A fermionic operatorial model for a network of agents with competitive and cooperative interactions

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Mathematical Aspects in Non Equilibrium Systems: from Micro to Macro
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Operatorial models

Consider an N–compartment/agent model $\mathcal{S}$, and associate to each compartment (or agent) an annihilation (a_j), a creation ($a_j^\dagger$), and a number ($\hat{n}_j = a_j^\dagger a_j$) fermionic operator. These actors satisfy the *CAR*'s:

$$\{a_i, a_j^\dagger\} = \delta_{i,j}\mathbb{I}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

The states of the system are vectors in the 2^N–dimensional Hilbert space $\mathbb{H}$ constructed as the linear span of the vectors

$$\varphi_{n_1,n_2,\ldots,n_N} := (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_N^\dagger)^{n_N}\varphi_{\mathbf{0}},$$

where $n_j \in \{0,1\}$ for all $j = 1,\ldots,N$, and $\varphi_{\mathbf{0}}$ is the *vacuum* of the theory, *i.e.*, a vector annihilated by all the operators a_j.

The vectors $\varphi_{n_1,\ldots,n_N}$ give an o.n. set of eigenstates of $\hat{n}_j$, say

$$\hat{n}_j\varphi_{n_1,\ldots,n_N} = n_j\varphi_{n_1,\ldots,n_N}, \quad \text{for all } j = 1,\ldots,N.$$

$$\hat{n}_j(a_j\varphi_{n_1,\ldots,n_N}) = \begin{cases} (n_j-1)(a_j\varphi_{n_1,\ldots,n_N}) & \text{if} \quad n_j = 1 \\ 0 & \text{if} \quad n_j = 0 \end{cases},$$

$$\hat{n}_j(a_j^\dagger\varphi_{n_1,\ldots,n_N}) = \begin{cases} (n_j+1)(a_j\varphi_{n_1,\ldots,n_N}) & \text{if} \quad n_j = 0 \\ 0 & \text{if} \quad n_j = 1 \end{cases}.$$

Assign the dynamics

Let H be the time–independent self–adjoint **Hamiltonian operator** embedding the main effects deriving from the interactions among the compartments/agents of the system.
For the dynamics of any operator X we use the **Heisenberg representation**,

$$X(t) = \exp(iHt)X\exp(-iHt),$$

or, equivalently,

$$\frac{dX(t)}{dt} = i[H, X(t)].$$

Once defined a vector state Φ representing the initial configuration of the system, we compute the mean values

$$n_j(t) = \langle \Phi, \hat{n}_j(t)\Phi \rangle, \qquad j = 1, \ldots, N,$$

$\langle \cdot, \cdot \rangle$ being the scalar product in $\mathbb{H}$.
These mean values can be interpreted as a measure of the **density** of the compartments (or of some **feature** of the agents) of the model.

Computational cost

From a computational point of view, once the Hamiltonian operator H has been assigned, in general, the dynamics is obtained by computing, at each instant t

$$\exp(iHt),$$

i.e., the exponential of a $2^N \times 2^N$ matrix.
The same task can be achieved by solving the system of $N4^N$ complex differential equations, say

$$\dot{a}_i(t) = i[H, a_i(t)], \qquad i = 1, \dots, N,$$

where each a_i is a $2^N \times 2^N$ matrix.

For large N (even not too large!), the computational cost is a serious problem, unless we restrict to quadratic Hamiltonians (the case we will face).

Quadratic Hamiltonian

The Hamiltonian we consider is

$$H = H_0 + H_I.$$

made of

- the **free part**,

$$H_0 = \sum_{j=1}^{N} \omega_j a_j^\dagger a_j,$$

  where the real constants ω_j can be interpreted as **inertia** parameters (the higher the values, the less the tendency of the corresponding degrees of freedom to change);

- the interaction part, H_I, embedding competition and/or cooperation,

$$H_I = \sum_{i,j=1, i<j}^{N} \left(\lambda_{i,j}(a_i^\dagger a_j + a_j^\dagger a_i) + \mu_{i,j}(a_i^\dagger a_j^\dagger + a_j a_i) \right),$$

  where the real constants $\lambda_{i,j}$ and $\mu_{i,j}$ measure the strength of the interactions.

A comment about Interaction

$$H_I = \sum_{i,j=1,i<j}^{N} \Big(\lambda_{i,j}(a_i^\dagger a_j + a_j^\dagger a_i) + \mu_{i,j}(a_i^\dagger a_j^\dagger + a_j a_i) \Big),$$

The term $a_i^\dagger a_j$ "creates" a particle for the i–th agent and "destroys" a particle for the j–th agent; the adjoint term swaps the roles of the two agents:
the loss (gain) of an agent is the gain (loss) of the other agent.

The term $a_i^\dagger a_j^\dagger$ creates a particle for both agents, and the adjoint part destroys a particle for both agents:
the gain (loss) of an agent is the gain (loss) of the other agent.

