Reduction of Balance Laws to Autonomous Conservation Laws by Means of Equivalence Transformations

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Lie Symmetries

Lie Symmetries of a DE are continuous transformations which map any solution of a DE into another solution of the same DE.

Use of Lie Symmetries of DEs

- lowering the order or solving by quadrature, for ODEs;
- determining particular solutions (invariant solutions) or generating new solutions, for ODEs and PDEs;
- building mappings between different DEs.

Given a system of PDEs

$$\mathbf{\Delta}(\mathbf{z}^k) = \mathbf{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = \mathbf{0} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ are the independent and the dependent variables and $\mathbf{u}^{(k)}$ denotes the set of all k-th partial derivatives.

One-parameter Lie group of transformations is equivalent to

$$\begin{aligned} \mathbf{x}^\star &= \mathbf{X}(\mathbf{x}, \mathbf{u}; a) = \mathbf{x} + a\boldsymbol{\xi}(\mathbf{x}, \mathbf{u}) + O(a^2), \\ \mathbf{u}^\star &= \mathbf{U}(\mathbf{x}, \mathbf{u}; a) = \mathbf{u} + a\boldsymbol{\eta}(\mathbf{x}, \mathbf{u}) + O(a^2), \end{aligned} \tag{2}$$

and it is characterized by its infinitesimal operator:

$$\Xi = \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \eta^A(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^A}, \tag{3}$$

$i = 1, \ldots, n, \qquad A = 1, \ldots, m.$

Prolongation of infinitesimal operator until k-th order

$$\Xi^{(k)} = \Xi^{(k-1)} + \eta^A_{[i_1,\dots,i_k]} \frac{\partial}{\partial u^A_{i_1,\dots,i_k}}, \tag{4}$$

where $u^A_{i_1,\dots,i_k} = \frac{\partial^k u^A}{\partial x_{i_1},\dots,\partial x_{i_k}}$ e $\eta^A_{[i_1,\dots,i_k]}$ defined recursively by

$$\eta^A_{[i_1,\dots,i_k]} = \frac{D\eta^A_{[i_1,\dots,i_{k-1}]}}{Dx_{i_k}} - u^A_{i_1,\dots,i_{k-1}j} \frac{D\xi_j}{Dx_{i_k}}. \tag{5}$$

with the Lie Derivative

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \frac{\partial u^A}{\partial x_i}\frac{\partial}{\partial u^A} + \frac{\partial^2 u^A}{\partial x_i \partial x_j}\frac{\partial}{\partial u^A_j} + \dots \tag{6}$$

Infinitesimal Criterion for DEs [S. Lie]

Let

$$\Xi = \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \eta^A(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^A}$$

be the infinitesimal operator corresponding to (2), and $\Xi^{(k)}$ the k-th extended infinitesimal operator.
The group (2) is admitted by the system (1) *if and only if*

$$\Xi^{(k)} \boldsymbol{\Delta}(\mathbf{z}^k) \Big|_{\boldsymbol{\Delta}=\mathbf{0}} = \mathbf{0}$$

Physical laws are often expressed mathematically by systems of PDEs in the form of:

Conservations laws

$$\sum_{i=1}^{n} \frac{\partial \mathbf{F}^i(\mathbf{u})}{\partial x_i} = \mathbf{0}$$

Balance laws

$$\sum_{i=1}^{n} \frac{\partial \mathbf{F}^i(\mathbf{u})}{\partial x_i} = \mathbf{G}(\mathbf{u})$$

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By expliciting the derivatives we get quasilinear first order systems:

- Homogeneous:

$$\sum_{i=1}^{n} \mathbf{A}^i(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{0}$$

- Nonhomogeneous:

$$\sum_{i=1}^{n} \mathbf{A}^i(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{G}(\mathbf{u})$$

Theorem [F. Oliveri, 2012]

A nonhomogeneous and nonautonomous first order quasilinear system

$$\sum_{i=1}^{n} \mathbf{A}^i(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{G}(\mathbf{x},\mathbf{u})$$

can be transformed by the invertible map

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{u},\mathbf{x})$$

into an autonomous and homogeneous first order quasilinear system if and only if it admits as subalgebra of its Lie symmetries an $(n+1)-$dimensional Lie algebra spanned by the vector fields

...

$$\Xi_i = \sum_{j=1}^{n} \xi_i^j(\mathbf{x})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_i^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A}, \qquad (i = 1,\dots,n),$$

$$\Xi_{n+1} = \sum_{j=1}^{n} \xi_{n+1}^j(\mathbf{x})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_{n+1}^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A},$$

provided that

$$[\Xi_i,\Xi_j] = 0, \qquad [\Xi_i,\Xi_{n+1}] = \Xi_i, \qquad i,j = 1,\dots,n.$$

Furthermore, it has to be

$$rank \parallel \xi_\alpha^i \parallel = n \qquad \alpha = 1,\dots,n+1, \quad i = 1,\dots,n$$

and the variables $\mathbf{w}$ are invariants of all $(n+1)$ operators.

