# Decoupling of first order quasilinear systems of PDEs

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## Problem

*When can a system like*

$$\frac{\partial \mathbf{u}}{\partial x_1} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0},$$

*where* $\mathbf{u} \equiv (u_1, \dots, u_n)^T \in \mathbb{R}^n$ *and* $a$ *is a* $n \times n$ *matrix with real entries depending on* $\mathbf{u}$*, be locally decoupled in some coordinates* $v_1(\mathbf{u}), \dots, v_n(\mathbf{u})$ *into* $k$ *non-interacting subsystems of some orders* $n_1, \dots, n_k$ *with* $n_1 + \cdots + n_k = n$*?*

# Necessary and Sufficient Conditions

## Theorem [Nijenhuis]

The necessary and sufficient condition for the complete decoupling of the system

$$\frac{\partial \mathbf{u}}{\partial x_1} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}$$

into $n$ non-interacting one-dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor[a]

$$N_{jik} = a_{\alpha i}\frac{\partial a_{jk}}{\partial u_\alpha} - a_{\alpha k}\frac{\partial a_{ji}}{\partial u_\alpha} + a_{j\alpha}\frac{\partial a_{\alpha i}}{\partial u_k} - a_{j\alpha}\frac{\partial a_{\alpha k}}{\partial u_i},$$

provided that all eigenvalues of $a$ $(a_{ij})$ are real and distinct.

[a] A. Nijenhuis. Proc. Kon. Ned. Akad. Amsterdam, **54**, 200–212, 1951.

## Necessary and Sufficient Conditions

Bogoyavlenskij[a] provided necessary and sufficient conditions for the decoupling of quasilinear first order systems into non–interacting diagonal blocks.

Tunitsky[b] established necessary and sufficient conditions for transforming quasilinear first order systems into triangular blocks (*i.e.*, partially decoupled systems).

[a] O. I. Bogoyavlenskij. Commun. Math. Phys., **269**, 545–556, 2007.
[b] D. V. Tunitsky. Sbornik: Mathematics, **204**, 438–462, 2013.

## Remark

Both the results obtained are based on Nijenhuis and Haantjes tensors.

## Notation

Let $\mathbf{U} \equiv (U_1, U_2, \dots, U_n)^T \in \mathbb{R}^n$. Let us relabel and group the components of $\mathbf{U}$ as follows:

$$\{\{U^{(1,1)}, \dots, U^{(1,n_1)}\}, \dots, \{U^{(k,1)}, \dots, U^{(k,n_k)}\}\},$$

with $k > 1$, and $n_1 + \dots + n_k = n$.
We set

$$\mathcal{U}_i = \bigcup_{r=1}^{i} \{U^{(r,1)}, \dots U^{(r,n_r)}\},$$

$$\overline{\mathcal{U}}_i = \bigcup_{r=i+1}^{k} \{U^{(r,1)}, \dots, U^{(r,n_r)}\}.$$

Let us denote with $m_i$ the number of the elements of $\mathcal{U}_i$, where $m_1 = n_1$ and $m_i = m_{i-1} + n_i$ for $i > 1$.

## Definition (Partially decoupled systems)

The first order quasilinear system

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0}, \qquad \mathbf{U} \in \mathbb{R}^n, \quad A \quad n \times n \text{ real matrix,} \tag{1}$$

is partially decoupled in $2 \leq k \leq n$ subsystems of some orders $n_1, \ldots, n_k$ $(n_1 + \ldots + n_k = n)$ if, relabelling and suitably arranging the components of $\mathbf{U}$ in $k$ subgroups and sorting the equations (1), we recognize $k$ subsystems such that the $i$-th subsystem $(i = 1, \ldots, k)$ involves at most the $m_i$ field variables of $\mathcal{U}_i$.

