Decoupling of Hyperbolic Quasilinear Systems

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Problem [Courant]

When can an autonomous and homogeneous quasilinear first order system of PDEs

$$\frac{\partial \mathbf{u}}{\partial x_1} + A(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0},$$

where $\mathbf{u} \in \mathbb{R}^n$, $A \in M_{n,n}(\mathbb{R})$, be decoupled — in some new coordinates $v_1(\mathbf{u}), \ldots, v_n(\mathbf{u})$ — into k non-interacting subsystems of some orders $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$?

Necessary and Sufficient Conditions

Theorem [Nijenhuis]

The necessary and sufficient condition for the complete decoupling of the system

$$\frac{\partial \mathbf{u}}{\partial x_1} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}$$

into n non-interacting one-dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor

$$N_{jik} = A_{\alpha i}\frac{\partial A_{jk}}{\partial u_\alpha} - A_{\alpha k}\frac{\partial A_{ji}}{\partial u_\alpha} + A_{j\alpha}\frac{\partial A_{\alpha i}}{\partial u_k} - A_{j\alpha}\frac{\partial A_{\alpha k}}{\partial u_i},$$

provided that all eigenvalues of A (A_{ij}) are real and distinct.

Necessary and Sufficient Conditions

Theorem [Bogoyavlenskij]

For a system of quasilinear PDEs to be locally reducible into k non-interacting subsystems of some orders $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$ it is necessary and sufficient that in the tangent spaces $T_x(\mathbb{R}^n)$ there exist k smooth distributions $L_{1x}, \ldots, L_{kx}$ of dimensions $n_1, \ldots, n_k$ such that $L_{1x} \oplus \cdots \oplus L_{kx} = T_x(\mathbb{R}^n)$ and the conditions

$$A(L_{ix}) \subset L_{ix}, \qquad N_A(L_{ix}, L_{ix}) \subset L_{ix}, \qquad N_A(L_{ix}, L_{rx}) = 0,$$

hold provided that the eigenvalues of the operator A in any two different subspaces L_{ix} and L_{rx} are different almost everywhere for $\mathbf{x} \in \mathbb{R}^n$. Here $i \neq r$; $i, r \in \{1, ..., k\}$.

General Decoupling Problem

Let us consider a hyperbolic system of PDEs

$$\frac{\partial \mathbf{u}}{\partial x_1} + p(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \qquad (\textit{source system})$$

where $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^n$, $p \in M_{n,n}(\mathbb{R})$, $\mathbf{g} \in \mathbb{R}^n$, $p_{ij}(\mathbf{x}, \mathbf{u})$ and $g_i(\mathbf{x}, \mathbf{u})$ functions of the indicated arguments.

Through an invertible transformation like

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{u} = \mathbf{h}(\mathbf{x}, \mathbf{U}),$$

or, equivalently,

$$\mathbf{x} = \mathbf{z}(\mathbf{X}), \qquad \mathbf{U} = \mathbf{H}(\mathbf{x}, \mathbf{u}),$$

the quasilinear form is preserved, and we obtain

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}) \qquad (\textit{target system})$$

Transformation

By defining

$$J = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} \end{bmatrix},$$

we get

$$P = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}(J_{11}I + J_{12}p)^{-1}(J_{21}I + J_{22}p)\nabla_{\mathbf{U}}\mathbf{h},$$

$$\mathbf{G} = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}(J_{11}I + J_{12}p)^{-1}\left(\mathbf{g} - \frac{\partial \mathbf{h}}{\partial x_1} - p\frac{\partial \mathbf{h}}{\partial x_2}\right) =$$

$$= \frac{1}{\det J}\left((J_{22}I - J_{12}P)\left((\nabla_{\mathbf{u}}\mathbf{H})\mathbf{g} + \frac{\partial \mathbf{H}}{\partial x_1}\right) - (J_{21}I - J_{11}P)\frac{\partial \mathbf{H}}{\partial x_2}\right),$$

$$\mathbf{L}^{(i)} = \mathbf{l}^{(i)}(\nabla_{\mathbf{u}}\mathbf{H})^{-1}, \qquad \mathbf{R}^{(i)} = (\nabla_{\mathbf{u}}\mathbf{H})\mathbf{r}^{(i)}, \qquad \Lambda_i = \frac{J_{21} + J_{22}\lambda_i}{J_{11} + J_{12}\lambda_i},$$

