Decoupling of $(1+1)-$Dimensional First Order Quasilinear Systems

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Problem

When an autonomous and homogeneous quasilinear first order system of PDEs

$$\frac{\partial u_l}{\partial x_1} = \sum_{m=1}^{n} A_m^l(u_1, \ldots, u_n)\frac{\partial u_m}{\partial x_2}, \qquad (l = 1, \ldots, n), \tag{1}$$

can be decoupled in some new coordinates $v_1(\mathbf{u}), \ldots, v_n(\mathbf{u})$ into k non-interacting subsystems

$$\frac{\partial v_{m_j+i}}{\partial x_1} = \sum_{l=1}^{n_j} \tilde{A}_{m_j+l}^{m_j+i}(v_{m_j+1}, \ldots, v_{m_j+n_j})\frac{\partial v_{m_j+l}}{\partial x_2} \tag{2}$$

$$(j = 1, \ldots, k, \quad i = 1, \ldots, n_j, \quad m_j = n_1 + \ldots + n_{j-1})$$

of some orders $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$?

Theorem 1 [Nijenhuis]

The necessary and sufficient condition for the complete decoupling of the system (1) into n non-interacting one-dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor

$$N^j_{ik} = A^\alpha_i \frac{\partial A^j_k}{\partial u_\alpha} - A^\alpha_k \frac{\partial A^j_i}{\partial u_\alpha} + A^j_\alpha \left(\frac{\partial A^\alpha_i}{\partial u_k} - \frac{\partial A^\alpha_k}{\partial u_i} \right),$$

provided that all eigenvalues of $A^i_j(\mathbf{u})$ are real and distinct.

Necessary condition for decoupling

A necessary condition for the decoupling of (1) into k non-interacting subsystems is that the polynomial $P_N(V, \lambda) = \det(N_V - \lambda)$ should have degree $n - k$ in variables V and should be a product of k factors.
V is a tangent vector, $V \in T_u(\mathbb{R}^n)$, and $(N_V)^i_j = N^i_{\alpha j} V^\alpha$.

NonAutonomus and NonHomogeneous Systems

Given a system of PDEs,

$$\sum_{i=1}^{n} p^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $p \in M_{m,m}(\mathbb{R})$, $\mathbf{g} \in \mathbb{R}^m$, $p_{ij}(\mathbf{x}, \mathbf{u})$ and $g_i(\mathbf{x}, \mathbf{u})$ arbitrary functions of the indicated arguments.
Through an invertible transformation preserving the quasilinear form we obtain

$$\sum_{i=1}^{n} P^i(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_i} = \mathbf{G}(\mathbf{X}, \mathbf{U}) \tag{4}$$

which will be analyzed for the decoupling problem.

2 × 2 Systems

Let us consider a 2 × 2 system of PDEs,

$$\frac{\partial \mathbf{u}}{\partial x_1} + p(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \qquad (\textit{source system}) \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^2$, $p \in M_{2,2}(\mathbb{R})$, $\mathbf{g} \in \mathbb{R}^2$, $p_{ij}(\mathbf{x}, \mathbf{u})$ and $g_i(\mathbf{x}, \mathbf{u})$ arbitrary functions of the indicated arguments.
Introducing $\mathbf{X} \equiv (X_1, X_2)$ and $\mathbf{U} \equiv (U_1, U_2)$ linked to $\mathbf{x}$ and $\mathbf{u}$ by

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{u} = \mathbf{F}(\mathbf{x}, \mathbf{U}), \tag{6}$$

the quasilinear form of (5) is preserved and we obtain

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}) \qquad (\textit{target system}) \tag{7}$$

Remark

The previous result is coherent with the equivalence transformations admitted by (5). In fact, their projection on the space (x_1, x_2, u_1, u_2) is generated by the following vector fields:

$$\begin{aligned} \Xi_1 &= f_1(x_1, x_2)\frac{\partial}{\partial x_1}, \qquad \Xi_2 = f_2(x_1, x_2)\frac{\partial}{\partial x_2}, \\ \Xi_3 &= f_3(x_1, x_2, u_1, u_2)\frac{\partial}{\partial u_1}, \qquad \Xi_4 = f_4(x_1, x_2, u_1, u_2)\frac{\partial}{\partial u_2}, \end{aligned} \tag{8}$$

with f_i $(i = 1, \ldots, 4)$ arbitrary functions of the indicated arguments.
The finite transformations corresponding to (8) provide a map like (6) preserving the differential structure of (5).

