Direct approach to approximate conservation laws

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General framework of conservations laws

Lie groups methods to construct conservation laws:

- Variational systems $\longrightarrow$ Noether's theorem [Noether, 1918];
- Extension of Noether's theorem with generalized symmetries [Boyer, 1967];
- Extension of Noether's theorem with differential operators [Vinogradov, 1984];
- Formal Lagrangian and self–adjointness [Ibragimov, 2007];
- Partial Lagrangians [Kara, Mahomed, 2006];
- ...

Not via Lie symmetries...

In general, for DEs not admitting a variational principle, conservation laws may be found through a direct approach [Anco, Bluman, 2002].

Remark

The approaches for the derivation of conservation laws for both variational and non–variational problems need some adaptations when dealing with differential equations involving terms of different orders of magnitude.

Notation

Consider a system of differential equations of order r involving a small parameter $\varepsilon \ll 1$

$$\Delta\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) = 0,$$

where:

- $\Delta \equiv (\Delta^1, \dots, \Delta^q)$ sufficiently smooth functions;
- $\mathrm{x} \equiv (x_1, \dots, x_n) \in X \subseteq \mathbb{R}^n$ the independent variables;
- $\mathrm{u}^{(0)} = \mathrm{u} \equiv (u_1, \dots, u_m) \in U \subseteq \mathbb{R}^m$ the dependent variables;
- $\mathrm{u}^{(r)} = \left\{ \frac{\partial^{|J|} u_\alpha}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} \,:\, \alpha = 1, \dots, m,\ |J| = 0, \dots, r \right\} \in U^{(r)} \subseteq \mathbb{R}^N$, with $N = m\binom{n+r}{r}$, the derivatives of u w.r.t. x up to the order r, and $|J| = j_1 + \cdots + j_n$.

Definition

An approximate conservation law of order r, compatible with the system, is a divergence expression

$$\sum_{i=1}^{n} D_i \left(\Phi^i \left(\mathrm{x}, \mathrm{u}^{(r-1)}; \varepsilon\right)\right) = 0,$$

holding for all solutions of the system, where $\Phi^i \left(\mathrm{x}, \mathrm{u}^{(r-1)}; \varepsilon\right)$ are the fluxes of the conservation law, and D_i is the Lie derivative.

Perturbed variational problems

Approximate conservation laws may be found by means of the approximate Noether's theorem[a], which establishes a correspondence between the approximate Lie symmetries[b] of the perturbed action integral and approximate conservation laws through an explicit formula involving the approximate infinitesimals and the perturbed Lagrangian itself.

[a] Gorgone, Oliveri, Mathematics, 2021
[b] Di Salvo, Gorgone, Oliveri, Nonlin. Dyn., 2018

Perturbed non–variational problems: direct method[1]

Given a system of DEs,

$$\Delta\left(x, u^{(r)}; \varepsilon\right) = 0,$$

we want to determine sets of non–singular (when evaluated on the solutions of the system) multipliers $\Lambda^{\nu}\left(x, u^{(r)}; \varepsilon\right)$ $(\nu = 1, \dots, q)$ provided that

$$\sum_{\nu=1}^{q}\left(\Lambda^{\nu}\left(x, u^{(r)}; \varepsilon\right)\Delta^{\nu}\left(x, u^{(r)}; \varepsilon\right)\right) \equiv \sum_{i=1}^{n} D_i\left(\Phi^i\left(x, u^{(r-1)}; \varepsilon\right)\right) = O(\varepsilon^{p+1})$$

is an approximate divergence expression holding for all solutions of the system.

Key aspects of the direct approach

- Any divergence expression is annihilated by the Euler operators associated to all dependent variables;
- All the sets of multipliers can be found algorithmically by solving a linear system of determining equations.