Quadratic Hamiltonian with Prey-Predator Interaction

The computational cost is drastically reduced if the Hamiltonian is quadratic; for instance, if

$$H = \sum_i \omega_i a_i^\dagger a_i + \sum_{i<j} \lambda_{ij}(a_i^\dagger a_j + a_j^\dagger a_i),$$

introducing

$$A(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \dots \\ a_N(t) \end{pmatrix}, \quad \Gamma = i \begin{pmatrix} -\omega_1 & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1N} \\ \lambda_{12} & -\omega_2 & \lambda_{23} & \cdots & \lambda_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{1N} & \lambda_{2N} & \lambda_{3N} & \cdots & -\omega_N \end{pmatrix},$$

we may write

$$\dot{A}(t) = \Gamma A(t) \quad \Rightarrow \quad A(t) = V(t)A(0), \qquad V(t) = \exp(\Gamma t)$$

If n_ℓ the initial density of the ℓ–th compartment, we can compute $n_\ell(t)$ by suitably using the components of the matrix $V(t)$. Therefore, the computational cost reduces to that needed for computing the exponential of the $N \times N$ matrix Γt.

The simplest example: 2 compartments/agents

Consider a system with 2 fermionic modes whose evolution is ruled by the Hamiltonian operator

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda(a_1^\dagger a_2 + a_2^\dagger a_1).$$

We are led to the differential equations

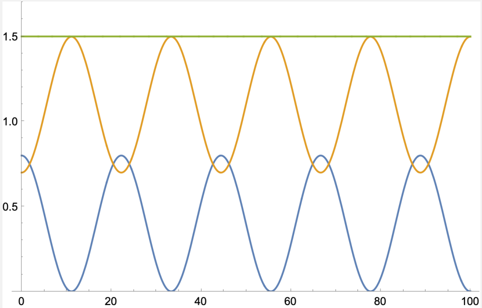
$$\dot{a}_1 = i(-\omega_1 a_1 + \lambda a_2),$$
$$\dot{a}_2 = i(-\omega_2 a_2 + \lambda a_1),$$

and the time evolution of the mean values of the number operators $n_1(t)$ and $n_2(t)$ is periodic.

Motion integral:

$[H, \hat{n}_1 + \hat{n}_2] = 0$.

Trivial dynamics



$\omega_1 = 0.7$, $\omega_2 = 0.5$, $\lambda = 0.2$, $n_1(0) = 0.8$, $n_2(0) = 0.7$.

3 Fermionic Modes

$$H = \sum_{k=1}^{3} \omega_k a_k^\dagger a_k + \lambda_{12}(a_1^\dagger a_2 + a_2^\dagger a_1) + \lambda_{13}(a_1^\dagger a_3 + a_3^\dagger a_1) + \lambda_{23}(a_2^\dagger a_3 + a_3^\dagger a_2).$$

We have to consider the differential equations

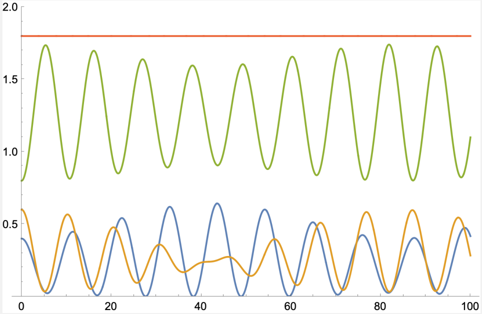
$$\begin{aligned} \dot{a}_1 &= i(-\omega_1 a_1 + \lambda_{12} a_2 + \lambda_{13} a_3), \\ \dot{a}_2 &= i(-\omega_2 a_2 + \lambda_{12} a_1 + \lambda_{23} a_3), \\ \dot{a}_3 &= i(-\omega_3 a_3 + \lambda_{13} a_1 + \lambda_{23} a_2), \end{aligned}$$

and the time evolution of the mean values $n_1(t)$, $n_2(t)$ and $n_3(t)$ of the number operators is in general quasiperiodic.

Motion integral:

$[H, \hat{n}_1 + \hat{n}_2 + \hat{n}_3] = 0$.

Trivial dynamics



$\omega_1 = 0.7$, $\omega_2 = 1/\sqrt{2}$, $\omega_3 = 0.3$, $\lambda_{12} = 0.1$, $\lambda_{13} = 0.15$, $\lambda_{23} = 0.2$, $n_1(0) = 0.4$, $n_2(0) = 0.6$, $n_2(0) = 0.8$.

More than quadratic Hamiltonians

The competitive interactions we can model with a quadratic Hamiltonian imply that the dynamics we can deduce is at most **quasiperiodic** (the system admits also a first integral).

Taking a Hamiltonian operator containing contributions of order greater than two, the Heisenberg equations of motion are **nonlinear** and we need to solve, for a fermionic model involving N modes, $N \cdot 4^N$ complex nonlinear equations (numerically).

Adding cooperative interactions doubles the number of the differential equations we have to manage, (and first integrals are lost!).

Therefore, calculations are **not readily implementable at all** unless very low values of N are considered.