Arbitrary Functions Transformation

By defining

$$W = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} \end{bmatrix}, \qquad J = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} \end{bmatrix},$$

the matrix P and the vector $\mathbf{G}$ involved in (7) have the form

$$\begin{aligned} P &= (\alpha I + \beta W^{-1} p W), \\ \mathbf{G} &= \frac{1}{\det J} \left((J_{22} I - J_{12} P) W^{-1} (\mathbf{g} - \frac{\partial \mathbf{F}}{\partial x_1}) + (J_{21} I - J_{11} P) W^{-1} \frac{\partial \mathbf{F}}{\partial x_2} \right), \end{aligned} \tag{9}$$

where

$$\alpha = \frac{1}{J_{11}} \left(J_{21} + \frac{\det[Jp] J_{12}}{\det[J_{11} I + J_{12} p]} \right), \qquad \beta = \frac{\det[J]}{\det[J_{11} I + J_{12} p]}.$$

Considering the target system

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}), \tag{10}$$

two main cases can be distinguished:

Considering the target system

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}), \tag{10}$$

two main cases can be distinguished:

Partial Decoupling

$$P_{12} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial G_1}{\partial U_2} = 0$$

Considering the target system

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Full Decoupling

$$P_{12} = P_{21} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial P_{22}}{\partial U_1} = \frac{\partial G_1}{\partial U_2} = \frac{\partial G_2}{\partial U_1} = 0$$

Considering the target system

$$\frac{\partial \mathbf{U}}{\partial X_1} + P(\mathbf{X}, \mathbf{U}) \frac{\partial \mathbf{U}}{\partial X_2} = \mathbf{G}(\mathbf{X}, \mathbf{U}), \tag{10}$$

two main cases can be distinguished:

Partial Decoupling

$$P_{12} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial G_1}{\partial U_2} = 0$$

Full Decoupling

$$P_{12} = P_{21} = \frac{\partial P_{11}}{\partial U_2} = \frac{\partial P_{22}}{\partial U_1} = \frac{\partial G_1}{\partial U_2} = \frac{\partial G_2}{\partial U_1} = 0$$

Since the transformation

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{u} = \mathbf{F}(\mathbf{x}, \mathbf{U}),$$

is invertible, in the following we will consider

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{U} = \mathbf{H}(\mathbf{x}, \mathbf{u}).$$

Partial Decoupling

1. $p_{12}p_{21} \neq 0$, from $P_{12} = 0$, $\dfrac{\partial P_{11}}{\partial U_2} = 0$ and $\dfrac{\partial G_1}{\partial U_2} = 0$ we get

$$\begin{cases} p_{12}\dfrac{\partial H_1}{\partial u_1} + (\lambda_\pm - p_{11})\dfrac{\partial H_1}{\partial u_2} = 0, \quad \left(\lambda_\pm = \dfrac{tr\,(p) \pm \sqrt{(tr\,(p))^2 - 4\det p}}{2}\right) \\ \dfrac{\partial \lambda_\mp}{\partial u_2}\dfrac{\partial H_1}{\partial u_1} - \dfrac{\partial \lambda_\mp}{\partial u_1}\dfrac{\partial H_1}{\partial u_2} = 0, \\ \left(\dfrac{\partial H_1}{\partial u_1}\right)^2 \dfrac{\partial g_1}{\partial u_2} + \dfrac{\partial H_1}{\partial u_1}\dfrac{\partial H_1}{\partial u_2}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - \left(\dfrac{\partial H_1}{\partial u_2}\right)^2 \dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{11}$$