[1] Bluman, Anco, Eur. J. Appl. Math., 2002

Approaches to approximate conservation laws

First method: without expansion of dependent variables

Given a sistem of differential equations involving a small parameter

$$\Delta\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) \equiv \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}\right) = O(\varepsilon^{p+1}),$$

an expansion of the Lagrange multipliers is considered,

$$\Lambda^\nu\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) \equiv \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^\nu_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}\right), \qquad \nu = 1, \dots, q,$$

and the following Euler operator in the algorithmic procedure is used:

$$E_{u_\alpha} = \frac{\partial}{\partial u_\alpha} - \sum_{i=1}^{n} D_i\left(\frac{\partial}{\partial u_{\alpha,i}}\right) + \dots + (-1)^s \sum_{i_1=1}^{n} \dots \sum_{i_s=i_{s-1}}^{n} D_{i_1} \dots D_{i_s}\left(\frac{\partial}{\partial u_{\alpha,i_1\dots i_s}}\right), \quad \alpha = 1, \dots, m.$$

Remark

This approach moves within the same framework of BGI[2] method for approximate symmetries.

[2] Baikov, Gazizov, Ibragimov, Mat. Sb., 1988

Approaches to approximate conservation laws

Second method: with expansion of dependent variables

Given a sistem of differential equations involving a small parameter

$$\Delta\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) = 0, \qquad (*)$$

the dependent variables are expanded in a perturbation series as done in usual perturbation analysis:

$$\mathrm{u}^{(r)}(\mathrm{x}; \varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}^{(r)}_{(k)}(\mathrm{x}) + O(\varepsilon^{p+1}),$$

with $\mathrm{u}^{(r)}_{(k)} \equiv \left(u^{(r)}_{(k)1}, \ldots, u^{(r)}_{(k)N}\right)$.

By separating at each order of approximation, we have a coupled system:

$$\widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}_{(0)}, \ldots, \mathrm{u}^{(r)}_{(k)}\right) = 0, \qquad k = 0, \ldots, p. \qquad (**)$$

Approaches to approximate conservation laws

... Multipliers and Euler operators

The approximate multipliers of system $(*)$ are defined as the exact multipliers

$$\Lambda^i\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)},\dots,\mathrm{u}^{(r)}_{(k)}\right),\qquad i,k=0,\dots,p$$

of system $(**)$ obtained from perturbations.
The following Euler operator is considered:

$$E_{u_{(k)\alpha}}=\frac{\partial}{\partial u_{(k)\alpha}}-\sum_{i=1}^{n}D_i\left(\frac{\partial}{\partial u_{(k)\alpha,i}}\right)+\dots+(-1)^s\sum_{i_1=1}^{n}\dots\sum_{i_s=i_{s-1}}^{n}D_{i_1}\dots D_{i_s}\left(\frac{\partial}{\partial u_{(k)\alpha,i_1\dots i_s}}\right),$$

with $k=0,\dots,p$ and $\alpha=1,\dots,m$.

Remark

This approach moves within the same framework of FS^3 method for approximate symmetries.

Fushchich and Shtelen, J. Phys. A., 1989

A consistent approach

Consider a system of DEs containing a small term ε:

$$\Delta\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) = 0.$$

- Assume the Lagrange multipliers to be dependent on the small parameter ε,

$$\Lambda^{\nu} = \Lambda^{\nu}\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right), \qquad \nu = 1, \ldots, q.$$

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- Expand the dependent variables in power series of the small parameter,

$$\mathrm{u}^{(r)}(\mathrm{x}; \varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}_{(k)}^{(r)}(\mathrm{x}) + O(\varepsilon^{p+1}),$$

  and the system write as

$$\Delta \equiv \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}_{(0)}^{(r)}, \ldots, \mathrm{u}_{(k)}^{(r)}\right) = O(\varepsilon^{p+1}).$$

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- Consider a consistent expansion of the Lagrange multipliers.
- Use a consistent definition of approximate Euler operators.

Expansions of multipliers

$$\Lambda^\nu\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^\nu_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}_{(0)}, \ldots, \mathrm{u}^{(r)}_{(k)}\right) + O(\varepsilon^{p+1}), \qquad \nu = 1, \ldots, q$$

where $\widetilde{\Lambda}^\nu_{(k)}$ $(k > 0)$ are suitable polynomials in $\mathrm{u}^{(r)}_{(1)}, \ldots, \mathrm{u}^{(r)}_{(k)}$ with coefficients given by $\Lambda^\nu_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}_{(0)}\right)$ $(k = 0, \ldots, p)$ and their derivatives with respect to $\mathrm{u}^{(r)}_{(0)}$.