## Definition (Partially decoupled systems)

The first order quasilinear system

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0}, \qquad \mathbf{U} \in \mathbb{R}^n, \quad A \quad n \times n \text{ real matrix}, \tag{1}$$

is partially decoupled in $2 \leq k \leq n$ subsystems of some orders $n_1, \ldots, n_k$ $(n_1 + \ldots + n_k = n)$ if, relabelling and suitably arranging the components of $\mathbf{U}$ in $k$ subgroups and sorting the equations (1), we recognize $k$ subsystems such that the $i$-th subsystem $(i = 1, \ldots, k)$ involves at most the $m_i$ field variables of $\mathcal{U}_i$.

## Definition (Fully decoupled systems)

The system (1) is fully decoupled in $2 \leq k \leq n$ subsystems of some orders $n_1, \ldots, n_k$ $(n_1 + \ldots + n_k = n)$ if we recognize $k$ subsystems such that the $i$-th subsystem $(i = 1, \ldots, k)$ involves exactly the $n_i$ field variables $\{U^{(i,1)}, \ldots, U^{(i,n_i)}\}$, and the various subsystems are non–interacting.

## Lemma

*Let*

$$A = \begin{bmatrix} A_1^1 & 0_2^1 & 0_3^1 & \dots & 0_{k-1}^1 & 0_k^1 \\ A_1^2 & A_2^2 & 0_3^2 & \dots & 0_{k-1}^2 & 0_k^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_1^{k-1} & A_2^{k-1} & A_3^{k-1} & \dots & A_{k-1}^{k-1} & 0_k^{k-1} \\ A_1^k & A_2^k & A_3^k & \dots & A_{k-1}^k & A_k^k \end{bmatrix},$$

*be a matrix with real eigenvalues and a complete set of eigenvectors. The matrices* $A_j^i$ $(i = 1, \dots, k,\ j = 1, \dots, i)$ *depend at most on the* $m_i$ *variables of* $\mathcal{U}_i$ *if and only if the set of the eigenvalues of* $A$ *(with corresponding left and right eigenvectors) can be divided into* $k$ *subsets*

$$\left\{\{\Lambda^{(1,1)}, \dots, \Lambda^{(1,n_1)}\}, \dots, \{\Lambda^{(k,1)}, \dots, \Lambda^{(k,n_k)}\}\right\},$$

$$\left\{\{\mathbf{L}^{(1,1)}, \dots, \mathbf{L}^{(1,n_1)}\}, \dots, \{\mathbf{L}^{(k,1)}, \dots, \mathbf{L}^{(k,n_k)}\}\right\},$$

$$\left\{\{\mathbf{R}^{(1,1)}, \dots, \mathbf{R}^{(1,n_1)}\}, \dots, \{\mathbf{R}^{(k,1)}, \dots, \mathbf{R}^{(k,n_k)}\}\right\},$$

...

where

$$\{\Lambda^{(i,1)},\dots,\Lambda^{(i,n_i)}\}$$

are the eigenvalues (counted with their multiplicity) of matrix $A_i^i$, provided that

$$\begin{aligned}
&\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,\\
&\mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0,\\
&i=1,\dots k-1,\quad \ell=1,\dots,i,\\
&\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta\quad\text{if}\quad i=\ell,\\
&j=i+1,\dots,k,\quad \gamma=1,\dots,n_j,
\end{aligned}$$

with

$$\nabla_{\mathbf{U}}\equiv\left(\frac{\partial}{\partial U^{(1,1)}},\dots,\frac{\partial}{\partial U^{(1,n_1)}},\dots,\frac{\partial}{\partial U^{(k,1)}},\dots,\frac{\partial}{\partial U^{(k,n_k)}}\right).$$