1. $p_{12}p_{21} \neq 0$, from $P_{12} = 0$, $\dfrac{\partial P_{11}}{\partial U_2} = 0$ and $\dfrac{\partial G_1}{\partial U_2} = 0$ we get

$$\begin{cases} p_{12}\dfrac{\partial H_1}{\partial u_1} + (\lambda_\pm - p_{11})\dfrac{\partial H_1}{\partial u_2} = 0, \quad \left(\lambda_\pm = \dfrac{tr\,(p) \pm \sqrt{(tr\,(p))^2 - 4\det p}}{2}\right) \\ \dfrac{\partial \lambda_\mp}{\partial u_2}\dfrac{\partial H_1}{\partial u_1} - \dfrac{\partial \lambda_\mp}{\partial u_1}\dfrac{\partial H_1}{\partial u_2} = 0, \\ \left(\dfrac{\partial H_1}{\partial u_1}\right)^2 \dfrac{\partial g_1}{\partial u_2} + \dfrac{\partial H_1}{\partial u_1}\dfrac{\partial H_1}{\partial u_2}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - \left(\dfrac{\partial H_1}{\partial u_2}\right)^2 \dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{11}$$

Considering the right eigenvectors $\mathbf{r}_\pm$ of matrix p and the gradient operator $\nabla_\mathbf{u}$ with respect the field components

$$\mathbf{r}_\pm = \begin{pmatrix} p_{12} \\ \lambda_\pm - p_{11} \end{pmatrix}, \qquad \nabla_\mathbf{u} = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right),$$

$$\begin{cases} \nabla_\mathbf{u} H_1 \cdot \mathbf{r}_\pm = 0, \qquad \nabla_\mathbf{u}\lambda_\mp \cdot \mathbf{r}_\pm = 0, \\ \left(\dfrac{\partial H_1}{\partial u_1}\right)^2 \dfrac{\partial g_1}{\partial u_2} + \dfrac{\partial H_1}{\partial u_1}\dfrac{\partial H_1}{\partial u_2}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - \left(\dfrac{\partial H_1}{\partial u_2}\right)^2 \dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{12}$$

The coefficients of the PARTIALLY DECOUPLED SYSTEM are

$$P_{11} = \frac{J_{21} + J_{22}\lambda_{\mp}}{J_{11} + J_{12}\lambda_{\mp}}, \qquad P_{12} = 0,$$
$$P_{21} = \beta \frac{\nabla_{\mathbf{u}} H_2 \cdot \mathbf{r}_{\mp}}{\frac{\partial H_1}{\partial u_2}}, \qquad P_{22} = \frac{J_{21} + J_{22}\lambda_{\pm}}{J_{11} + J_{12}\lambda_{\pm}},$$
$$G_1 = \frac{(J_{21} - J_{11}P_{11})\frac{\partial H_1}{\partial x_2} + (J_{22} - J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1} + g_2\frac{\partial H_1}{\partial u_2} - \frac{\partial H_1}{\partial x_1})}{\det J},$$
$$G_2 = \frac{J_{22}(g_s\frac{\partial H_2}{\partial u_s} - \frac{\partial H_2}{\partial x_1}) + J_{21}\frac{\partial H_2}{\partial x_2} - P_{2k}(J_{11}\frac{\partial H_k}{\partial x_2} + J_{12}(g_s\frac{\partial H_k}{\partial u_s} - \frac{\partial H_k}{\partial x_1}))}{\det J}, \tag{13}$$

with $k, s = 1, 2$, where the convention of sum over repeated indices has been used.