Given spaces $X \equiv \mathbb{R}^n(\mathbf{x})$, $Y \equiv \mathbb{R}^m(\mathbf{u})$, $Z = X \times Y \equiv \mathbb{R}^{n+m}(\mathbf{z})$ we can consider firs order systems of DEs

$$\boldsymbol{\Delta}(\mathbf{z}^1) = 0$$

where $\mathbf{z}^1 = (\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) \equiv (x_i, u_A, u_{A,i})$, $u_{A,i} = \frac{\partial u_A}{\partial x_i}$.
If arbitrary functions $(p_1, p_2, \ldots, p_l) \in \mathbb{R}^l$ are involved, $(p_r : Z \to \mathbb{R})$ that we assume to be continuously differentiable, we can characterize a class $\mathcal{E}(\boldsymbol{p})$ of first order DEs

$$\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}; \boldsymbol{p}) = 0$$

whose elements are given once $\boldsymbol{p}$ is assigned.

Equivalence transformations for a DE in a given class

Transformations which map a DE in a class into another DE in the same class.

Definition (Equivalence transformations)

A one–parameter Lie group of equivalence transformations (E.T.) of a family $\mathcal{E}(\boldsymbol{p})$ of PDEs is a one–parameter Lie group of transformations given by

$$X_i = X_i(\mathbf{x}, \mathbf{u}, \boldsymbol{p}; a); \qquad i = 1, \dots n;$$
$$U_A = U_A(\mathbf{x}, \mathbf{u}, \boldsymbol{p}; a); \qquad A = 1, \dots m;$$
$$P_k = P_k(\mathbf{x}, \mathbf{u}, \boldsymbol{p}; a); \qquad k = 1, \dots l;$$

which maps a class $\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \boldsymbol{p}) = \mathbf{0} \in \mathcal{E}(\boldsymbol{p})$ of differential equations into itself.

E.T.'s properties :

1. An E.T. operates only on arbitrary elements p_k of the structure of the system.
2. E.T. generate a group called the equivalence group.

[1] Ovsiannikov, L. V.,Academic Press, New York, 1982.

In the augmented space $A = X \times Y \times W$, where $W \equiv \mathbb{R}^l(\boldsymbol{p})$, the equivalence operator Ξ is:

$$\Xi = \xi_i \partial_{x_i} + \eta^A \partial_{u_A} + \mu^k \partial_{p_k} \qquad \begin{cases} i = 1, \dots, n; \\ A = 1, \dots, m; \\ k = 1, \dots, l; \end{cases}$$

where $\xi_i = \xi_i(\mathbf{x}, \mathbf{u})$, $\eta^A = \eta^A(\mathbf{x}, \mathbf{u})$ and $\mu^k = \mu^k(\mathbf{x}, \mathbf{u}, \boldsymbol{p})$. The first prolongation of Ξ is

$$\Xi^{(1)} = \Xi + \eta^A_{[i]} \partial_{u_{A,i}} + \mu^k_{[i]} \partial_{p_{k,i}}$$

with

$$\eta^A_{[i]} = \frac{D\eta^A}{Dx_i} - u_{A,j} \frac{D\xi_j}{Dx_i}; \qquad \mu^k_{[i]} = \frac{\tilde{D}\mu^k}{\tilde{D}z_i} - p_{k,j} \frac{\tilde{D}\xi_j}{\tilde{D}z_i} - p_{k,A} \frac{\tilde{D}\eta^A}{\tilde{D}z_i}$$

where the *Lie derivatives* have been introduced

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_{A,i} \frac{\partial}{\partial u_A} \qquad \frac{\tilde{D}}{\tilde{D}z_i} = \frac{\partial}{\partial z_i} + p_{k,i} \frac{\partial}{\partial p_k}.$$