The aim is to **enrich** the dynamics without rendering the problem computationally hard!

Modified descriptions of the dynamics

An extended version of the operatorial models consists in considering the evolution of system $\mathcal{S}$ depending on its Hamiltonian, its initial conditions, and some *external/internal action* acting repeatedly on the system itself or rather on the model: (H, ρ)-**induced dynamics**[1].

A **rule** is nothing else than a set of conditions mapping *some* input values into new ones. It may be intended to modify repeatedly **the values of the parameters**, *i.e.*, the model is repeatedly *adjusted* according to the current state (or its variation) of the system.

The use of a rule enriches the description of the dynamics still with a **quadratic Hamiltonian** by taking into account some effects occurring during the time evolution of the system.

The combined effect of H and ρ may produce the convergence of the system to some **asymptotic equilibria**.

[1] Bagarello, Di Salvo, Gargano, Oliveri, Physica A, 2018.

(H, ρ)-induced dynamics: How to do.

Consider a self-adjoint time-independent quadratic Hamiltonian operator $H^{(1)}$. According to Heisenberg view, in a time interval of length $\tau > 0$:

- compute the evolution of annihilation and creation operators, and, choosing an initial condition for the mean values of the number operators, obtain their time evolution (our observables);
- according to the values of the observables at time τ, or to their variations in the time interval $[0, \tau]$, we modify some of the parameters involved in $H^{(1)}$;
- we get a new Hamiltonian operator $H^{(2)}$, having the same functional form as $H^{(1)}$, but (in general) with different values of (some of) the involved parameters;
- follow the continuous evolution of the system under the action of this new Hamiltonian for the next time interval of length τ. And so on.

No stop & go!

At each step, we do not restart the evolution of the system from a new initial condition, but simply continue to follow the evolution with the only difference that for $t \in](k-1)\tau, k\tau]$ a new Hamiltonian $H^{(k)}$ rules the process.

ρ as a map in the space of the parameters of H

Split the time interval $[0, T]$ into $n = T/\tau$ subintervals of length τ. In the k-th subinterval $](k-1)\tau, k\tau]$ consider an Hamiltonian $H^{(k)}$ ruling the dynamics. The global dynamics arises from a sequence of Hamiltonian operators:

$$H^{(1)} \xrightarrow{\tau} H^{(2)} \xrightarrow{\tau} H^{(3)} \xrightarrow{\tau} \ldots \xrightarrow{\tau} H^{(n)}.$$

- Follow the dynamics of the observables X_i $(i = 1, \ldots, N)$ governed by $H^{(k)}$ in a time interval of length $\tau > 0$,

$$X_i(t) = \exp(iH^{(k)}t) X_i \exp(-iH^{(k)}t).$$

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- Change the values of the parameters on the basis of the mean values $x_i(\tau)$ $(i = 1, \ldots, N)$ so obtaining $H^{(k+1)}$.

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- Change the values of the parameters on the basis of the mean values $x_i(\tau)$ $(i = 1, \ldots, N)$ so obtaining $H^{(k+1)}$.
- **Continue** the evolution as ruled by $H^{(k+1)}$, and so on.

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- The parameters entering the model are stepwise (in time) constant.

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- **Continue** the evolution as ruled by $H^{(k+1)}$, and so on.
- The parameters entering the model are stepwise (in time) constant.
- Often, a sort of **synchronization** of the oscillations is reached, and the evolution may admit asymptotic equilibrium states.

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- The parameters entering the model are stepwise (in time) constant.
- Often, a sort of **synchronization** of the oscillations is reached, and the evolution may admit asymptotic equilibrium states.

The global evolution is obtained by glueing the local evolutions.

(H, ρ)–induced dynamics – Example

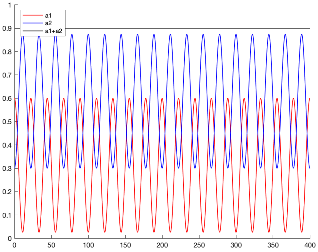
$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda(a_1^\dagger a_2 + a_2^\dagger a_1),$$
$$\omega_1 = 0.4,\ \omega_2 = 0.5,\ \lambda = 0.2,\ n_1(0) = 0.7,\ n_2(0) = 0.6.$$

The **rule** consists in the replacements at times $k\tau$ ($k \in \mathbb{N}$)

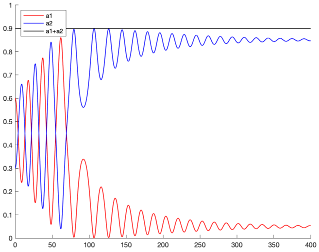
$$\begin{cases} \omega_1 \mapsto \omega_1(1 + \delta_1^{(k)}), & \delta_1^{(k)} = n_1(k\tau) - n_1((k-1)\tau), \\ \omega_2 \mapsto \omega_2(1 + \delta_2^{(k)}), & \delta_2^{(k)} = n_2(k\tau) - n_2((k-1)\tau). \end{cases}$$

The rule modify only the *inertia* of the compartments/agents! If the mean value of a number operator increases in a subinterval, then its inertia increases.
Because of the existence of the first integral, ω_1 and ω_2 undergo opposite variations (one is creasing and the other one decreasing).
In some sense, the actors of the system **change their attitudes** as a consequence of the evolution of the system.