Partial Decoupling

2. $p_{12} \neq 0$ and $p_{21} = 0$, PARTIAL DECOUPLING is obtained if

$$\frac{\partial H_1}{\partial u_1} = 0, \qquad \frac{\partial p_{22}}{\partial u_1} = 0, \qquad \frac{\partial g_2}{\partial u_1} = 0, \tag{14}$$

or

$$\begin{cases} p_{12}\dfrac{\partial H_1}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial H_1}{\partial u_2} = 0, \\ p_{12}\dfrac{\partial p_{11}}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial p_{11}}{\partial u_2} = 0, \\ (p_{11} - p_{22})^2\dfrac{\partial g_1}{\partial u_2} + p_{12}(p_{11} - p_{22})\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - p_{12}^2\dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{15}$$

2. $p_{12} \neq 0$ and $p_{21} = 0$, PARTIAL DECOUPLING is obtained if

$$\frac{\partial H_1}{\partial u_1} = 0, \qquad \frac{\partial p_{22}}{\partial u_1} = 0, \qquad \frac{\partial g_2}{\partial u_1} = 0, \tag{14}$$

or

$$\begin{cases} p_{12}\dfrac{\partial H_1}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial H_1}{\partial u_2} = 0, \\ p_{12}\dfrac{\partial p_{11}}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial p_{11}}{\partial u_2} = 0, \\ (p_{11} - p_{22})^2\dfrac{\partial g_1}{\partial u_2} + p_{12}(p_{11} - p_{22})\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - p_{12}^2\dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{15}$$

3. $p_{12} = p_{21} = 0$, we get

$$\frac{\partial H_1}{\partial u_1} = 0, \qquad \frac{\partial p_{22}}{\partial u_1} = 0, \qquad \frac{\partial g_2}{\partial u_1} = 0, \tag{16}$$

or

$$\frac{\partial H_1}{\partial u_2} = 0, \qquad \frac{\partial p_{11}}{\partial u_2} = 0, \qquad \frac{\partial g_1}{\partial u_2} = 0. \tag{17}$$

1. $p_{12}p_{21} \neq 0$, from
$P_{12} = P_{21} = \dfrac{\partial P_{11}}{\partial U_2} = \dfrac{\partial P_{22}}{\partial U_1} = \dfrac{\partial G_1}{\partial U_2} = \dfrac{\partial G_2}{\partial U_1} = 0$ we get

$$\begin{cases} p_{12}\dfrac{\partial H_1}{\partial u_1} + (\lambda_\pm - p_{11})\dfrac{\partial H_1}{\partial u_2} = 0, \\ p_{21}\dfrac{\partial H_2}{\partial u_2} + (\lambda_\mp - p_{22})\dfrac{\partial H_2}{\partial u_1} = 0, \\ \dfrac{\partial \lambda_\mp}{\partial u_1}\dfrac{\partial H_1}{\partial u_2} - \dfrac{\partial \lambda_\mp}{\partial u_2}\dfrac{\partial H_1}{\partial u_1} = 0, \\ \dfrac{\partial \lambda_\pm}{\partial u_1}\dfrac{\partial H_2}{\partial u_2} - \dfrac{\partial \lambda_\pm}{\partial u_2}\dfrac{\partial H_2}{\partial u_1} = 0, \\ \left(\dfrac{\partial H_1}{\partial u_1}\right)^2 \dfrac{\partial g_1}{\partial u_2} + \dfrac{\partial H_1}{\partial u_1}\dfrac{\partial H_1}{\partial u_2}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - \left(\dfrac{\partial H_1}{\partial u_2}\right)^2 \dfrac{\partial g_2}{\partial u_1} = 0, \\ \left(\dfrac{\partial H_2}{\partial u_1}\right)^2 \dfrac{\partial g_1}{\partial u_2} + \dfrac{\partial H_2}{\partial u_1}\dfrac{\partial H_2}{\partial u_2}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) - \left(\dfrac{\partial H_2}{\partial u_2}\right)^2 \dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{18}$$