# Sketch of the proof: necessary condition

Let us assume that $A^i_j$ ($i = 1, \dots, k$, $j = 1, \dots, i$) depend at most on the elements of the set $\mathcal{U}_i$.
Let us group the components of a right (left, resp.) eigenvector $\mathbf{R}^{(r,\alpha)}$ ($\mathbf{L}^{(r,\alpha)}$, resp.) as follows:

$$\mathbf{R}^{(r,\alpha)} = \begin{pmatrix} \mathbf{R}_1^{(r,\alpha)} \\ \mathbf{R}_2^{(r,\alpha)} \\ \dots \\ \mathbf{R}_k^{(r,\alpha)} \end{pmatrix}, \qquad \mathbf{L}^{(r,\alpha)} = \left(\mathbf{L}_1^{(r,\alpha)}, \mathbf{L}_2^{(r,\alpha)}, \dots, \mathbf{L}_k^{(r,\alpha)}\right).$$

If $r < k$, $\mathbf{L}^{(r,\alpha)} = \left(\mathbf{L}_1^{(r,\alpha)}, \dots, \mathbf{L}_r^{(r,\alpha)}, \mathbf{0}_{r+1}, \dots, \mathbf{0}_k\right)$ and $\mathbf{L}_s^{(r,\alpha)}$ ($s = 1, \dots, r$) depend at most on the elements of $\mathcal{U}_r$.
If $r > 1$, $\mathbf{R}^{(r,\alpha)} = \left(\mathbf{0}_1, \dots, \mathbf{0}_{r-1}, \mathbf{R}_r^{(r,\alpha)}, \dots, \mathbf{R}_k^{(r,\alpha)}\right)^T$ and $\mathbf{R}_r^{(r,\alpha)}$ depend at most on the elements of $\mathcal{U}_r$. As a consequence, the structure conditions are trivially satisfied.

# Sketch of the proof: sufficient condition

Let us assume the structure conditions hold.
Let be $\Lambda$ one of the eigenvalues of the matrix $A_r^r$ $(1 \leq r < k)$, and

$$\nabla_{\mathbf{U}} \equiv \left(\nabla_{\mathbf{U}^{(1)}}, \dots, \nabla_{\mathbf{U}^{(k)}}\right),$$

where

$$\nabla_{\mathbf{U}^{(i)}} \equiv \left(\frac{\partial}{\partial U^{(i,1)}}, \dots, \frac{\partial}{\partial U^{(i,n_i)}}\right), \qquad i = 1, \dots, k.$$

For $j > r$, it is

$$\begin{aligned}
(\nabla_{\mathbf{U}^{(r+1)}}\Lambda)\cdot\mathbf{R}_{r+1}^{(r+1,\gamma)} + (\nabla_{\mathbf{U}^{(r+2)}}\Lambda)\cdot\mathbf{R}_{r+2}^{(r+1,\gamma)} + \dots + (\nabla_{\mathbf{U}^{(k)}}\Lambda)\cdot\mathbf{R}_{k}^{(r+1,\gamma)} &= 0,\\
(\nabla_{\mathbf{U}^{(r+2)}}\Lambda)\cdot\mathbf{R}_{r+2}^{(r+2,\gamma)} + \dots + (\nabla_{\mathbf{U}^{(k)}}\Lambda)\cdot\mathbf{R}_{k}^{(r+2,\gamma)} &= 0,\\
&\dots\\
(\nabla_{\mathbf{U}^{(k)}}\Lambda)\cdot\mathbf{R}_{k}^{(k,\gamma)} &= 0,
\end{aligned}$$

and it follows that

$$\frac{\partial\Lambda}{\partial U^{(r+1,1)}} = \dots = \frac{\partial\Lambda}{\partial U^{(r+1,n_{r+1})}} = \dots = \frac{\partial\Lambda}{\partial U^{(k,1)}} = \dots = \frac{\partial\Lambda}{\partial U^{(k,n_k)}} = 0.$$

# Sketch of the proof: sufficient condition

From the relations defining the right eigenvectoros, taking into account that $\Lambda^{(r,\alpha)}$ are independent of $\overline{\mathcal{U}}_r$ and using

$$\begin{aligned}
&\mathbf{L}^{(i,\alpha)} \cdot \left( (\nabla_{\mathbf{U}} \mathbf{R}^{(\ell,\beta)}) \mathbf{R}^{(j,\gamma)} \right) = 0, \\
&i = 1, \dots k-1, \quad \ell = 1, \dots, i, \\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \quad \text{if} \quad i = \ell, \\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j,
\end{aligned}$$

it is proved that all entries of matrices $A_s^r$, where $s = 1, \dots, r$, and $r = 1, \dots, k-1$, are independent of $\overline{\mathcal{U}}_r$.