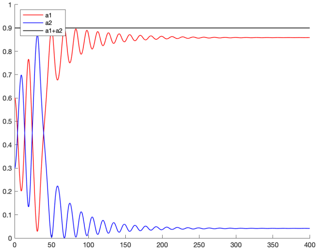
2 modes - Some plots



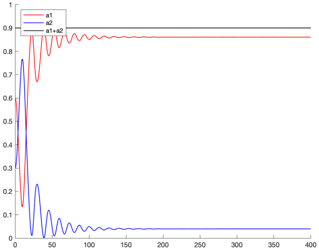
No rule



$\tau = 1$

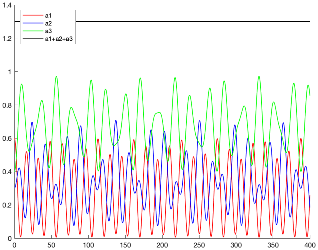


$\tau = 2$

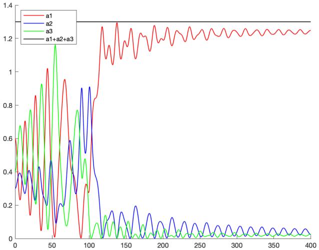


$\tau = 4$

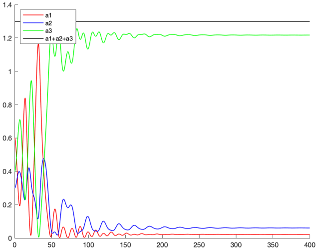
3 modes - Some plots



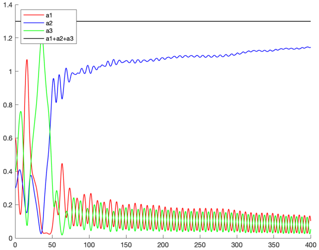
No rule



$\tau = 1$



$\tau = 2$



$\tau = 4$

Quadratic Hamiltonians: competition and cooperation

$$H = \sum_{j=1}^{N} \omega_j \, a_j^\dagger \, a_j + \sum_{1 \le j < k \le N} \lambda_{j,k} \left(a_j \, a_k^\dagger + a_k \, a_j^\dagger \right) + \sum_{1 \le j < k \le N} \mu_{j,k} \left(a_j^\dagger \, a_k^\dagger + a_k \, a_j \right),$$

The dynamical equations we have to solve:

$$\dot{a}_j = i \left(-\omega_j a_j + \sum_{1 \le \ell < j} \lambda_{\ell,j} a_\ell + \sum_{j < k \le N} \lambda_{j,k} a_k + \sum_{1 \le \ell < j} \mu_{\ell,j} a_\ell^\dagger - \sum_{j < k \le N} \mu_{j,k} a_k^\dagger \right),$$

$$\dot{a}_j^\dagger = i \left(\omega_j a_j^\dagger - \sum_{1 \le \ell < j} \lambda_{\ell,j} a_\ell^\dagger - \sum_{j < k \le N} \lambda_{j,k} a_k^\dagger - \sum_{1 \le \ell < j} \mu_{\ell,j} a_\ell + \sum_{j < k \le N} \mu_{j,k} a_k \right).$$

Quadratic Hamiltonians: competition and cooperation

Setting $A = \left(a_1, \ldots, a_N, a_1^\dagger, \ldots, a_N^\dagger\right)^T$, and defining the square matrix of order $2N$

$$\Gamma = \begin{bmatrix} \Gamma_0 & \Gamma_1 \\ -\Gamma_1 & -\Gamma_0 \end{bmatrix},$$

where the symmetric block Γ_0 and the antisymmetric block Γ_1 are

$$\Gamma_0 = \begin{bmatrix} -\omega_1 & \lambda_{1,2} & \cdots & \cdots & \lambda_{1,N} \\ \lambda_{1,2} & -\omega_2 & \lambda_{2,3} & \cdots & \lambda_{2,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{1,N}, & \lambda_{2,N} & \cdots & \lambda_{N-1,N} & -\omega_N \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0 & -\mu_{1,2} & \cdots & \cdots & -\mu_{1,N} \\ \mu_{1,2} & 0 & -\mu_{2,3} & \cdots & -\mu_{2,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{1,N}, & \mu_{2,N} & \cdots & -\mu_{N-1,N} & 0 \end{bmatrix},$$

The evolutions equations read

$$\frac{d\mathbf{A}}{dt} = \mathrm{i}\Gamma\mathbf{A}, \qquad \mathbf{A}(0) = \mathbf{A}^0,$$

and we have a formal solution

$$\mathbf{A}(t) = \mathcal{B}(t)\,\mathbf{A}^0, \qquad \mathcal{B}(t) = \exp\left(\mathrm{i}\Gamma t\right).$$