Considering the right eigenvectors $\mathbf{r}_{\pm}$ of matrix p and the gradient operator $\nabla_{\mathbf{u}}$ with respect the field components, relations (18) can be rewritten as

$$\begin{cases} \nabla_{\mathbf{u}} H_1 \cdot \mathbf{r}_{\pm} = 0, \qquad \nabla_{\mathbf{u}} H_2 \cdot \mathbf{r}_{\mp} = 0, \\ \nabla_{\mathbf{u}} \lambda_- \cdot \mathbf{r}_+ = 0, \qquad \nabla_{\mathbf{u}} \lambda_+ \cdot \mathbf{r}_- = 0, \\ \left(\frac{\partial H_1}{\partial u_1}\right)^2 \frac{\partial g_1}{\partial u_2} + \frac{\partial H_1}{\partial u_1}\frac{\partial H_1}{\partial u_2}\left(\frac{\partial g_2}{\partial u_2} - \frac{\partial g_1}{\partial u_1}\right) - \left(\frac{\partial H_1}{\partial u_2}\right)^2 \frac{\partial g_2}{\partial u_1} = 0, \\ \left(\frac{\partial H_2}{\partial u_1}\right)^2 \frac{\partial g_1}{\partial u_2} + \frac{\partial H_2}{\partial u_1}\frac{\partial H_2}{\partial u_2}\left(\frac{\partial g_2}{\partial u_2} - \frac{\partial g_1}{\partial u_1}\right) - \left(\frac{\partial H_2}{\partial u_2}\right)^2 \frac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{19}$$

The coefficients of the FULLY DECOUPLED SYSTEM are

$$P_{11} = \frac{J_{21} + J_{22}\lambda_{\mp}}{J_{11} + J_{12}\lambda_{\mp}}, \qquad P_{12} = P_{21} = 0, \qquad P_{22} = \frac{J_{21} + J_{22}\lambda_{\pm}}{J_{11} + J_{12}\lambda_{\pm}},$$
$$G_1 = \frac{(J_{21} - J_{11}P_{11})\frac{\partial H_1}{\partial x_2} + (J_{22} - J_{12}P_{11})(g_1\frac{\partial H_1}{\partial u_1} + g_2\frac{\partial H_1}{\partial u_2} - \frac{\partial H_1}{\partial x_1})}{\det J},$$
$$G_2 = \frac{(J_{21} - J_{11}P_{22})\frac{\partial H_2}{\partial x_2} + (J_{22} - J_{12}P_{22})(g_1\frac{\partial H_2}{\partial u_1} + g_2\frac{\partial H_2}{\partial u_2} - \frac{\partial H_2}{\partial x_1})}{\det J}. \tag{20}$$

2. $p_{12} \neq 0,\ p_{21} = 0,$ FULL DECOUPLING is obtained if

$$\begin{cases} \dfrac{\partial H_i}{\partial u_1} = 0, \qquad (i \neq j, \quad i,j = 1,2) \\ p_{12}\dfrac{\partial H_j}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial H_j}{\partial u_2} = 0, \\ \dfrac{\partial p_{22}}{\partial u_1} = 0, \qquad \dfrac{\partial g_2}{\partial u_1} = 0, \\ p_{12}\dfrac{\partial p_{11}}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial p_{11}}{\partial u_2} = 0, \\ p_{12}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) + (p_{11} - p_{22})\dfrac{\partial g_1}{\partial u_2} = 0. \end{cases} \tag{21}$$

2. $p_{12} \neq 0,\ p_{21} = 0,$ FULL DECOUPLING is obtained if

$$\begin{cases} \dfrac{\partial H_i}{\partial u_1} = 0, \qquad (i \neq j, \quad i,j = 1,2) \\ p_{12}\dfrac{\partial H_j}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial H_j}{\partial u_2} = 0, \\ \dfrac{\partial p_{22}}{\partial u_1} = 0, \qquad \dfrac{\partial g_2}{\partial u_1} = 0, \\ p_{12}\dfrac{\partial p_{11}}{\partial u_1} - (p_{11} - p_{22})\dfrac{\partial p_{11}}{\partial u_2} = 0, \\ p_{12}\left(\dfrac{\partial g_2}{\partial u_2} - \dfrac{\partial g_1}{\partial u_1}\right) + (p_{11} - p_{22})\dfrac{\partial g_1}{\partial u_2} = 0. \end{cases} \tag{21}$$