I identity matrix, $\mathbf{l}^{(i)}$ and $\mathbf{r}^{(i)}$ left and right eigenvector of p corresponding to the eigenvalue λ_i, $\mathbf{L}^{(i)}$ and $\mathbf{R}^{(i)}$ left and right eigenvector of P corresponding to the eigenvalue Λ_i (i=1,...,n)

Case $n = 2$ (Two field variables)

By considering the target system

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U})\frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}),$$

we have two main cases:

1. **PARTIAL DECOUPLING**

$$P_{12} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial G_1}{\partial U_2} = 0$$

2. **FULL DECOUPLING**

$$P_{12} = P_{21} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial P_{22}}{\partial U_1} = \frac{\partial G_1}{\partial U_2} = \frac{\partial G_2}{\partial U_1} = 0$$

Case $n = 2$

1. PARTIAL DECOUPLING

- $P_{12} = 0 \Longrightarrow R_1^{(2)} = (\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}^{(2)} = 0;$
- Since $R_1^{(2)} = 0$ means $\frac{\partial \mathbf{h}}{\partial U_2} \parallel \mathbf{r}^{(2)}$ and $P_{11} = \Lambda_1 = \dfrac{J_{21} + J_{22}\lambda_1}{J_{11} + J_{12}\lambda_1}$,

$$\frac{\partial P_{11}}{\partial U_2} = \frac{\partial \Lambda_1}{\partial U_2} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_1) \cdot \mathbf{r}^{(2)} = 0;$$

- Since

$$G_1 = \frac{(J_{11}P_{11} - J_{21})\frac{\partial H_1}{\partial x_2} + (J_{22} - J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1} + g_2\frac{\partial H_1}{\partial u_2} + \frac{\partial H_1}{\partial x_1})}{\det J},$$

$$\frac{\partial G_1}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}} H_1) \cdot \mathbf{g} + \frac{\partial H_1}{\partial x_1} + \lambda_1 \frac{\partial H_1}{\partial x_2}\right) \cdot \mathbf{r}^{(2)} = 0.$$

Case $n = 2$

1. **PARTIAL DECOUPLING CONDITIONS**

$$\begin{cases} (\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}^{(2)} = 0, \\ \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_1)\cdot\mathbf{g} + \dfrac{\partial H_1}{\partial x_1} + \lambda_1\dfrac{\partial H_1}{\partial x_2}\right)\cdot\mathbf{r}^{(2)} = 0. \end{cases}$$

From

$$(\nabla_{\mathbf{u}}H_1)\cdot\mathbf{r}^{(2)} = 0,$$

we get

$$U_1 = H_1(x_1, x_2, u_1, u_2).$$

Case $n = 2$

2. FULL DECOUPLING

- $P_{12} = 0 \Longrightarrow R_1^{(2)} = (\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}^{(2)} = 0;$
- $P_{21} = 0 \Longrightarrow R_2^{(1)} = (\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}^{(1)} = 0;$
- Since $R_1^{(2)} = R_2^{(1)} = 0$ mean $\frac{\partial \mathbf{h}}{\partial U_1} \parallel \mathbf{r}^{(1)}$ and $\frac{\partial \mathbf{h}}{\partial U_2} \parallel \mathbf{r}^{(2)}$,

$$P_{11} = \Lambda_1 = \frac{J_{21} + J_{22}\lambda_1}{J_{11} + J_{12}\lambda_1} \text{ and } P_{22} = \Lambda_2 = \frac{J_{21} + J_{22}\lambda_2}{J_{11} + J_{12}\lambda_2},$$

$$\frac{\partial P_{11}}{\partial U_2} = \frac{\partial \Lambda_1}{\partial U_2} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_1) \cdot \mathbf{r}^{(2)} = 0;$$

$$\frac{\partial P_{22}}{\partial U_1} = \frac{\partial \Lambda_2}{\partial U_2} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_2) \cdot \mathbf{r}^{(1)} = 0;$$