Expansions of multipliers

$$\Lambda^\nu\left(\mathrm{x},\mathrm{u}^{(r)};\varepsilon\right)=\sum_{k=0}^{p}\varepsilon^k\widetilde{\Lambda}^\nu_{(k)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)},\dots,\mathrm{u}^{(r)}_{(k)}\right)+O(\varepsilon^{p+1}),\qquad \nu=1,\dots,q$$

where $\widetilde{\Lambda}^\nu_{(k)}$ $(k>0)$ are suitable polynomials in $\mathrm{u}^{(r)}_{(1)},\dots,\mathrm{u}^{(r)}_{(k)}$ with coefficients given by $\Lambda^\nu_{(k)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)}\right)$ $(k=0,\dots,p)$ and their derivatives with respect to $\mathrm{u}^{(r)}_{(0)}$.

In fact:

$$\widetilde{\Lambda}^\nu_{(0)}=\Lambda^\nu_{(0)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)}\right)=\Lambda^\nu\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)};0\right),\qquad \widetilde{\Lambda}^\nu_{(k+1)}=\frac{1}{k+1}\mathcal{R}[\widetilde{\Lambda}^\nu_{(k)}],$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives defined as

$$\mathcal{R}\left[\frac{\partial^{|\tau|}\Lambda^\nu_{(k)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}\right]=\frac{\partial^{|\tau|}\Lambda^\nu_{(k+1)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}+\sum_{i=1}^{N}\frac{\partial}{\partial u^{(r)}_{(0)i}}\left(\frac{\partial^{|\tau|}\Lambda^\nu_{(k)}\left(\mathrm{x},\mathrm{u}^{(r)}_{(0)}\right)}{\partial u^{(r)\tau_1}_{(0)1}\dots\partial u^{(r)\tau_N}_{(0)N}}\right)u^{(r)}_{(1)i},$$

$$\mathcal{R}[u^{(r)}_{(k)j}]=(k+1)u^{(r)}_{(k+1)j},$$

where $k\geq 0$, $j=1,\dots,N$, $|\tau|=\tau_1+\dots+\tau_N$.

Approximate conservation laws

Given a system

$$\Delta\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) \equiv \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}_{(0)}, \dots, \mathrm{u}^{(r)}_{(k)}\right) = O(\varepsilon^{p+1}),$$

an approximate conservation law of order r, compatible with the system, is an approximate divergence expression

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} D_i\left(\widetilde{\Phi}^i_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r-1)}_{(0)}, \dots, \mathrm{u}^{(r-1)}_{(k)}\right)\right)\right) = O(\varepsilon^{p+1}),$$

holding for all solutions of the system, where

$$\sum_{k=0}^{p} \varepsilon^k \widetilde{\Phi}^i_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r-1)}_{(0)}, \dots, \mathrm{u}^{(r-1)}_{(k)}\right), \qquad i = 1, \dots, n$$

are the expansions at order p of the fluxes $\Phi^i\left(\mathrm{x}, \mathrm{u}^{(r-1)}; \varepsilon\right)$ of the conservation law, and

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{k=0}^{p}\sum_{\alpha=1}^{m}\left(u_{(k)\alpha,i}\frac{\partial}{\partial u_{(k)\alpha}} + \sum_{j=1}^{n} u_{(k)\alpha,ij}\frac{\partial}{\partial u_{(k)\alpha,j}} + \dots\right)$$

is the approximate Lie derivative, with $u_{(k)\alpha,i} = \dfrac{\partial u_{(k)\alpha}}{\partial x_i}$, $u_{(k)\alpha,ij} = \dfrac{\partial^2 u_{(k)\alpha}}{\partial x_i \partial x_j}$, ...