Q.E.D.

## Remark

The relations

$$\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,\quad \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0,$$
$$i=1,\dots k-1,\quad \ell=1,\dots,i,$$
$$\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta\quad\text{if}\quad i=\ell,$$
$$j=i+1,\dots,k,\quad \gamma=1,\dots,n_j,$$

can be counted as

$$\sum_{i=1}^{k-1} n_i m_i(n-m_i),$$

and this is exactly the number of conditions we need to impose such that the entries of matrices $A^i_j$ $(i=1,\dots,k-1,\, j=1,\dots,i)$ are independent of $\overline{\mathcal{U}}_i$.

## Remark

If matrix $A$ has a lower triangular block structure then, since the first $m_{j-1}$ components of $\mathbf{R}^{(j,\gamma)}$ are vanishing, and $j > i \geq \ell$, the first $m_i$ components of the vector $(\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}$ can not be non vanishing; therefore, it is identically

$$
\begin{aligned}
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right) = 0,\\
&i = 1, \dots k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \quad \text{if} \quad i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned}
$$

Consequently, the structure conditions may be written as well as

$$
\begin{aligned}
&\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)} = 0,\\
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right) = 0.
\end{aligned}
$$

## Theorem (Partial decoupling)

*The first order quasilinear system*

$$\frac{\partial \mathbf{u}}{\partial x_1} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}, \qquad \mathbf{u} \in \mathbb{R}^n, \quad a \quad n \times n \textit{ real matrix},$$

*assumed to be hyperbolic in the $x_1$–direction, can be transformed by a smooth (locally) invertible transformation*

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \textit{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u}),$$

*into a system like*

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0},$$

*where $A = (\nabla_{\mathbf{u}}\mathbf{H})\, a\, (\nabla_{\mathbf{u}}\mathbf{H})^{-1}$ is a lower triangular block matrix*

$$A = \begin{bmatrix} A_1^1 & 0_2^1 & 0_3^1 & \dots & 0_{k-1}^1 & 0_k^1 \\ A_1^2 & A_2^2 & 0_3^2 & \dots & 0_{k-1}^2 & 0_k^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_1^{k-1} & A_2^{k-1} & A_3^{k-1} & \dots & A_{k-1}^{k-1} & 0_k^{k-1} \\ A_1^k & A_2^k & A_3^k & \dots & A_{k-1}^k & A_k^k \end{bmatrix},$$

### ...

if and only if the set of the eigenvalues of $a$ (with corresponding left and right eigenvectors) can be divided into $k$ subsets provided that

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,\\
&\mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right)=0,\\
&i=1,\dots k-1,\quad \ell=1,\dots,i,\\
&\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta\quad\text{if}\quad i=\ell,\\
&j=i+1,\dots,k,\quad \gamma=1,\dots,n_j.
\end{aligned}$$

Moreover, the decoupling variables $U^{(i,\alpha)}=H^{(i,\alpha)}(\mathbf{u})$ $(i=1,\dots,k-1,\ \alpha=1,\dots,n_i)$ are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,$$

where $j=i+1,\dots,k$, $\gamma=1,\dots,n_j$.