Case $n = 2$

2. FULL DECOUPLING

- Since

$$G_1 = \frac{(J_{11}P_{11} - J_{21})\frac{\partial H_1}{\partial x_2} + (J_{22} - J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1} + g_2\frac{\partial H_1}{\partial u_2} + \frac{\partial H_1}{\partial x_1})}{\det J},$$

$$G_2 = \frac{(J_{11}P_{22} - J_{21})\frac{\partial H_2}{\partial x_2} + (J_{22} - J_{12}P_{22})(g_1\frac{\partial H_2}{\partial u_1} + g_2\frac{\partial H_2}{\partial u_2} + \frac{\partial H_2}{\partial x_1})}{\det J},$$

$$\frac{\partial G_1}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_1)\cdot\mathbf{g} + \frac{\partial H_1}{\partial x_1} + \lambda_1\frac{\partial H_1}{\partial x_2}\right)\cdot\mathbf{r}^{(2)} = 0,$$

$$\frac{\partial G_2}{\partial U_1} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_2)\cdot\mathbf{g} + \frac{\partial H_2}{\partial x_1} + \lambda_2\frac{\partial H_2}{\partial x_2}\right)\cdot\mathbf{r}^{(1)} = 0.$$

Case $n = 2$

2 FULL DECOUPLING CONDITIONS

$$\begin{cases} (\nabla_{\mathbf{u}}\lambda_i)\cdot \mathbf{r}^{(j)} = 0, \qquad i,j = 1,2, \qquad i \neq j, \\ \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_i)\cdot \mathbf{g} + \frac{\partial H_i}{\partial x_1} + \lambda_i \frac{\partial H_i}{\partial x_2}\right)\cdot \mathbf{r}^{(j)} = 0. \end{cases}$$

From

$$(\nabla_{\mathbf{u}}H_i)\cdot \mathbf{r}^{(j)} = 0, \qquad i,j = 1,2, \qquad i \neq j,$$

we get

$$U_1 = H_1(x_1, x_2, u_1, u_2),$$
$$U_2 = H_2(x_1, x_2, u_1, u_2).$$

Case $n = 2$: Example

Let us consider a Galilean system, where u_1 is a velocity, of the form:

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} + u_1 \dfrac{\partial u_1}{\partial x_2} + f_1(u_2) \dfrac{\partial u_2}{\partial x_2} = g_1(u_2), \\ \dfrac{\partial u_2}{\partial x_1} + f_2(u_2) \dfrac{\partial u_1}{\partial x_2} + u_1 \dfrac{\partial u_2}{\partial x_2} = g_2(u_2), \end{cases}$$

f_i, g_i, $(i = 1, 2)$ functions of their arguments.
Integrating partial decoupling conditions, we get

$$\int \sqrt{\frac{f_1}{f_2}} du_2 = \sqrt{f_1 f_2} - k_1, \qquad g_1 = k_2 - \sqrt{\frac{f_1}{f_2}} g_2,$$

$$H_1 = H_1 \left(u_1 + \int \sqrt{\frac{f_1}{f_2}} du_2 \right),$$

k_1 and k_2 being arbitrary constants.