Now, let us define the vector (**not a pure state!**)

$$\boldsymbol{\varphi} = \sum_{k=1}^{N} \sqrt{n_k^0}\mathbf{e}_\ell, \qquad \ell = 2^{k-1} + 1,$$

i.e., fix the initial condition. If $B_{j,k}$ is the generic entry of matrix $\mathcal{B}(t)$, we have

$$a_k^\dagger(t) = \sum_{j=1}^{N} \left(B_{k+N,j}a_j^0 + B_{k+N,j+N}{a_j^0}^\dagger\right),$$

$$a_k(t) = \sum_{j=1}^{N} \left(B_{k,j}a_j^0 + B_{k,j+N}{a_j^0}^\dagger\right),$$

whereupon the formula

$$n_k(t) = \left\langle \boldsymbol{\varphi}, a_k^\dagger(t)a_k(t)\boldsymbol{\varphi}\right\rangle.$$

Mean values of the number operators

Using the CARs, we get the mean values of the number operators at time t:

$$n_k(t) = \sum_{i=1}^{N} \varphi_i^2 \sum_{\ell=1}^{N} B_{k,f(\ell,k)} B_{k+N,g(\ell,k)}$$
$$+ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \varphi_i \varphi_j \left(B_{k,i} B_{k+N,j+N} + B_{k,j} B_{k+N,i+N} \right.$$
$$\left. - B_{k,i+N} B_{k+N,j} - B_{k,j+N} B_{k+N,i} \right),$$

where

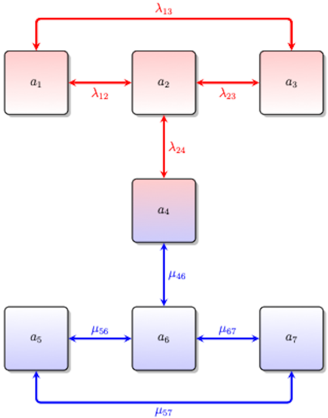
$$f(\ell,i) = \begin{cases} i & \text{if} \quad i = \ell, \\ i+N & \text{if} \quad i \neq \ell, \end{cases}$$

$$g(\ell,i) = \begin{cases} i+N & \text{if} \quad i = \ell, \\ i & \text{if} \quad i \neq \ell. \end{cases}$$

Toy Model with 7 interacting agents[2]

Consider a network of some interacting agents among which both cooperative and competitive effects occur (**the mean values of the number operators measure the wealth of the agents**).

$$H = \sum_{k=1}^{7} \omega_k a_k^\dagger a_k + \sum_{k=1}^{6} \sum_{\ell=k+1}^{7} \left(\lambda_{k\ell}(a_k a_\ell^\dagger + a_\ell a_k^\dagger) + \mu_{k\ell}(a_k^\dagger a_\ell^\dagger + a_\ell a_k) \right).$$



The system $\mathcal{S}$ consists of

- a subsystem (a_1, a_2, a_3) interacting in a competitive way;
- a subsystem (a_4, a_5, a_6) interacting in a cooperative way;
- an **opportunist agent** (a_4) competing with a_2 and cooperating with a_6.

[2] Gorgone, Inferrera, Oliveri, IJTP, 2023.

Toy Model with 7 interacting agents

For the considered model, the evolution equations read:

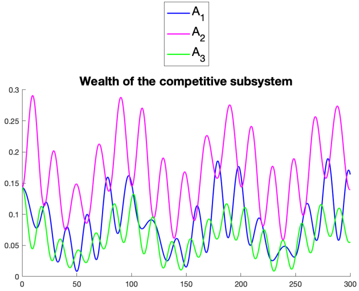
$$\begin{aligned}
\dot{a}_1 &= \mathrm{i}\left(-\omega_1 a_1 + \lambda_{12} a_2 + \lambda_{13} a_3\right), \\
\dot{a}_2 &= \mathrm{i}\left(-\omega_2 a_2 + \lambda_{12} a_1 + \lambda_{23} a_3 + \lambda_{24} a_4\right), \\
\dot{a}_3 &= \mathrm{i}\left(-\omega_3 a_3 + \lambda_{13} a_1 + \lambda_{23} a_2\right), \\
\dot{a}_4 &= \mathrm{i}\left(-\omega_4 a_4 + \lambda_{24} a_2 - \mu_{46} a_6^\dagger\right), \\
\dot{a}_5 &= \mathrm{i}\left(-\omega_5 a_5 - \mu_{56} a_6^\dagger - \mu_{57} a_7^\dagger\right), \\
\dot{a}_6 &= \mathrm{i}\left(-\omega_6 a_6 + \mu_{46} a_4^\dagger + \mu_{56} a_5^\dagger - \mu_{67} a_7^\dagger\right), \\
\dot{a}_7 &= \mathrm{i}\left(-\omega_7 a_7 + \mu_{57} a_5^\dagger + \mu_{67} a_6^\dagger\right),
\end{aligned}$$

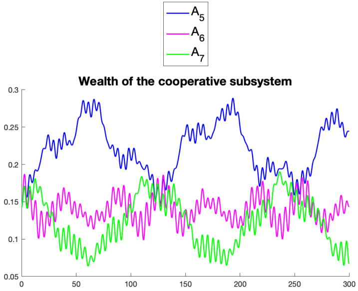
together with their adjoints.