# Sketch of the proof

By introducing $\mathbf{U} = \mathbf{H}(\mathbf{u})$ such that

$$\left(\nabla_{\mathbf{u}} H^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0, \qquad i = 1, \dots, k-1,\ \alpha = 1, \dots, n_i,$$
$$j = i+1, \dots, k,\ \gamma = 1, \dots, n_j,$$

we obtain that $A$ is a lower triangular block matrix.
Since

$$\lambda^{(i,\alpha)} = \Lambda^{(i,\alpha)}, \qquad \nabla_{\mathbf{u}}(\cdot) = \nabla_{\mathbf{U}}(\cdot)(\nabla_{\mathbf{u}}\mathbf{H}),$$
$$\mathbf{l}^{(i,\alpha)} = \mathbf{L}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H}), \qquad \mathbf{r}^{(i,\alpha)} = (\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(i,\alpha)},$$

we have

$$0 = \left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)} =$$
$$= \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)}.$$

$$
\begin{aligned}
0 &= \mathbf{l}^{(i,\alpha)} \cdot \left( (\nabla_{\mathbf{u}} \mathbf{r}^{(\ell,\beta)}) \mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}} \mathbf{r}^{(j,\gamma)}) \mathbf{r}^{(\ell,\beta)} \right) = \\
&= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}} \mathbf{H}) \left( \nabla_{\mathbf{u}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(\ell,\beta)} \right) (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(j,\gamma)} \right. \\
&\qquad \left. - \nabla_{\mathbf{u}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(j,\gamma)} \right) (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(\ell,\beta)} \right) = \\
&= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}} \mathbf{H}) \left( \nabla_{\mathbf{U}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(\ell,\beta)} \right) (\nabla_{\mathbf{u}} \mathbf{H}) (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(j,\gamma)} \right. \\
&\qquad \left. - \nabla_{\mathbf{U}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(j,\gamma)} \right) (\nabla_{\mathbf{u}} \mathbf{H}) (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(\ell,\beta)} \right) = \\
&= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}} \mathbf{H}) \left( \nabla_{\mathbf{U}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(\ell,\beta)} \right) \mathbf{R}^{(j,\gamma)} \right. \\
&\qquad \left. - \nabla_{\mathbf{U}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \mathbf{R}^{(j,\gamma)} \right) \mathbf{R}^{(\ell,\beta)} \right) = \\
&= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}} \mathbf{H}) \left( \nabla_{\mathbf{U}} \left( (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \right) \left( \mathbf{R}^{(\ell,\beta)} \mathbf{R}^{(j,\gamma)} - \mathbf{R}^{(j,\gamma)} \mathbf{R}^{(\ell,\beta)} \right) \right. \\
&\qquad \left. + (\nabla_{\mathbf{u}} \mathbf{H})^{-1} \left( (\nabla_{\mathbf{U}} \mathbf{R}^{(\ell,\beta)}) \mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j,\gamma)}) \mathbf{R}^{(\ell,\beta)} \right) \right) = \\
&= \mathbf{L}^{(i,\alpha)} \cdot \left( (\nabla_{\mathbf{U}} \mathbf{R}^{(\ell,\beta)}) \mathbf{R}^{(j,\gamma)} \right) - \mathbf{L}^{(i,\alpha)} \left( (\nabla_{\mathbf{U}} \mathbf{R}^{(j,\gamma)}) \mathbf{R}^{(\ell,\beta)} \right) = \\
&= \mathbf{L}^{(i,\alpha)} \cdot \left( (\nabla_{\mathbf{U}} \mathbf{R}^{(\ell,\beta)}) \mathbf{R}^{(j,\gamma)} \right).
\end{aligned}
$$

Therefore:

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0\quad\Leftrightarrow\quad\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,$$

and

$$\mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right)=0\quad\Leftrightarrow$$
$$\Leftrightarrow\quad\mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0.$$

Q.E.D.