Case $n = 2$: Example

By choosing

$$U_1 = u_1 + \sqrt{f_1 f_2} - k_1, \qquad U_2 = u_2,$$

and replacing its inverse transformation in the source system, *i.e.*

$$u_1 = U_1 - \sqrt{f_1 f_2} + k_1, \qquad u_2 = U_2,$$

we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial U_1}{\partial x_1} + (U_1 + k_1)\dfrac{\partial U_1}{\partial x_2} = k_2, \\ \dfrac{\partial U_2}{\partial x_1} + f_2(U_2)\dfrac{\partial U_1}{\partial x_2} + (U_1 - 2\sqrt{f_1(U_2)f_2(U_2)} + k_1)\dfrac{\partial U_2}{\partial x_2} = g_2(U_2). \end{cases}$$

Case $n = 2$: Example

The system so considered, when

$$f_2(u_2) = u_2, \qquad g_1(u_2) = g_2(u_2) = 0,$$

represents the one–dimensional Euler equations of a barotropic fluid with density u_2 and pressure

$$p(u_2) = \int u_2 f_1(u_2)\, du_2.$$

The constitutive law

$$p(u_2) = \frac{k}{3} u_2^3, \qquad k \text{ constant},$$

allows to satisfy the decoupling condition and, in the new variables

$$U_1 = u_1 + \sqrt{k}\, u_2, \qquad U_2 = u_2,$$

we have a partially decoupled system

$$\frac{\partial U_1}{\partial x_1} + U_1 \frac{\partial U_1}{\partial x_2} = 0, \qquad \frac{\partial U_2}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} + (U_1 - 2\sqrt{k}\, U_2) \frac{\partial U_2}{\partial x_2} = 0.$$

Case $n = 3$ (Three field variables)

PARTIAL DECOUPLING – 1a

$$P_{13} = P_{23} = \frac{\partial P_{11}}{\partial U_3} = \frac{\partial P_{12}}{\partial U_3} = \frac{\partial P_{21}}{\partial U_3} = \frac{\partial P_{22}}{\partial U_3} = \frac{\partial G_1}{\partial U_3} = \frac{\partial G_2}{\partial U_3} = 0$$

PARTIAL DECOUPLING – 1b

$$P_{31} = P_{32} = \frac{\partial P_{33}}{\partial U_1} = \frac{\partial P_{33}}{\partial U_2} = \frac{\partial G_3}{\partial U_1} = \frac{\partial G_3}{\partial U_2} = 0$$

FULL DECOUPLING IN TWO SUBSYSTEMS

$$P_{13} = P_{23} = \frac{\partial P_{11}}{\partial U_3} = \frac{\partial P_{12}}{\partial U_3} = \frac{\partial P_{21}}{\partial U_3} = \frac{\partial P_{22}}{\partial U_3} = \frac{\partial G_1}{\partial U_3} = \frac{\partial G_2}{\partial U_3} = 0$$

$$P_{31} = P_{32} = \frac{\partial P_{33}}{\partial U_1} = \frac{\partial P_{33}}{\partial U_2} = \frac{\partial G_3}{\partial U_1} = \frac{\partial G_3}{\partial U_2} = 0$$

Case $n = 3$

1. PARTIAL DECOUPLING – 1a

- $P_{13} = 0 \Longrightarrow R_1^{(3)} = (\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}^{(3)} = 0;$
- $P_{23} = 0 \Longrightarrow R_2^{(3)} = (\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}^{(3)} = 0;$
- $\dfrac{\partial P_{11}}{\partial U_3} = \dfrac{\partial P_{12}}{\partial U_3} = \dfrac{\partial P_{21}}{\partial U_3} = \dfrac{\partial P_{22}}{\partial U_3} = 0$ are equivalent to $\dfrac{\partial \Lambda_1}{\partial U_3} = \dfrac{\partial \Lambda_2}{\partial U_3} = \dfrac{\partial}{\partial U_3}\left(\dfrac{R_2^{(1)}}{R_1^{(1)}}\right) = \dfrac{\partial}{\partial U_3}\left(\dfrac{R_1^{(2)}}{R_2^{(2)}}\right) = 0;$