Finite transformations

In $A = X \times Y \times W$, given an equivalence operator Ξ,

$$\Xi = \xi_i \partial_{x_i} + \eta^A \partial_{u_A} + \mu^k \partial_{p_k}$$

where $i = 1, \ldots, n$, $A = 1, \ldots, m$, $k = 1, \ldots, l$,
Lie's equations (a is the parameter) are

$$\frac{dX_i}{da} = \xi_i(\mathbf{X}, \mathbf{U}), \quad \frac{dU_A}{da} = \eta^A(\mathbf{X}, \mathbf{U}), \quad \frac{dP_k}{da} = \mu^k(\mathbf{X}, \mathbf{U}, \boldsymbol{P}),$$
$$\mathbf{X}(0) = \mathbf{x}, \quad \mathbf{U}(0) = \mathbf{u}, \quad \boldsymbol{P}(0) = \boldsymbol{p}.$$

Integrating them, we get a finite transformation which maps the class into itself.
The integration of Lie's equations in $Z = X \times Y$ give an E.T. mapping a system in the class into another system in the same class with in general different coefficients (arbitrary functions).

[2] Oliveri, F., Speciale, M. P., Acta Appl. Math. **122**, 447–460 2012.

3D Balance equations

Consider the class $\mathcal{E}(\mathbf{p},\mathbf{g})$ with $\mathbf{p}=(p_1,\dots,p_{12})$ and $\mathbf{g}=(g_2,\dots,g_5)$ of systems

$$
\begin{aligned}
&\partial_{x_1}u_1+\partial_{x_2}u_2+\partial_{x_3}u_3+\partial_{x_4}u_4=0,\\
&\partial_{x_1}u_2+\partial_{x_2}p_1+\partial_{x_3}p_2+\partial_{x_4}p_3=g_2,\\
&\partial_{x_1}u_3+\partial_{x_2}p_4+\partial_{x_3}p_5+\partial_{x_4}p_6=g_3,\\
&\partial_{x_1}u_4+\partial_{x_2}p_7+\partial_{x_3}p_8+\partial_{x_4}p_9=g_4,\\
&\partial_{x_1}u_5+\partial_{x_2}p_{10}+\partial_{x_3}p_{11}+\partial_{x_4}p_{12}=g_5,
\end{aligned}
\tag{7}
$$

where $p_i\equiv p_i(x_1,x_2,x_3,x_4,u_1,u_2,u_3,u_4,u_5)$ and $g_k\equiv g_k(x_1,x_2,x_3,x_4,u_1,u_2,u_3,u_4,u_5)$ $(i=1,\dots,12,\ k=1,\dots,5)$ are arbitrary continuously differentiable functions of the indicated arguments.

[3] Gorgone M., Oliveri F., Speciale M. P., Acta Appl. Math. 2014.

By computing (from Lie's equations) the finite transformations,

$$X_i = X_i(\mathbf{x}, \mathbf{u}; a), \quad U_k = U_k(\mathbf{x}, \mathbf{u}; a), \quad (i = 1, \dots, 4,\ k = 1, \dots, 5)$$

being $\mathbf{x} \equiv (x_1, x_2, x_3, x_4)$, $\mathbf{u} \equiv (u_1, u_2, u_3, u_4, u_5)$ and a the parameter of group, we obtain a system having same differential structure, in particular, a system of conservation laws:

$$\begin{aligned}
&\partial_{X_1} U_1 + \partial_{X_2} U_2 + \partial_{X_3} U_3 + \partial_{X_4} U_4 = 0,\\
&\partial_{X_1} U_2 + \partial_{X_2} P_1 + \partial_{X_3} P_2 + \partial_{X_4} P_3 = 0,\\
&\partial_{X_1} U_3 + \partial_{X_2} P_4 + \partial_{X_3} P_5 + \partial_{X_4} P_6 = 0,\\
&\partial_{X_1} U_4 + \partial_{X_2} P_7 + \partial_{X_3} P_8 + \partial_{X_4} P_9 = 0,\\
&\partial_{X_1} U_5 + \partial_{X_2} P_{10} + \partial_{X_3} P_{11} + \partial_{X_4} P_{12} = 0,
\end{aligned} \tag{8}$$

where $P_k \equiv P_k(U_1, U_2, U_3, U_4, U_5), \qquad k = 1, \dots, 12.$

Symmetries on $A \equiv \mathbb{R}^4 \times \mathbb{R}^5 \times \mathbb{R}^{16}$

$$\Xi_1 = f(x_1)\partial_{x_1} - f'(x_1)\left(\sum_{i=2}^{4} u_i\partial_{u_i} + 2\sum_{j=1}^{9} p_j\partial_{p_j} + \sum_{k=10}^{12} p_k\partial_{p_k} + 2\sum_{l=2}^{4} g_l\partial_{g_l} + g_5\partial_{g_5}\right)$$
$$-f''(x_1)(u_2\partial_{g_1} + u_3\partial_{g_2} + u_4\partial_{g_3}),$$