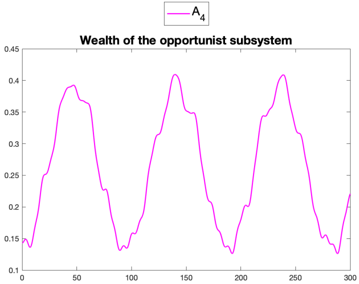
Parameters used in the numerical simulations:

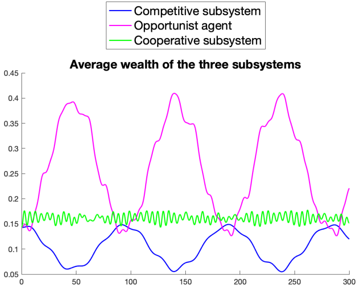
$$\omega_1 = 0.5, \quad \omega_2 = 0.55, \quad \omega_3 = 0.6, \quad \omega_4 = 0.3, \quad \omega_5 = 0.65, \quad \omega_6 = 0.7, \quad \omega_7 = 0.75,$$
$$\lambda_{12} = 0.1, \quad \lambda_{13} = 0.1, \quad \lambda_{23} = 0.1, \quad \lambda_{24} = 0.01,$$
$$\mu_{46} = 0.01, \quad \mu_{56} = 0.1, \quad \mu_{57} = 0.1, \quad \mu_{67} = 0.1.$$

Heisenberg dynamics with no rule









Insert the rules

- Fix τ and divide the time interval $[0, T]$, where we study the evolution of the system, in subintervals of length τ.
- Set the initial value of the inertia parameters:

$$\omega_1 = 0.5,\ \omega_2 = 0.55,\ \omega_3 = 0.6,\ \omega_4 = 0.3,$$
$$\omega_5 = 0.65,\ \omega_6 = 0.7,\ \omega_7 = 0.75.$$

- After the time τ has elapsed, change the inertia parameters with the following rule:

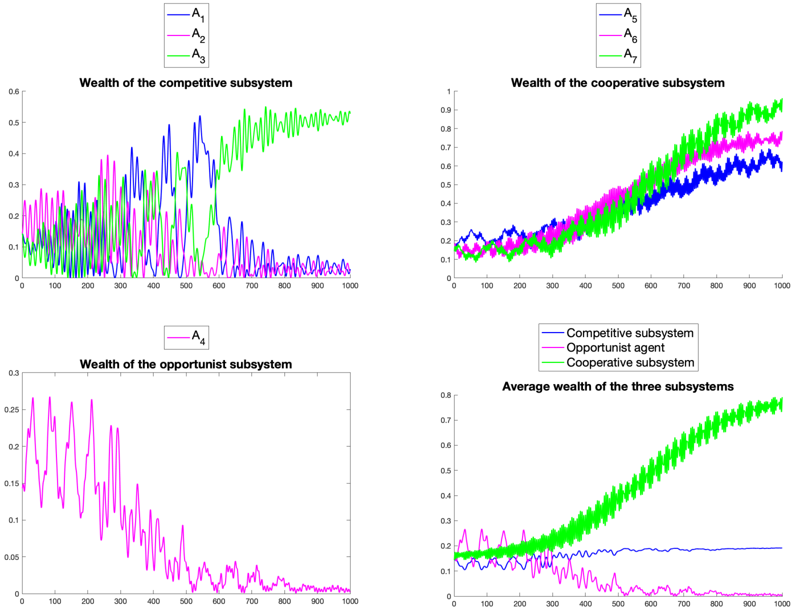
$$\omega_j \mapsto \omega_j(1 + \delta_j^{(k)}),$$

where

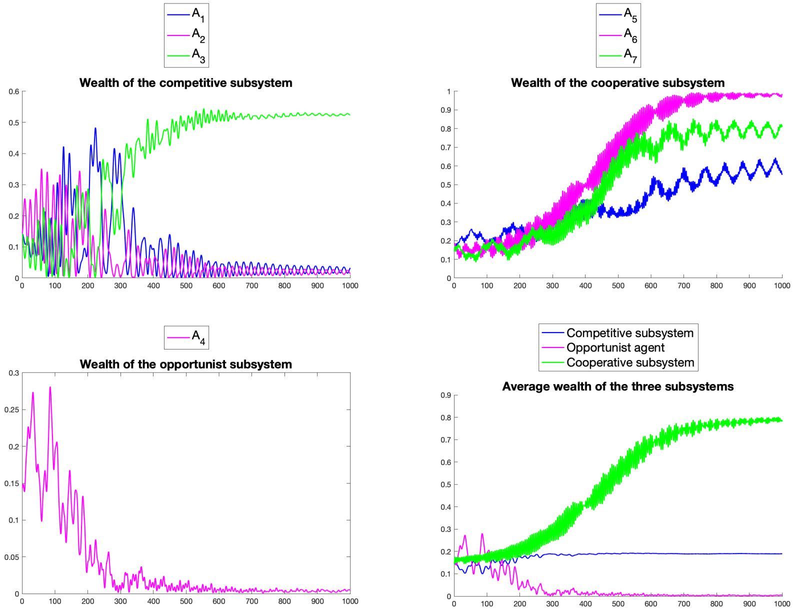
$$\delta_j^{(k)} = n_j(k\tau) - n_j((k-1)\tau), \qquad j = 1, \dots, 7.$$

