1, x_2, u_1, u_2, u_3)
\end{aligned}
$$

Case $m = 3$

2. PARTIAL DECOUPLING – 1b

- $P_{31} = 0 \Longrightarrow R_3^{(1)} = \nabla_{\mathbf{u}} H_3 \cdot \mathbf{r}^{(1)} = 0;$
- $P_{32} = 0 \Longrightarrow R_3^{(2)} = \nabla_{\mathbf{u}} H_3 \cdot \mathbf{r}^{(2)} = 0;$
- Since $R_3^{(1)} = R_3^{(2)} = 0$ mean both $\frac{\partial \mathbf{h}}{\partial U_2}$ and $\frac{\partial \mathbf{h}}{\partial U_3}$ belong to the plane spanned by $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$, $P_{33} = \Lambda_3 = \dfrac{J_{21} + J_{22}\lambda_3}{J_{11} + J_{12}\lambda_3}$,

$$\frac{\partial P_{33}}{\partial U_1} = \frac{\partial \Lambda_3}{\partial U_1} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_3 \cdot \mathbf{r}^{(1)} = 0,$$
$$\frac{\partial P_{33}}{\partial U_2} = \frac{\partial \Lambda_3}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}} \lambda_3 \cdot \mathbf{r}^{(2)} = 0;$$

Case $m = 3$

2. PARTIAL DECOUPLING – 1b

Since

$$\mathbf{G} = \frac{1}{\det J}\Big((J_{22}I - J_{12}P)(\nabla_{\mathbf{u}}\mathbf{H}) \cdot \mathbf{g} + (J_{22}I - J_{12}P)\frac{\partial \mathbf{H}}{\partial x_1} - (J_{21}I - J_{11}P)\frac{\partial \mathbf{H}}{\partial x_2}\Big),$$

$$\frac{\partial G_3}{\partial U_1} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}} H_3 \cdot \mathbf{g} + \frac{\partial H_3}{\partial x_1} + \lambda_3 \frac{\partial H_3}{\partial x_2}\right) \cdot \mathbf{r}^{(1)} = 0,$$

$$\frac{\partial G_3}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}} H_3 \cdot \mathbf{g} + \frac{\partial H_3}{\partial x_1} + \lambda_3 \frac{\partial H_3}{\partial x_2}\right) \cdot \mathbf{r}^{(2)} = 0.$$

Case $m = 3$

2 PARTIAL DECOUPLING CONDITIONS – 1b

$$\begin{cases} \nabla_{\mathbf{u}}\lambda_3 \cdot \mathbf{r}^{(i)} = 0, \\ \nabla_{\mathbf{u}}\left(\nabla_{\mathbf{u}}H_3 \cdot \mathbf{g} + \dfrac{\partial H_3}{\partial x_1} + \lambda_3 \dfrac{\partial H_3}{\partial x_2}\right) \cdot \mathbf{r}^{(i)} = 0, \\ \nabla_{\mathbf{u}}H_3 \cdot \mathbf{r}^{(i)} = 0, \qquad i = 1, 2 \end{cases}$$

$$U_3 = H_3(x_1, x_2, u_1, u_2, u_3)$$

3 FULL DECOUPLING CONDITIONS

Merging conditions of the two previous cases.

Case $m = 3$: Example

Let us consider a Galilean system, where u_1 is a velocity, of the form

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} + (u_1 + p_{11})\dfrac{\partial u_1}{\partial x_2} + p_{12}\dfrac{\partial u_2}{\partial x_2} + p_{13}\dfrac{\partial u_3}{\partial x_2} = g_1, \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{p_{11}p_{22}}{p_{12}}\dfrac{\partial u_1}{\partial x_2} + (u_1 + p_{22})\dfrac{\partial u_2}{\partial x_2} + p_{23}\dfrac{\partial u_3}{\partial x_2} = g_2, \\ \dfrac{\partial u_3}{\partial x_1} + (u_1 + p_{33})\dfrac{\partial u_3}{\partial x_2} = g_3, \end{cases}$$

p_{ij}, g_i, $(i, j = 1, \ldots, 3)$ functions of (u_2, u_3).

Case $m = 3$: Example

By taking $p_{13} = \frac{u_2}{u_3} p_{12}$ and integrating partial decoupling conditions, we get

$$
\begin{aligned}
H_1 &= H_1(u_1, u_2 u_3), & H_2 &= H_2(u_1, u_2 u_3),\\
p_{11} &= q_{11} - q_{22}, & p_{12} &= \frac{q_{12}}{u_2},\\
p_{22} &= q_{22}, & p_{23} &= \frac{u_2}{u_3}(q_{22} - p_{33}),\\
g_1 &= c_1, & g_2 &= c_2 - \frac{u_2}{u_3} g_3,
\end{aligned}
$$

q_{11}, q_{12}, q_{22}, c_1, c_2 functions of $(u_2 u_3)$.
If we choose

$$
U_1 = u_1 + u_2 u_3, \qquad U_2 = u_2 u_3, \qquad U_3 = u_3,
$$

and replace its inverse transformation in the source system, *i.e.*

$$
u_1 = U_1 - U_2, \qquad u_2 = \frac{U_2}{U_3}, \qquad u_3 = U_3,
$$

Case $m = 3$: Example

we obtain this partially decoupled system

$$
\begin{cases}
\dfrac{\partial U_1}{\partial X_1} + \left(U_1 + U_2\left(\dfrac{q_{22}}{q_{12}}(q_{11}-q_{22}) - 1\right) + q_{11} - q_{22}\right)\dfrac{\partial U_1}{\partial X_2} \\
\quad + \left(U_2\dfrac{q_{22}}{q_{12}}(q_{22}-q_{11}) + \dfrac{q_{12}}{U_2} + q_{22} + q_{12} - q_{11}\right)\dfrac{\partial U_2}{\partial X_2} = c_1 + U_2\, c_2, \\
\dfrac{\partial U_2}{\partial X_1} + \left(U_2\dfrac{q_{22}}{q_{12}}(q_{11}-q_{22})\right)\dfrac{\partial U_1}{\partial X_2} \\
\quad + \left(U_1 + U_2\left(\dfrac{q_{22}}{q_{12}}(q_{22}-q_{11}) - 1\right) + q_{22}\right)\dfrac{\partial U_2}{\partial X_2} = U_2\, c_2, \\
\dfrac{\partial U_3}{\partial X_1} + \left(U_1 - U_2 + p_{33}\left(\dfrac{U_2}{U_3}, U_3\right)\right)\dfrac{\partial U_3}{\partial X_2} = g_3\left(\dfrac{U_2}{U_3}, U_3\right)
\end{cases}
$$

where the functions q_{ij} and c_i depend only on U_2.

THANKS