Since $R_1^{(3)} = R_2^{(3)} = 0$ mean both $\dfrac{\partial \mathbf{h}}{\partial U_3} \parallel \mathbf{r}^{(3)}$ and

$$\nabla_{\mathbf{u}} H_k = \alpha_k(\mathbf{x}, \mathbf{u})\, \mathbf{l}^{(1)} + \beta_k(\mathbf{x}, \mathbf{u})\, \mathbf{l}^{(2)}, \quad k = 1, 2,$$

(α_k and β_k functions such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$)

Case $n = 3$

1. PARTIAL DECOUPLING – 1a

we get

$$\frac{\partial \Lambda_1}{\partial U_3} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_1) \cdot \mathbf{r}^{(3)} = 0,$$

$$\frac{\partial \Lambda_2}{\partial U_3} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_2) \cdot \mathbf{r}^{(3)} = 0,$$

$$\frac{\partial}{\partial U_3}\left(\frac{R_2^{(1)}}{R_1^{(1)}}\right) = 0 \Longrightarrow \mathbf{l}^{(2)} \cdot \left((\nabla_{\mathbf{u}} \mathbf{r}^{(3)})\mathbf{r}^{(1)} - (\nabla_{\mathbf{u}} \mathbf{r}^{(1)})\mathbf{r}^{(3)}\right) = 0,$$

$$\frac{\partial}{\partial U_3}\left(\frac{R_1^{(2)}}{R_2^{(2)}}\right) = 0 \Longrightarrow \mathbf{l}^{(1)} \cdot \left((\nabla_{\mathbf{u}} \mathbf{r}^{(3)})\mathbf{r}^{(2)} - (\nabla_{\mathbf{u}} \mathbf{r}^{(2)})\mathbf{r}^{(3)}\right) = 0;$$

Case $n = 3$

1. PARTIAL DECOUPLING – 1a

- Since

$$\mathbf{G} = \frac{1}{\det J}\Bigg((J_{22}I - J_{12}P)(\nabla_{\mathbf{u}}\mathbf{H})\mathbf{g} + (J_{22}I - J_{12}P)\frac{\partial \mathbf{H}}{\partial x_1} - (J_{21}I - J_{11}P)\frac{\partial \mathbf{H}}{\partial x_2}\Bigg),$$

$$\frac{\partial G_1}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}} H_1)\cdot \mathbf{g} + \frac{\partial H_1}{\partial x_1} + \lambda_1 \frac{\partial H_1}{\partial x_2}\right)\cdot \mathbf{r}^{(3)} = 0,$$

$$\frac{\partial G_2}{\partial U_3} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}} H_2)\cdot \mathbf{g} + \frac{\partial H_2}{\partial x_1} + \lambda_2 \frac{\partial H_2}{\partial x_2}\right)\cdot \mathbf{r}^{(3)} = 0.$$

Case $n = 3$

1. PARTIAL DECOUPLING CONDITIONS – 1a

$$\begin{cases} (\nabla_{\mathbf{u}}\lambda_i) \cdot \mathbf{r}^{(3)} = 0, \qquad i,j = 1,2 \qquad i \neq j, \\ \mathbf{l}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(3)})\mathbf{r}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j)})\mathbf{r}^{(3)} \right) = 0, \\ \nabla_{\mathbf{u}} \left((\nabla_{\mathbf{u}} H_i) \cdot \mathbf{g} + \frac{\partial H_i}{\partial x_1} + \lambda_i \frac{\partial H_i}{\partial x_2} \right) \cdot \mathbf{r}^{(3)} = 0. \end{cases}$$