The dynamics of the system drastically changes!

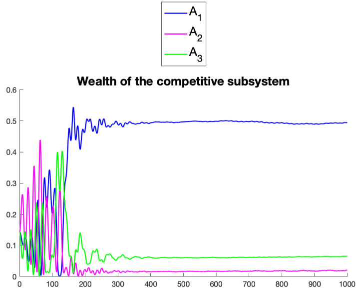
Heisenberg dynamics with rule ($\tau = 1$)

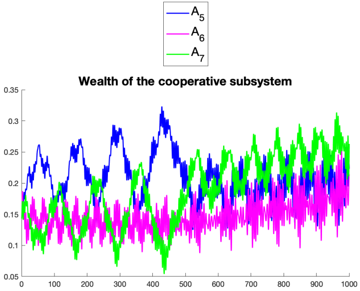


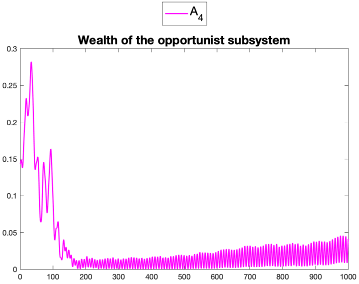
Heisenberg dynamics with rule ($\tau = 2$)

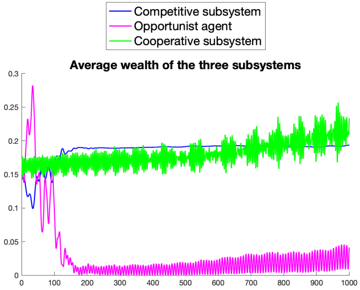


Heisenberg dynamics with rule ($\tau = 4$)