$$\Xi_i = f_i(\mathbf{x})\partial_{x_i} + \sum_{k=1}^{4} u_k\left(\partial_{x_k} f_i(\mathbf{x})\partial_{u_i} - \partial_{x_i} f_i(\mathbf{x})\partial_{u_k}\right) - \sum_{i=1}^{9} \partial_{x_i} f_i(\mathbf{x}) p_k\partial_{p_k} + \ldots, \quad i = 2, 3, 4,$$

$$\vdots$$

$$\Xi_{12} = \sum_{k=1}^{4}(u_k\partial_{u_k} + g_{k+1}\partial_{g_{k+1}}) + \sum_{i=1}^{9} p_i\partial_{p_i},$$

$$\Xi_{12+i} = f_{12+i}(\mathbf{x})\partial_{p_i} + \ldots \quad i = 1, \ldots, 12$$

with f_i $(i = 1, \ldots, 24)$ arbitrary functions of the indicated variables

Projection on $Z \equiv \mathbb{R}^4 \times \mathbb{R}^5$

$$
\begin{aligned}
\Xi_1 &= f_1(x_1)\partial_{x_1} - f_1'(x_1)(u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4}),\\
\Xi_i &= f_i(\mathbf{x})\partial_{x_i} + \sum_{k=1}^{4} u_k\left(\partial_{x_k} f_i(\mathbf{x})\partial_{u_i} - \partial_{x_i} f_i(\mathbf{x})\partial_{u_k}\right), \quad (i = 2, 3, 4),\\
\Xi_{4+i} &= u_i f_{4+i}(\mathbf{x})\partial_{u_5}, \qquad (i = 1, \dots, 5),\\
\Xi_{10} &= f_{10}(\mathbf{x})\partial_{u_5}, \quad \Xi_{11} = \sum_{k=1}^{4} f_{10+k}(\mathbf{x})\partial_{u_k}, \quad \Xi_{12} = \sum_{k=1}^{4} u_k\partial_{u_k},
\end{aligned}
\tag{9}
$$

E.T. by Ξ_1

By considering the operator Ξ_1, in terms of capital variables (X_i, U_k)

$$\Xi_1 = f(X_1)\partial_{X_1} - f'(X_1)\left(U_2\partial_{U_2} + U_3\partial_{U_3} + U_4\partial_{U_4}\right), \tag{10}$$

where $f(X_1) = f_1(X_1)$, we get the finite transformation

$$x_1 = \tilde{x}_1(X_1; a), \qquad x_2 = X_2, \qquad x_3 = X_3, \qquad x_4 = X_4,$$
$$u_1 = U_1, \quad u_2 = U_2\frac{f(X_1)}{f(x_1)}, \quad u_3 = U_3\frac{f(X_1)}{f(x_1)}, \quad u_4 = U_4\frac{f(X_1)}{f(x_1)}, \quad u_5 = U_5, \tag{11}$$

$\tilde{x}_1(X_1; a)$ being such that $\partial_{X_1}\tilde{x}_1 = \frac{f(x_1)}{f(X_1)}$, whereupon we may write

$$U_1 = u_1, \quad U_2 = u_2\partial_{X_1}\tilde{x}_1, \quad U_3 = u_3\partial_{X_1}\tilde{x}_1, \quad U_4 = u_4\partial_{X_1}\tilde{x}_1, \quad U_5 = u_5. \tag{12}$$

Summarizing, the system

$$
\begin{aligned}
&\partial_{X_1} U_1 + \partial_{X_2} U_2 + \partial_{X_3} U_3 + \partial_{X_4} U_4 = 0,\\
&\partial_{X_1} U_2 + \partial_{X_2} P_1 + \partial_{X_3} P_2 + \partial_{X_4} P_3 = 0,\\
&\partial_{X_1} U_3 + \partial_{X_2} P_4 + \partial_{X_3} P_5 + \partial_{X_4} P_6 = 0,\\
&\partial_{X_1} U_4 + \partial_{X_2} P_7 + \partial_{X_3} P_8 + \partial_{X_4} P_9 = 0,\\
&\partial_{X_1} U_5 + \partial_{X_2} P_{10} + \partial_{X_3} P_{11} + \partial_{X_4} P_{12} = 0,
\end{aligned}
\tag{13}
$$