Approximate multipliers

Functions $\Lambda^\nu(\mathrm{x}, \mathrm{u}^{(s)}; \varepsilon)$ $(\nu = 1, \dots, q)$ are approximate multipliers depending on s–th order derivatives if, after expanding in pertubation series of ε up to the order p, *i.e.*,

$$\Lambda^\nu\left(\mathrm{x}, \mathrm{u}^{(s)}; \varepsilon\right) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^\nu_{(k)}\left(\mathrm{x}, \mathrm{u}^{(s)}_{(0)}, \dots, \mathrm{u}^{(s)}_{(k)}\right) + O(\varepsilon^{p+1}), \qquad \nu = 1, \dots, q,$$

the relation

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right) - \sum_{i=1}^{n} D_i \widetilde{\Phi}^i_{(k)}\right) = O(\varepsilon^{p+1})$$

holds for arbitrary $\mathrm{u}^{(s)}_{(\ell)}(\mathrm{x})$ and some suitable functions $\widetilde{\Phi}^i_{(k)}\left(\mathrm{x}, \mathrm{u}^{(s-1)}_{(0)}, \dots, \mathrm{u}^{(s-1)}_{(k)}\right)$.

Then, if $\Lambda^\nu\left(\mathrm{x}, \mathrm{u}^{(s)}; \varepsilon\right)$ are non–singular, an approximate conservation law can be recovered:

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^\nu_{(\ell)}\widetilde{\Delta}^\nu_{(k-\ell)}\right)\right) \equiv \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} D_i \widetilde{\Phi}^i_{(k)}\right) = O(\varepsilon^{p+1}).$$

Theorem

The non–singular approximate multipliers

$$\Lambda^{\nu}\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right), \qquad \nu = 1, \dots, q$$

yield an approximate conservation law iff the set of relations

$$E_{u_{(0)\alpha}}\left(\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^{\nu}_{(\ell)}\widetilde{\Delta}^{\nu}_{(k-\ell)}\right)\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha = 1, \dots, m$$

holds for arbitrary $\mathrm{u}^{(r)}_{(k)}(\mathrm{x})$ $(k = 0, \dots, p)$, where

$$E_{u_{(0)\alpha}} = \frac{\partial}{\partial u_{(0)\alpha}} - \sum_{i=1}^{n} D_i\left(\frac{\partial}{\partial u_{(0)\alpha,i}}\right) + \dots + (-1)^r \sum_{i_1=1}^{n} \dots \sum_{i_r=i_{r-1}}^{n} D_{i_1} \dots D_{i_r}\left(\frac{\partial}{\partial u_{(0)\alpha,i_1\dots i_r}}\right)$$

are the approximate Euler operators.

Approximate direct method with the consistent approach: algorithm

- Expand the dependent variables in power series of ε: $\mathrm{u}(\mathrm{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}_{(k)}(\mathrm{x}) + O(\varepsilon^{p+1})$;
- Expand in perturbation series of ε the multipliers, so obtaining:

$$\Lambda^{\nu}\left(\mathrm{x}, \mathrm{u}^{(r)}; \varepsilon\right) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^{\nu}_{(k)}\left(\mathrm{x}, \mathrm{u}^{(r)}_{(0)}, \dots, \mathrm{u}^{(r)}_{(k)}\right) + O(\varepsilon^{p+1}), \qquad \nu = 1, \dots, q;$$

- Apply the approximate Euler operators, *i.e.*,

$$E_{u_{(0)\alpha}}\left(\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^{\nu}_{(\ell)}\widetilde{\Delta}^{\nu}_{(k-\ell)}\right)\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha = 1, \dots, m;$$

- Separate the resulting conditions at each order in ε, and split into an overdetermined system for the unknown approximate multipliers;
- Insert the recovered approximate multipliers in

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{\ell=0}^{k}\sum_{\nu=1}^{q}\left(\widetilde{\Lambda}^{\nu}_{(\ell)}\widetilde{\Delta}^{\nu}_{(k-\ell)}\right) - \sum_{i=1}^{n} D_i \widetilde{\Phi}^{i}_{(k)}\right) = O(\varepsilon^{p+1})$$

and, if possible, find the approximate fluxes.