3. $p_{12} = p_{21} = 0,$ we get

$$\begin{cases} \dfrac{\partial H_1}{\partial u_i} = 0, \qquad \dfrac{\partial p_{11}}{\partial u_2} = 0, \qquad \dfrac{\partial g_1}{\partial u_2} = 0, \qquad (i \neq j, \quad i,j = 1,2) \\ \dfrac{\partial H_2}{\partial u_j} = 0, \qquad \dfrac{\partial p_{22}}{\partial u_1} = 0, \qquad \dfrac{\partial g_2}{\partial u_1} = 0. \end{cases} \tag{22}$$

Galilean Systems

Let us require the invariance of system

$$\frac{\partial \mathbf{u}}{\partial x_1} + p(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

with respect the vector fields:

$$\begin{aligned} \Xi_1 &= x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_1}, \\ \Xi_2 &= x_1 \frac{\partial}{\partial x_1} - u_1 \frac{\partial}{\partial u_1} + a\, u_2 \frac{\partial}{\partial u_2}, \\ \Xi_3 &= x_2 \frac{\partial}{\partial x_2} + u_1 \frac{\partial}{\partial u_1} + b\, u_2 \frac{\partial}{\partial u_2}, \end{aligned} \tag{23}$$

where a, b are arbitrary constants.

. . .

By requiring invariance with respect (23) we get the following functional form for p_{ij} and g_i:

$$\begin{aligned} p_{11} &= u_1 + x_1^{-1}\xi\hat{p}_{11}(\omega), & p_{12} &= x_1^{-(a+2)}\xi^{2-b}\hat{p}_{12}(\omega), \\ p_{21} &= x_1^a\xi^b\hat{p}_{21}(\omega), & p_{22} &= u_1 + x_1^{-1}\xi\hat{p}_{22}(\omega), \\ g_1 &= x_1^{-2}\xi\hat{g}_2(\omega), & g_2 &= x_1^{a-1}\xi^b\hat{g}_2(\omega), \end{aligned} \tag{24}$$

where $\xi = x_2 - u_1x_1$, $\omega = \frac{u_2}{x_1^a\xi^b}$ and $\lambda_\pm = u_1 + \frac{\xi}{x_1}\hat{\lambda}_\pm$.

Constraints Partial Decoupling

1. $$x_1^{-a-1}\xi^{1-b}\hat{p}_{12}\frac{\partial H_1}{\partial u_1}+(\hat{\lambda}_\pm-\hat{p}_{11})\frac{\partial H_1}{\partial u_2}=0,$$
2. $$\left(\hat{\lambda}_\pm-\hat{p}_{11}+b\omega\hat{p}_{12}\right)\frac{\partial\hat{\lambda}_\mp}{\partial\omega}+\hat{p}_{12}=0,$$
3. $$(x_1^{-a-1}\xi^{1-b})^2\frac{\partial\hat{g}_1}{\partial\omega}\left(\frac{\partial H_1}{\partial u_1}\right)^2+$$
$$+x_1^{-a-1}\xi^{1-b}\left(\frac{\partial\hat{g}_2}{\partial\omega}-b\omega\frac{\partial\hat{g}_1}{\partial\omega}+\hat{g}_1\right)\frac{\partial H_1}{\partial u_1}\frac{\partial H_1}{\partial u_2}+$$
$$+b\left(\hat{g}_2-\omega\frac{\partial\hat{g}_2}{\partial\omega}\right)\left(\frac{\partial H_1}{\partial u_2}\right)^2=0.$$

THANKS FOR YOUR ATTENTION