is mapped by

$$
\begin{cases}
x_1 = \tilde{x}_1(X_1; a), \quad x_2 = X_2, \quad x_3 = X_3, \quad x_4 = X_4,\\
U_1 = u_1, \quad U_2 = u_2 \partial_{X_1}\tilde{x}_1, \quad U_3 = u_3 \partial_{X_1}\tilde{x}_1, \quad U_4 = u_4 \partial_{X_1}\tilde{x}_1, \quad U_5 = u_5,
\end{cases}
\tag{14}
$$

with $\partial_{X_1}\tilde{x}_1 = \frac{f(x_1)}{f(X_1)}$, to

$$
\begin{aligned}
&\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 + \partial_{x_4} u_4 = 0,\\
&\partial_{x_1} u_2 + \partial_{x_2} p_1 + \partial_{x_3} p_2 + \partial_{x_4} p_3 = g_2,\\
&\partial_{x_1} u_3 + \partial_{x_2} p_4 + \partial_{x_3} p_5 + \partial_{x_4} p_6 = g_3,\\
&\partial_{x_1} u_4 + \partial_{x_2} p_7 + \partial_{x_3} p_8 + \partial_{x_4} p_9 = g_4,\\
&\partial_{x_1} u_5 + \partial_{x_2} p_{10} + \partial_{x_3} p_{11} + \partial_{x_4} p_{12} = g_5,
\end{aligned}
\tag{15}
$$

with following links between new and old (small and capital) functions

$$
\begin{aligned}
p_k &= \frac{P_k}{(\partial_{X_1}\tilde{x}_1)^2}, \qquad k = 1,\dots,12,\\
g_i &= -u_i \frac{\partial^2_{X_1 X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}, \qquad i = 2,\dots 4,\\
g_5 &= 0,
\end{aligned} \tag{16}
$$

where $P_k = P_k(u_1, u_2\partial_{X_1}\tilde{x}_1, u_3\partial_{X_1}\tilde{x}_1, u_4\partial_{X_1}\tilde{x}_1, u_5)$, $k = 1,\dots,12$.

Remark

By the inverse transformation, a system of balance equations is mapped to a system of conservation laws with P_k, $(k = 1,\dots,12)$, given by (16).

E.T. by Ξ_i

Operators Ξ_i, written in terms of $\mathbf{X}$ and $\mathbf{U}$

$$\Xi_i = f(\mathbf{X})\partial_{X_i} + \sum_{k=1}^{4}\left(U_k\partial_{X_k} f(\mathbf{X})\partial_{U_i} - U_k\partial_{X_i} f(\mathbf{X})\partial_{U_k}\right), \qquad (i = 2, 3, 4),$$

where $f(\mathbf{X}) = f_i(\mathbf{X})$, lead us to finite transformation

$$x_k = \begin{cases} X_k, & k = 1, \dots, 4,\ k \neq i \\ \tilde{x}_k(\mathbf{X}; a), & k = i, \end{cases}$$

$$u_k = \begin{cases} U_k \dfrac{1}{\partial_{X_i}\tilde{x}_i}, & k = 1, \dots, 4,\ k \neq i, \\ U_k + \dfrac{1}{\partial_{X_i}\tilde{x}_i} \displaystyle\sum_{j=1, j\neq i}^{4} U_j \partial_{X_k}\tilde{x}_j, & k = i, \\ U_k, & k = 5, \end{cases}$$

where $\tilde{x}_i(\mathbf{X}; a)$ is such that $\partial_{X_k}\tilde{x}_i = \begin{cases} f(\mathbf{x}) \displaystyle\int_0^a \frac{\partial_{X_k} f(\mathbf{x})}{f(\mathbf{x})} da, & k \neq i \\ \dfrac{f(\mathbf{x})}{f(\mathbf{X})}, & k = i. \end{cases}$

and by introducing the matrix $J_4 = [\partial_{x_k} \tilde{x}_j]$ $(j, k = 1, \dots 4)$ and by defining

$$J = \begin{bmatrix} J_4, & 0, \\ 0 & \partial_{x_i} \tilde{x}_i, \end{bmatrix} \quad \text{and} \quad A = \frac{J}{\partial_{x_i} \tilde{x}_i} \tag{17}$$

we may write

$$\mathbf{u} = A\mathbf{U}.$$

Moreover, by defining the matrices

$$q = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & 0 \\ u_2 & p_1 & p_2 & p_3 & 0 \\ u_3 & p_4 & p_5 & p_6 & 0 \\ u_4 & p_7 & p_8 & p_9 & 0 \\ u_5 & p_{10} & p_{11} & p_{12} & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & 0 \\ U_2 & P_1 & P_2 & P_3 & 0 \\ U_3 & P_4 & P_5 & P_6 & 0 \\ U_4 & P_7 & P_8 & P_9 & 0 \\ U_5 & P_{10} & P_{11} & P_{12} & 0 \end{bmatrix},$$

we can write the systems of balance equations and of conservation laws, respectively

$$\frac{\partial q_{ij}}{\partial x_j} = g_i \qquad \frac{\partial Q_{ij}}{\partial X_j} = 0 \tag{18}$$