Approximate direct method – Applications

By means of the approximate direct method, approximate conservation laws have been determined for:

- Perturbed Van der Pool equation:

$$\ddot{u} + u - \varepsilon\left(1 - u^2\right)\dot{u} = 0;$$

- Perturbed KdV–Burgers equation:

$$u_{,t} + uu_{,x} + u_{,xxx} - \varepsilon u_{,xx} = 0;$$

- A perturbed nonlinear wave equation:

$$u_{,xx} - \frac{1}{c^2}u_{,tt} - \lambda u^3 - \varepsilon f(u) = 0;$$

- Two perturbed nonlinear Schrödinger equations:

$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon|p|^4 p = 0;$$

$$\mathrm{i}p_{,t} + \frac{1}{2}p_{,xx} + |p|^2 p + \mathrm{i}\varepsilon\left(\beta_1 p_{,xxx} + \beta_2|p|^2 p_{,x} + \beta_3 p(|p|^2)_{,x}\right) = 0;$$

- The generalized Kaup–Newell equation:

$$u_{,t} - \frac{1}{2}u_{,xx} + uvu_{,x} + \frac{1}{2}u^2 v_{,x} + 2\varepsilon uu_{,x} = 0,$$

$$v_{,t} + \frac{1}{2}v_{,xx} + uvv_{,x} + \frac{1}{2}v^2 u_{,x} + 2\varepsilon(vu_{,x} + uv_{,x}) = 0.$$

Perturbed nonlinear second order Schrödinger equation:

$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon |p|^4 p = 0,$$

with $p \equiv p(t, x; \varepsilon)$ the complex-valued envelope of the wave. By decomposing into real and imaginary parts:

$$\begin{aligned}\Delta^1 &= u_{,t} + v_{,xx} + 2v(u^2 + v^2) - \varepsilon v \left(u^2 + v^2\right)^2 = 0,\\ \Delta^2 &= v_{,t} - u_{,xx} - 2u(u^2 + v^2) + \varepsilon u \left(u^2 + v^2\right)^2 = 0.\end{aligned}$$

Expand $u(t, x; \varepsilon)$ and $v(t, x; \varepsilon)$ at first order in ε and look for the approximate multipliers of the form

$$\Lambda^\nu = \Lambda^\nu_{(0)} + \varepsilon \left(\Lambda^\nu_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0)}} u_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0)}} v_{(1)} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0),x}} u_{(1),x} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0),x}} v_{(1),x} + \frac{\partial \Lambda^\nu_{(0)}}{\partial u_{(0),xx}} u_{(1),xx} + \frac{\partial \Lambda^\nu_{(0)}}{\partial v_{(0),xx}} v_{(1),xx}\right),$$

where $\Lambda^\nu_{(k)} \equiv \Lambda^\nu_{(k)} \left(t, x, u_{(0)}, v_{(0)}, u_{(0),x}, v_{(0),x}, u_{(0),xx}, v_{(0),xx}\right)$ $(k = 0, 1$ and $\nu = 1, 2)$.
By solving the approximate determining equations

$$E_{u_{(0)}} \left(\Lambda^1 \Delta^1 + \Lambda^2 \Delta^2\right) = 0, \qquad E_{v_{(0)}} \left(\Lambda^1 \Delta^1 + \Lambda^2 \Delta^2\right) = 0,$$

where $E_{w_{(0)}} = \frac{\partial}{\partial w_{(0)\alpha}} - D_t \left(\frac{\partial}{\partial w_{(0),t}}\right) - D_x \left(\frac{\partial}{\partial w_{(0),x}}\right) + D_t D_t \left(\frac{\partial}{\partial w_{(0),tt}}\right) + D_t D_x \left(\frac{\partial}{\partial w_{(0),tx}}\right) + D_x D_x \left(\frac{\partial}{\partial w_{(0),xx}}\right)$,

we obtain the sets of approximate multipliers with the corresponding approximate conservation laws.