## Theorem (Full decoupling)

*For a hyperbolic system of first order H. and A. quasilinear PDEs to be locally reducible into k non–interacting subsystems of some orders $n_1, \dots, n_k$, with $n_1 + \dots + n_k = n$, it is necessary and sufficient that its characteristic velocities (with corresponding left and right eigenvectors) can be divided into k subsets provided that*

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)} = 0, \quad \mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right) = 0,$$
$$\forall\, i,j = 1,\dots k, \quad i \neq j, \quad \ell = 1,\dots,i, \quad \gamma = 1,\dots,n_j,$$
$$\alpha = 1,\dots,n_i, \quad \beta = 1,\dots,n_\ell, \quad \alpha \neq \beta \quad \textit{if} \quad i = \ell.$$

*Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$ are found from*

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)} = 0,$$

*where $i,j = 1,\dots,k$, $i \neq j$, $\alpha = 1,\dots,n_i$, $\gamma = 1,\dots,n_j$. The coefficient matrix for a fully decoupled system results in block diagonal form (diagonal if $k = n$).*

# One–dimensional isentropic gas dynamics equations

Governing equation:

$$\frac{\partial \mathbf{u}}{\partial t} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

with

$$\mathbf{u} = \begin{bmatrix} \rho \\ v \\ s \end{bmatrix}, \qquad a = \begin{bmatrix} v & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial s} \\ 0 & 0 & v \end{bmatrix},$$

$\rho(t,x)$ mass density, $v(t,x)$ velocity, $s(t,x)$ entropy, and $p(\rho,s)$ pressure.
Eigenvalues of matrix $a$:

$$\lambda_{1,2} = v \pm \sqrt{\frac{\partial p}{\partial \rho}}, \qquad \lambda_3 = v,$$

and associated left (right resp.) eigenvectors

$$\mathbf{l}_{1,2} = \left(\sqrt{\frac{\partial p}{\partial \rho}}, \pm\rho, \frac{\rho}{s}\sqrt{\frac{\partial p}{\partial \rho}}\right), \qquad \mathbf{l}_3 = (0,0,1),$$

# One–dimensional isentropic gas dynamics equations

$$\mathbf{r}_{1,2} = \begin{pmatrix} \rho \\ \pm\sqrt{\frac{\partial p}{\partial \rho}} \\ 0 \end{pmatrix}, \qquad \mathbf{r}_3 = \begin{pmatrix} \rho \\ 0 \\ -s \end{pmatrix}.$$

The relations

$$(\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_2 = 0, \qquad (\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_3 = 0,$$
$$(\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_1 = 0, \qquad (\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_3 = 0,$$

are satisfied with the constitutive law

$$p(\rho, s) = p_0\rho^3 s^2 + f(s),$$

with $p_0$ constant and $f$ function of its argument.

# One–dimensional isentropic gas dynamics equations

Using

$$(\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}_2 = 0, \qquad (\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}_3 = 0,$$
$$(\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}_1 = 0, \qquad (\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}_3 = 0,$$

we choose the new field variables as

$$\begin{aligned} U_1 &= H_1(\rho, v, s) = v + \sqrt{3p_0}\rho s, \\ U_2 &= H_2(\rho, v, s) = v - \sqrt{3p_0}\rho s, \\ U_3 &= H_3(\rho, v, s) = s, \end{aligned}$$

and obtain the following partially decoupled system

$$\begin{cases} \dfrac{\partial U_1}{\partial t} + U_1 \dfrac{\partial U_1}{\partial x} = 0, \\ \dfrac{\partial U_2}{\partial t} + U_2 \dfrac{\partial U_2}{\partial x} = 0, \\ \dfrac{\partial U_3}{\partial t} + \dfrac{1}{2}(U_1 + U_2) \dfrac{\partial U_3}{\partial x} = 0. \end{cases}$$

# Model of a travelling threadline

Governing equations[1]:

$$\frac{\partial \mathbf{u}}{\partial t} + a(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ V^x \\ v \\ \varepsilon \end{pmatrix}, \qquad a = \begin{bmatrix} V^x & \rho & 0 & 0 \\ \frac{-T'}{\rho(1+\varepsilon^2)} & V^x & 0 & \frac{\varepsilon}{1+\varepsilon^2)^2}\left(T' + \frac{T}{m}\right) \\ 0 & 0 & 2V^x & (V^x)^2 - \frac{T}{m(1+\varepsilon^2)} \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$\rho$ mass density, $V^x$ and $v$ the components of velocity, $\epsilon$ transverse displacement and $T(m)$ tension, $\rho = m\sqrt{1+\epsilon^2}$, $T'(m) < 0$.