Summarizing

$$\frac{\partial Q_{ij}}{\partial X_j} = 0 \quad \text{is mapped to} \quad \frac{\partial q_{ij}}{\partial x_j} = g_i \tag{19}$$

by finite transformations

$$x_k = \begin{cases} X_k, & k = 1, \dots, 4, \; k \neq i \\ \tilde{x}_k(\mathbf{X}; a), & k = i, \end{cases}, \qquad \mathbf{U} = A^{-1}\mathbf{u} \tag{20}$$

with following links between new and old (small and capital) functions

$$\begin{aligned} q &= AQJ^T, \\ g_i &= \sum_{j=1}^{5} A_{ij} \sum_{\ell=1}^{5} \left(\sum_{k=1}^{4} u_k \frac{\partial^2 R_{\ell j}}{\partial U_\ell \partial X_k} - \sum_{k=1}^{5} u_k \frac{\partial^2 R_{\ell j}}{\partial U_k \partial X_\ell} \right), \quad i = 2, \dots, 5, \end{aligned} \tag{21}$$

where $R_{\ell j}$ is the generic entry of the matrix JQ^T,

Physical applications

We construct the finite transformations corresponding of linear combinations of Ξ_1, Ξ_2, Ξ_3 and Ξ_4.

$$\Xi = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4$$

and we assume

$$f_2(\mathbf{X}) = n_1(X_1)X_2 + n_2(X_1)X_3, \quad f_3(\mathbf{X}) = -n_2(X_1)X_2 + n_1(X_1)X_3,$$
$$f_4(\mathbf{X}) = n_3(X_1), \tag{22}$$

with $n_i(X_1)$, $(i = 1, \ldots, 3)$, arbitrary functions of X_1.

Integration of Lie's equations provides:

$$
\begin{aligned}
x_1 &= \tilde{x}_1(X_1; a), \qquad x_4 = \tilde{x}_4(X_1, X_4; a) = X_4 + m_3,\\
x_2 &= \tilde{x}_2(X_1, X_2, X_3; a) = \exp(m_1)(X_2 \cos(m_2) + X_3 \sin(m_2)),\\
x_3 &= \tilde{x}_3(X_1, X_2, X_3; a) = \exp(m_1)(-X_2 \sin(m_2) + X_3 \cos(m_2)),
\end{aligned} \tag{23}
$$

where

$$
m_i(X_1; a) = \int_{X_1}^{x_1} \frac{n_i(s)}{f_1(s)} ds, \qquad \tilde{n}_i(X_1; a) = n_i(x_1) - n_i(X_1), \quad i = 1, 2, 3,
$$

$$
\partial_{X_1}\tilde{x}_1 = \frac{f_1(x_1)}{f_1(X_1)}, \qquad \partial_{X_1}\tilde{x}_2 = \frac{\tilde{n}_1(X_1; a)x_2 + \tilde{n}_2(X_1; a)x_3}{f_1(X_1)},
$$

$$
\partial_{X_1}\tilde{x}_3 = \frac{-\tilde{n}_2(X_1; a)x_2 + \tilde{n}_1(X_1; a)x_3}{f_1(X_1)}, \qquad \partial_{X_1}\tilde{x}_4 = \frac{\tilde{n}_3(X_1; a)}{f_1(X_1)}.
$$

by which

$$
A = \frac{1}{\exp(2m_1)\partial_{X_1}\tilde{x}_1}
\begin{bmatrix}
\partial_{X_1}\tilde{x}_1 & 0 & 0 & 0 & 0\\
\partial_{X_1}\tilde{x}_2 & \exp(m_1)\cos(m_2) & \exp(m_1)\sin(m_2) & 0 & 0\\
\partial_{X_1}\tilde{x}_3 & -\exp(m_1)\sin(m_2) & \exp(m_1)\cos(m_2) & 0 & 0\\
\partial_{X_1}\tilde{x}_4 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & \partial_{X_1}\tilde{x}_1
\end{bmatrix}
$$

and we get

$$
u = AU \tag{24}
$$

and

$$q = AQJ^T,$$

$$g_i = \sum_{j=1}^{5} A_{ij} \sum_{\ell=1}^{5} \left(\sum_{k=1}^{4} u_k \frac{\partial^2 R_{\ell j}}{\partial U_\ell \partial X_k} - \sum_{k=1}^{5} u_k \frac{\partial^2 R_{\ell j}}{\partial U_k \partial X_\ell} \right), \qquad i = 2, \dots, 5, \tag{25}$$

where $R_{\ell j}$ is the generic entry of the matrix JQ^T.
In conclusion, the system of conservation laws is mapped to the system of balance equations.