By solving

$$(\nabla_{\mathbf{u}} H_i) \cdot \mathbf{r}^{(3)} = 0, \qquad i,j = 1,2,$$

we obtain

$$U_1 = H_1(x_1, x_2, u_1, u_2, u_3),$$
$$U_2 = H_2(x_1, x_2, u_1, u_2, u_3).$$

Case $n = 3$

2 PARTIAL DECOUPLING – 1b

- $P_{31} = 0 \Longrightarrow R_3^{(1)} = (\nabla_{\mathbf{u}} H_3) \cdot \mathbf{r}^{(1)} = 0;$
- $P_{32} = 0 \Longrightarrow R_3^{(2)} = (\nabla_{\mathbf{u}} H_3) \cdot \mathbf{r}^{(2)} = 0;$
- Since $R_3^{(1)} = R_3^{(2)} = 0$ mean both $\frac{\partial \mathbf{h}}{\partial U_2}$ and $\frac{\partial \mathbf{h}}{\partial U_3}$ belong to the plane spanned by $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$, $P_{33} = \Lambda_3 = \dfrac{J_{21} + J_{22}\lambda_3}{J_{11} + J_{12}\lambda_3}$,

$$\frac{\partial P_{33}}{\partial U_1} = \frac{\partial \Lambda_3}{\partial U_1} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_3) \cdot \mathbf{r}^{(1)} = 0,$$
$$\frac{\partial P_{33}}{\partial U_2} = \frac{\partial \Lambda_3}{\partial U_2} = 0 \Longrightarrow (\nabla_{\mathbf{u}} \lambda_3) \cdot \mathbf{r}^{(2)} = 0;$$

Case $n = 3$

2 PARTIAL DECOUPLING – 1b

- Since

$$\mathbf{G} = \frac{1}{\det J}\Big((J_{22}I - J_{12}P)(\nabla_{\mathbf{u}}\mathbf{H})\mathbf{g} + (J_{22}I - J_{12}P)\frac{\partial \mathbf{H}}{\partial x_1} - (J_{21}I - J_{11}P)\frac{\partial \mathbf{H}}{\partial x_2}\Big),$$

$$\frac{\partial G_3}{\partial U_1} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_3)\cdot\mathbf{g} + \frac{\partial H_3}{\partial x_1} + \lambda_3\frac{\partial H_3}{\partial x_2}\right)\cdot\mathbf{r}^{(1)} = 0,$$

$$\frac{\partial G_3}{\partial U_2} = 0 \Longrightarrow \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_3)\cdot\mathbf{g} + \frac{\partial H_3}{\partial x_1} + \lambda_3\frac{\partial H_3}{\partial x_2}\right)\cdot\mathbf{r}^{(2)} = 0.$$

Case $n = 3$

2 PARTIAL DECOUPLING CONDITIONS – 1b

$$\begin{cases} (\nabla_{\mathbf{u}}\lambda_3)\cdot\mathbf{r}^{(i)} = 0, \qquad i = 1,2, \\ \nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_3)\cdot\mathbf{g} + \dfrac{\partial H_3}{\partial x_1} + \lambda_3\dfrac{\partial H_3}{\partial x_2}\right)\cdot\mathbf{r}^{(i)} = 0. \end{cases}$$

By solving

$$(\nabla_{\mathbf{u}}H_3)\cdot\mathbf{r}^{(i)} = 0, \qquad i = 1,2,$$

we get

$$U_3 = H_3(x_1, x_2, u_1, u_2, u_3)$$

3 FULL DECOUPLING CONDITIONS

Merging conditions of the two previous cases.

Case $n = 3$: Example

Let us consider a Galilean system, where u_1 is a velocity, of the form

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} + (u_1 + p_{11})\dfrac{\partial u_1}{\partial x_2} + p_{12}\dfrac{\partial u_2}{\partial x_2} + p_{13}\dfrac{\partial u_3}{\partial x_2} = g_1, \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{p_{11}p_{22}}{p_{12}}\ \dfrac{\partial u_1}{\partial x_2} + (u_1 + p_{22})\dfrac{\partial u_2}{\partial x_2} + p_{23}\dfrac{\partial u_3}{\partial x_2} = g_2, \\ \dfrac{\partial u_3}{\partial x_1} + (u_1 + p_{33})\dfrac{\partial u_3}{\partial x_2} = g_3, \end{cases}$$

$p_{ij}, g_i, \ (i, j = 1, \ldots, 3)$ functions of (u_2, u_3).