[1] W. F. Ames, S. Y. Lee, J. N. Zaiser. Int. J. Non-Linear Mech., **3**, 449–469, 1968.

# Model of a travelling threadline

Eigenvalues of matrix $a$

$$\lambda_{1,2} = V^x \pm \left(\frac{-T'}{1+\varepsilon^2}\right)^{1/2}, \qquad \lambda_{3,4} = V^x \pm \left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2},$$

with associated left and right eigenvectors

$$\mathbf{l}_{1,2} = \left(\pm\frac{\sqrt{-(1+\epsilon^2)T'}}{\rho\epsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\epsilon^2}}\right)}, \frac{1+\epsilon^2}{\epsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\epsilon^2}}\right)}, \frac{1}{V^x \mp \sqrt{\frac{-T'}{1+\epsilon^2}}}, 1\right),$$

$$\mathbf{l}_{3,4} = \left(0, 0, \rho, \rho V^x \pm \sqrt{\frac{\rho T}{(1+\epsilon^2)^{1/2}}}\right),$$

$$\mathbf{r}_{1,2} = \begin{pmatrix} \rho \\ \pm\left(\frac{-T'}{1+\varepsilon^2}\right)^{1/2} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_{3,4} = \begin{pmatrix} \frac{\rho\varepsilon}{1+\varepsilon^2} \\ \pm\left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2}\frac{\varepsilon}{1+\varepsilon^2} \\ -\left(V^x \pm \frac{T}{m(1+\varepsilon^2)}\right)^{1/2} \\ 1 \end{pmatrix}.$$

# Model of a travelling threadline

The structure conditions

$$(\nabla_{\mathbf{u}}\lambda_i)\cdot\mathbf{r}_j = 0,$$
$$\mathbf{l}_i\cdot((\nabla_{\mathbf{u}}\mathbf{r}_\ell)\mathbf{r}_j - (\nabla_{\mathbf{u}}\mathbf{r}_j)\mathbf{r}_\ell) = 0, \qquad i,\ell = 1,2, \quad i\neq\ell, \quad j = 3,4,$$

are satisfied with the following constitutive law

$$T(m) = \frac{k}{m}, \qquad k \text{ constant.}$$

By introducing $\mathbf{U} = \mathbf{H}(\mathbf{u})$ such that

$$(\nabla_{\mathbf{u}}H_i)\cdot\mathbf{r}_j = 0, \qquad i, = 1,2, \quad j = 3,4,$$

*i.e.*,

$$\left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial\epsilon} = 0, \quad \left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial\epsilon} = 0,$$
$$\left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial\epsilon} = 0, \quad \left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial\epsilon} = 0,$$

# Model of a travelling threadline

we get

$$H_1 = H_1\left(\rho, V^x\right), \qquad H_2 = H_2\left(\rho, V^x\right).$$

By choosing the identity transformation, we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + V^x \dfrac{\partial \rho}{\partial x} + \rho \dfrac{\partial V^x}{\partial x} = 0, \\ \dfrac{\partial V^x}{\partial t} + \dfrac{k}{\rho^3}\dfrac{\partial \rho}{\partial x} + V^x \dfrac{\partial V^x}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + 2V^x \dfrac{\partial v}{\partial x} + \left( (V^x)^2 - \dfrac{k}{\rho^2} \right) \dfrac{\partial \epsilon}{\partial x} = 0, \\ \dfrac{\partial \epsilon}{\partial t} - \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

# THANKS