$$\frac{\partial Q_{ij}}{\partial X_j} = 0 \quad \text{is mapped to} \quad \frac{\partial q_{ij}}{\partial x_j} = g_i. \tag{26}$$

Example (3D unsteady Euler equations)

With the choices

$$U_1 = \rho, \quad U_2 = \rho u, \quad U_3 = \rho v, \quad U_4 = \rho w, \quad U_5 = \rho S,$$

$$P_1 = \frac{U_2^2}{U_1} + p(U_1, U_5), \quad P_2 = P_4 = \frac{U_2 U_3}{U_1}, \quad P_3 = P_7 = \frac{U_2 U_4}{U_1},$$

$$P_5 = \frac{U_3^2}{U_1} + p(U_1, U_5), \quad P_6 = P_8 = \frac{U_3 U_4}{U_1}, \quad P_9 = \frac{U_4^2}{U_1} + p(U_1, U_5),$$

$$P_{10} = \frac{U_2 U_5}{U_1}, \quad P_{11} = \frac{U_3 U_5}{U_1}, \quad P_{12} = \frac{U_4 U_5}{U_1},$$

ρ being the mass density, (u, v, w) the component of the velocity, S the entropy, and $p(\rho, S)$ the pressure,

Example (3D unsteady Euler equations)

we get the 3D unsteady flow of an ideal fluid subject to no extraneous force:

$$\begin{aligned}
&\partial_{x_1} U_1 + \partial_{x_2} U_2 + \partial_{x_3} U_3 + \partial_{x_4} U_4 = 0,\\
&\partial_{x_1} U_2 + \partial_{x_2}\left(\frac{U_2^2}{U_1} + p(U_1, U_5)\right) + \partial_{x_3}\left(\frac{U_2 U_3}{U1}\right) + \partial_{x_4}\left(\frac{U_2 U_4}{U_1}\right) = 0,\\
&\partial_{x_1} U_3 + \partial_{x_2}\left(\frac{U_2 U_3}{U_1}\right) + \partial_{x_3}\left(\frac{U_3^2}{U_1} + p(U_1, U_5)\right) + \partial_{x_4}\left(\frac{U_3 U_4}{U_1}\right) = 0,\\
&\partial_{x_1} U_4 + \partial_{x_2}\left(\frac{U_2 U_4}{U_1}\right) + \partial_{x_3}\left(\frac{U_3 U_4}{U_1}\right) + \partial_{x_4}\left(\frac{U_4^2}{U_1} + p(U_1, U_5)\right) = 0,\\
&\partial_{x_1} U_5 + \partial_{x_2}\left(\frac{U_2 U_5}{U_1}\right) + \partial_{x_3}\left(\frac{U_3 U_5}{U_1}\right) + \partial_{x_4}\left(\frac{U_4 U_5}{U_1}\right) = 0.
\end{aligned} \tag{27}$$

Example (3D unsteady Euler equations)

Arbitrary functions:

$$p_1 = \frac{u_2^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{x_1}\tilde{x}_1)^2}, \quad p_2 = p_4 = \frac{u_2 u_3}{u_1}, \quad p_3 = p_7 = \frac{u_2 u_4}{u_1},$$

$$p_5 = \frac{u_3^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{x_1}\tilde{x}_1)^2}, \quad p_6 = p_8 = \frac{u_3 u_4}{u_1},$$

$$p_9 = \frac{u_4^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{x_1}\tilde{x}_1)^2}, \quad p_{10} = \frac{u_2 u_5}{u_1}, \quad p_{11} = \frac{u_3 u_5}{u_1}, \quad p_{12} = \frac{u_4 u_5}{u_1},$$