Case $n=3$: Example

By taking $p_{13}=\frac{u_2}{u_3}p_{12}$ and integrating partial decoupling conditions, we get

$$\begin{aligned} H_1 &= H_1(u_1, u_2u_3), & H_2 &= H_2(u_1, u_2u_3), \\ p_{11} &= q_{11}-q_{22}, & p_{12} &= \frac{q_{12}}{u_2}, \\ p_{22} &= q_{22}, & p_{23} &= \frac{u_2}{u_3}(q_{22}-p_{33}), \\ g_1 &= c_1, & g_2 &= u_2\left(c_2-\frac{g_3}{u_3}\right), \end{aligned}$$

q_{11}, q_{12}, q_{22}, c_1, c_2 functions of (u_2u_3).
If we choose

$$U_1=u_1+u_2u_3, \qquad U_2=u_2u_3, \qquad U_3=u_3,$$

and replace its inverse transformation in the source system, *i.e.*

$$u_1=U_1-U_2, \qquad u_2=\frac{U_2}{U_3}, \qquad u_3=U_3,$$

Case $n = 3$: Example

we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial U_1}{\partial x_1} + \left(U_1 + U_2 \left(\dfrac{q_{22}}{q_{12}} (q_{11} - q_{22}) - 1 \right) + q_{11} - q_{22} \right) \dfrac{\partial U_1}{\partial x_2} \\ \quad + \left(U_2 \dfrac{q_{22}}{q_{12}} (q_{22} - q_{11}) + \dfrac{q_{12}}{U_2} + 2q_{22} - q_{11} \right) \dfrac{\partial U_2}{\partial x_2} = c_1 + U_2\, c_2, \\ \dfrac{\partial U_2}{\partial x_1} + \left(U_2 \dfrac{q_{22}}{q_{12}} (q_{11} - q_{22}) \right) \dfrac{\partial U_1}{\partial x_2} \\ \quad + \left(U_1 + U_2 \left(\dfrac{q_{22}}{q_{12}} (q_{22} - q_{11}) - 1 \right) + q_{22} \right) \dfrac{\partial U_2}{\partial x_2} = U_2\, c_2, \\ \dfrac{\partial U_3}{\partial x_1} + \left(U_1 - U_2 + p_{33} \left(\dfrac{U_2}{U_3}, U_3 \right) \right) \dfrac{\partial U_3}{\partial x_2} = g_3 \left(\dfrac{U_2}{U_3}, U_3 \right) \end{cases}$$

where the functions q_{ij} and c_i depend only on U_2.

Back to the Courant Problem...

Decoupling Theorem

For a first order quasilinear system

$$\frac{\partial \mathbf{u}}{\partial x_1} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}, \qquad \mathbf{u} \in \mathbb{R}^n, \quad a(\mathbf{u})\ n \times n \text{ matrix},$$

assumed to be hyperbolic in the x_1–direction, can be transformed by a (locally) invertible transformation

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \text{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u})$$

such that

$$(\nabla_{\mathbf{u}} H_i) \cdot \mathbf{r}^{(j)} = 0, \qquad i = 1, \ldots, k, \quad j = k+1, \ldots, n,$$

into a system like

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0},$$

Back to the Courant Problem...