- $$\Lambda_1^1 = v_{(0),xx} + 2v_{(0)}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon\left(v_{(1),xx} - v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(u_{(0)}^2 v_{(1)} + 2u_{(0)}u_{(1)}v_{(0)} + 3v_{(0)}^2 v_{(1)})\right),$$
$$\Lambda_1^2 = -u_{(0),xx} - 2u_{(0)}(u_{(0)}^2 + v_{(0)}^2) - \varepsilon\left(u_{(1),xx} - u_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(v_{(0)}^2 u_{(1)} + 2u_{(0)}v_{(0)}v_{(1)} + 3u_{(0)}^2 u_{(1)})\right),$$
with
$$\begin{aligned} &D_t\left(\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 - (u_{(0)}^2 + v_{(0)}^2)^2\right) + \varepsilon\left(\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),x}\right.\right.\\ &\left.\left.+\left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),x} - 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right)\\ &+D_x\left(-(u_{(0),t}u_{(0),x} + v_{(0),t}v_{(0),x}) - \varepsilon\left(u_{(0),x}u_{(1),t} + v_{(0),x}v_{(1),t}\right.\right.\\ &\left.\left.+\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),t} + \left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),t}\right)\right) = 0. \end{aligned}$$

- $$\Lambda_2^1 = u_{(0),x} + \varepsilon u_{(1),x}, \qquad \Lambda_2^2 = v_{(0),x} + \varepsilon v_{(1),x},$$
with
$$\begin{aligned} &D_t\left(v_{(0)}u_{(0),x} + \varepsilon\left((tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),x} + tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),x} + v_{(0)}u_{(1),x}\right)\right)\\ &+D_x\left(-\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 + (u_{(0)}^2 + v_{(0)}^2)^2\right) - v_{(0)}u_{(0),t}\right.\\ &-\varepsilon\left(u_{(0),x}u_{(1),x} + v_{(0),x}v_{(1),x} + (tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),t}\right.\\ &\left.\left.+\ tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),t} + v_{(0)}u_{(1),t} + 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right) = 0. \end{aligned}$$

- $$\Lambda_3^1 = 2tu_{(0),x} + xv_{(0)} + \varepsilon\left(2tu_{(1),x} + xv_{(1)}\right), \qquad \Lambda_3^2 = 2tv_{(0),x} - xu_{(0)} + \varepsilon\left(2tv_{(1),x} - xu_{(1)}\right),$$

  with

$$\begin{aligned}
&D_t\left(2tv_{(0)}u_{(0),x} + \frac{x}{2}(u_{(0)}^2 + v_{(0)}^2)\right.\\
&\left.+\,\varepsilon\left(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),x} + t^2v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2v_{(0),x} + 2tv_{(0)}u_{(1),x} + x(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right)\\
+&D_t\left(-t\left(u_{(0),x}^2 + v_{(0),x}^2 + 2v_{(0)}u_{(0),t} + (u_{(0)}^2 + v_{(0)}^2)^2\right) - x(v_{(0)}u_{(0),x} - u_{(0)}v_{(0),x}) - u_{(0)}v_{(0)}\right.\\
&-\varepsilon\left(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),t} + (2tu_{(1),x} + xv_{(1)})u_{(0),x} + (2tv_{(1),x} - xu_{(1)})v_{(0),x}\right.\\
&+t^2v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2v_{(0),t} + x(v_{(0)}u_{(1),x} - u_{(0)}v_{(1),x})\\
&\left.\left.+\,2tv_{(0)}u_{(1),t} + 4t(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)}) + u_{(0)}v_{(1)} + v_{(0)}u_{(1)}\right)\right) = 0.
\end{aligned}$$

- $$\Lambda_4^1 = v_{(0)} + \varepsilon v_{(1)}, \qquad \Lambda_4^2 = -u_{(0)} - \varepsilon u_{(1)},$$

  with

$$\begin{aligned}
&D_t\left(\frac{1}{2}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\\
+&D_x\left(u_{(0)}v_{(0),x} - v_{(0)}u_{(0),x} + \varepsilon(u_{(1)}v_{(0),x} - v_{(1)}u_{(0),x} + u_{(0)}v_{(1),x} - v_{(0)}u_{(1),x})\right) = 0.
\end{aligned}$$

THANKS