$$g_2 = 2\frac{\partial_{x_1} m_1 u_2 + \partial_{x_1} m_2 u_3}{\partial_{x_1}\tilde{x}_1} - \frac{\partial^2_{x_1 x_1}\tilde{x}_1}{(\partial_{x_1}\tilde{x}_1)^2} u_2 + \frac{x_2(\partial^2_{x_1 x_1} m_1 - (\partial_{x_1} m_1)^2 + (\partial_{x_1} m_2)^2) + x_3(\partial^2_{x_1 x_1} m_2 - 2\partial_{x_1} m_1 \partial_{x_1} m_2)}{(\partial_{x_1}\tilde{x}_1)^2} u_1,$$

$$g_3 = 2\frac{\partial_{x_1} m_1 u_3 - \partial_{x_1} m_2 u_2}{\partial_{x_1}\tilde{x}_1} - \frac{\partial^2_{x_1 x_1}\tilde{x}_1}{(\partial_{x_1}\tilde{x}_1)^2} u_3 + \frac{-x_2(\partial^2_{x_1 x_1} m_2 - 2\partial_{x_1} m_1 \partial_{x_1} m_2) + x_3(\partial^2_{x_1 x_1} m_1 - (\partial_{x_1} m_1)^2 + (\partial_{x_1} m_2)^2)}{(\partial_{x_1}\tilde{x}_1)^2} u_1,$$

$$g_4 = -\frac{\partial^2_{x_1 x_1}\tilde{x}_1}{(\partial_{x_1}\tilde{x}_1)^2} u_4 + \frac{\partial^2_{x_1 x_1}\tilde{x}_4}{(\partial_{x_1}\tilde{x}_1)^2} u_1, \qquad g_5 = 2\frac{\partial_{x_1} m_1}{\partial_{x_1}\tilde{x}_1} u_5.$$

Example (3D unsteady Euler equations)

In order to preserve the autonomous form, *i.e.*, $p_i(\mathbf{u}) = P_i(\mathbf{U})$, $(i = 1, \dots, 12)$, we choose

$$\partial_{X_1} \tilde{x}_1 = 1, \quad \Rightarrow x_1 = X_1 + a, \quad \text{and} \quad m_1(X_1; a) = 0$$

and to have a physical meaning of source terms g_k $(k = 2, \dots 5)$ we take

$$m_2(X_1; a) = \omega X_1 + X_{1_0} \qquad m_3(X_1; a) = \frac{g X_1^2}{2} + a_1 X_1 + a_0$$

where ω, a_0, a_1, a_2, g and X_{1_0} are constants.
In conclusion we characterize the finite transformation *a posteriori* with

$$\begin{aligned} & f_1(X_1) = \text{const} \\ & f_2(\mathbf{X}) = \omega X_3, \quad f_3(\mathbf{X}) = -\omega X_2, \quad f_4(\mathbf{X}) = g X_1 + a_1, \end{aligned} \tag{28}$$

Example (3D unsteady Euler equations)

We get the system

$$
\begin{aligned}
&\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 + \partial_{x_4} u_4 = 0,\\
&\partial_{x_1} u_2 + \partial_{x_2}\left(\frac{u_2^2}{u_1} + p(u_1,u_5)\right) + \partial_{x_3}\left(\frac{u_2 u_3}{u_1}\right) + \partial_{x_4}\left(\frac{u_2 u_4}{u_1}\right) = 2\omega u_3 - \omega^2 x_2 u_1,\\
&\partial_{x_1} u_3 + \partial_{x_2}\left(\frac{u_2 u_3}{u_1}\right) + \partial_{x_3}\left(\frac{u_3^2}{u_1} + p(u_1,u_5)\right) + \partial_{x_4}\left(\frac{u_3 u_4}{u_1}\right) = -2\omega u_2 + \omega^2 x_3 u_1,\\
&\partial_{x_1} u_4 + \partial_{x_2}\left(\frac{u_2 u_4}{u_1}\right) + \partial_{x_3}\left(\frac{u_3 u_4}{u_1}\right) + \partial_{x_4}\left(\frac{u_4^2}{u_1} + p(u_1,u_5)\right) = g u_1,\\
&\partial_{x_1} u_5 + \partial_{x_2}\left(\frac{u_2 u_5}{u_1}\right) + \partial_{x_3}\left(\frac{u_3 u_5}{u_1}\right) + \partial_{x_4}\left(\frac{u_4 u_5}{u_1}\right) = 0.
\end{aligned}
\tag{29}
$$

describes an ideal gas in a non–inertial frame rotating with constant angular velocity ω around the vertical x_4–axis and subject to gravity.

Remark

Inverting the procedure:
System (29) can be transformed in a form where the gravity and apparent forces disappear.

Thank you for your attention