...

where $A = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}\, a\, \nabla_{\mathbf{U}}\mathbf{h}$ is a block matrix having the form

$$A = \begin{bmatrix} B_{(k,k)} & 0_{(k,n-k)} \\ C_{(n-k,k)} & D_{(n-k,n-k)} \end{bmatrix},$$

$B_{(k,k)}$ being a $k \times k$ matrix with entries depending at most on $U_1, \dots, U_k$, $0_{(k,n-k)}$ a $k \times (n-k)$ matrix of zeros, $C_{(n-k,k)}$ and $D_{(n-k,n-k)}$ $(n-k) \times k$ and $(n-k) \times (n-k)$ matrices, respectively, with entries depending in principle on all components of $\mathbf{U}$,
it is necessary and sufficient that the following conditions hold:

$$\begin{aligned} &(\nabla_{\mathbf{u}}\lambda_i) \cdot \mathbf{r}^{(j)} = 0, \\ &\mathbf{l}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell)})\mathbf{r}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j)})\mathbf{r}^{(\ell)} \right) = 0, \\ &i, \ell = 1, \dots, k, \qquad i \neq \ell, \qquad j = k+1, \dots, n. \end{aligned}$$

Example: Moving Threadline

Let us consider the motion equations for a moving threadline, where ρ is the mass density, u and v the components of velocity, ϵ the transverse displacement and $T(m)$ the tension:

$$\begin{cases} \dfrac{\partial \rho}{\partial x_1} + \dfrac{\partial}{\partial x_2}(\rho u) = 0, \\ \dfrac{\partial u}{\partial x_1} + u\dfrac{\partial u}{\partial x_2} - \dfrac{1}{\rho}\dfrac{\partial}{\partial x_2}\left(\dfrac{T}{\sqrt{1+\epsilon^2}}\right) = 0, \\ \dfrac{\partial v}{\partial x_1} + 2u\dfrac{\partial v}{\partial x_2} + \left(u^2 - \dfrac{T}{\rho\sqrt{1+\epsilon^2}}\right)\dfrac{\partial \epsilon}{\partial x_2} = 0, \\ \dfrac{\partial \epsilon}{\partial x_1} - \dfrac{\partial v}{\partial x_2} = 0, \end{cases}$$

being $\rho = m\sqrt{1+\epsilon^2}$, $T'(m) < 0$.

...

The constitutive law

$$T(m) = \frac{k^2}{m}, \qquad k \text{ constant},$$

allow us to satisfy the structure conditions

$$(\nabla_{\mathbf{u}}\lambda_i)\cdot\mathbf{r}^{(j)} = 0, \qquad i,\ell = 1,2, \qquad i \neq \ell, \qquad j = 3,4,$$
$$\mathbf{l}^{(i)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell)})\mathbf{r}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j)})\mathbf{r}^{(\ell)}\right) = 0,$$

and, by integrating the transformation conditions

$$(\nabla_{\mathbf{u}}H_i)\cdot\mathbf{r}^{(j)} = 0, \qquad i = 1,2, \qquad j = 3,4,$$

i.e

$$\left(u + \frac{k}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \epsilon} = 0, \quad \left(u + \frac{k}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \epsilon} = 0,$$
$$\left(u - \frac{k}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \epsilon} = 0, \quad \left(u - \frac{k}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \epsilon} = 0,$$

...

we get

$$H_1 = H_1\left(\rho, u\right), \qquad H_2 = H_2\left(\rho, u\right).$$

By choosing the identity transformation, we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial \rho}{\partial x_1} + u\dfrac{\partial \rho}{\partial x_2} + \rho\dfrac{\partial u}{\partial x_2} = 0, \\ \dfrac{\partial u}{\partial x_1} + \dfrac{k^2}{\rho^3}\dfrac{\partial \rho}{\partial x_2} + u\dfrac{\partial u}{\partial x_2} = 0, \\ \dfrac{\partial v}{\partial x_1} + 2u\dfrac{\partial v}{\partial x_2} + \left(u^2 - \dfrac{k^2}{\rho^2}\right)\dfrac{\partial \epsilon}{\partial x_2} = 0, \\ \dfrac{\partial \epsilon}{\partial x_1} - \dfrac{\partial v}{\partial x_2} = 0. \end{cases}$$

Thanks