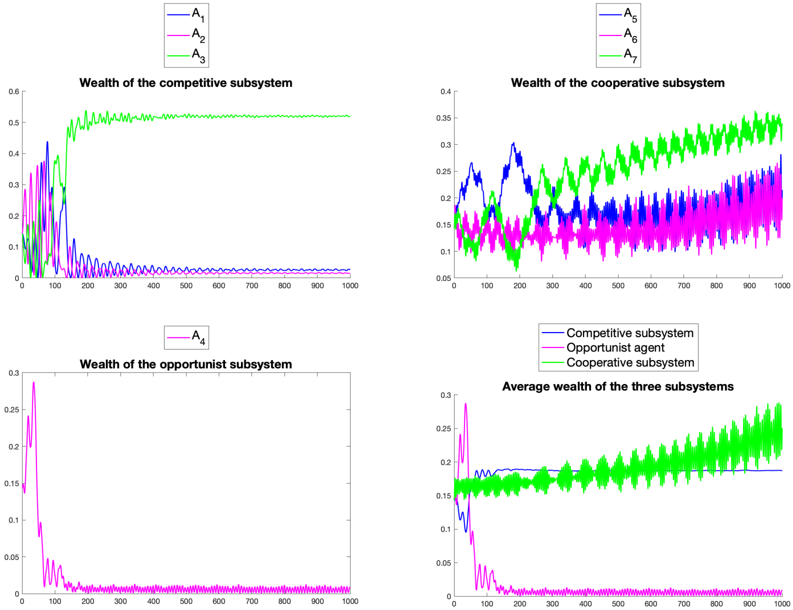




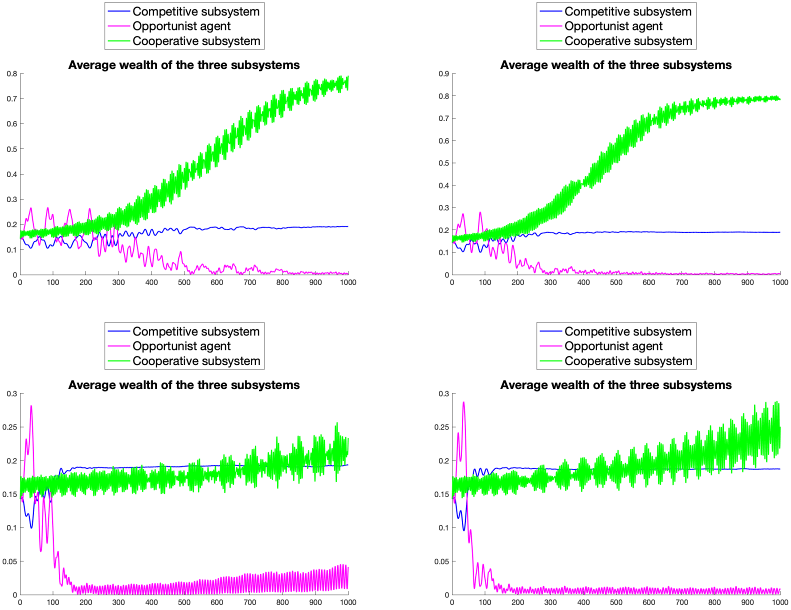




Heisenberg dynamics with rule ($\tau = 8$)



Comparison among the different cases



$\tau = 1$ (top left), $\tau = 2$ (top right), $\tau = 4$ (bottom left), $\tau = 8$ (bottom right).

A little bit more complex model[3]

Let us consider a system made by N agents; each agent is located in a cell of a one–dimensional torus partitioned in N cells, so that the cell 1 is adjacent to the cell N. The distance between adjacent cells is 1, whereupon the maximum distance between the cells is $d_{\mathsf{max}} = \lfloor N/2 \rfloor$.
Let us choose randomly:

- N_1 agents (the competitive subgroup) interacting each other with a competitive mechanism;
- N_2 agents (the cooperative subgroup) interacting each other with a cooperative mechanism;
- $N_3 = N - N_1 - N_2$ opportunist agents, *i.e.*, each opportunist agent has a competitive interaction with an agent of the competitive subgroup, and a cooperative interaction with an agent of the cooperative subgroup; moreover, the opportunist agents compete each other.

Each agent has an inertia parameter randomly chosen in the range between 0.5 and 0.7.

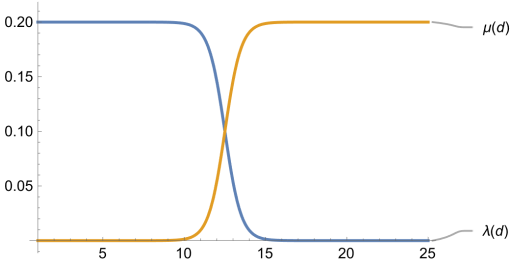
[3] Gorgone, Inferrera, Oliveri, IJTP, 2023

A little bit more complex model

When two agents are interacting in some way, the competition parameter $\lambda_{j,k}$ decreases with the distance $d(j,k)$ between the cells j and k; on the contrary, the cooperation parameter $\mu_{j,k}$ increases with $d(j,k)$:

$$\lambda_{j,k} = \lambda \left(1 - \tanh\left(d(j,k) - d_{\mathsf{max}}/2\right)\right),$$
$$\mu_{j,k} = \mu \left(1 + \tanh\left(d(j,k) - d_{\mathsf{max}}/2\right)\right),$$

where $\lambda = \mu = 0.1$:



Numerical results

The numerical results are obtained either using the standard Heisenberg view or the $(\mathcal{H},\rho)$–induced dynamics approach.

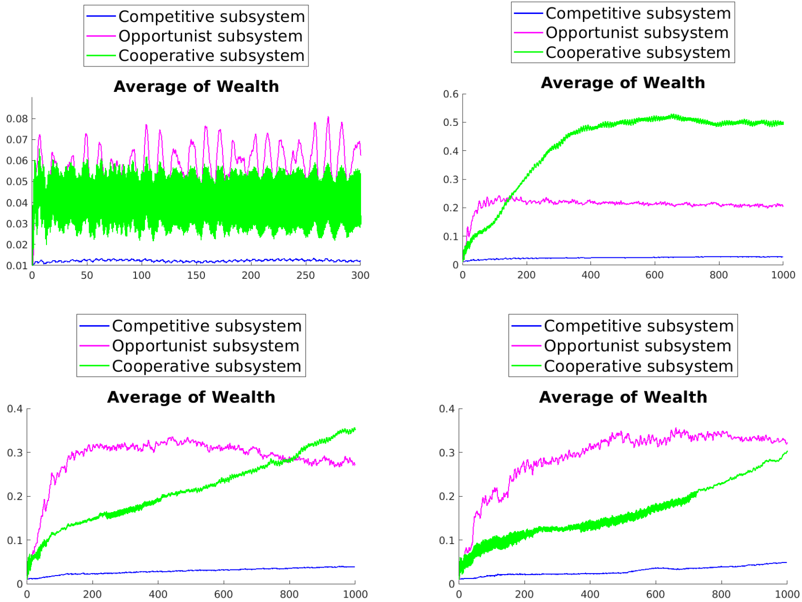
In all the simulations all the agents start with the same initial amount of wealth, say $1/N$; moreover, $N_1 = N_2$ (same number of cooperative and competitive agents), and five different values of N_3 (the number of opportunist agents) are considered.

To fix the rule, let us define

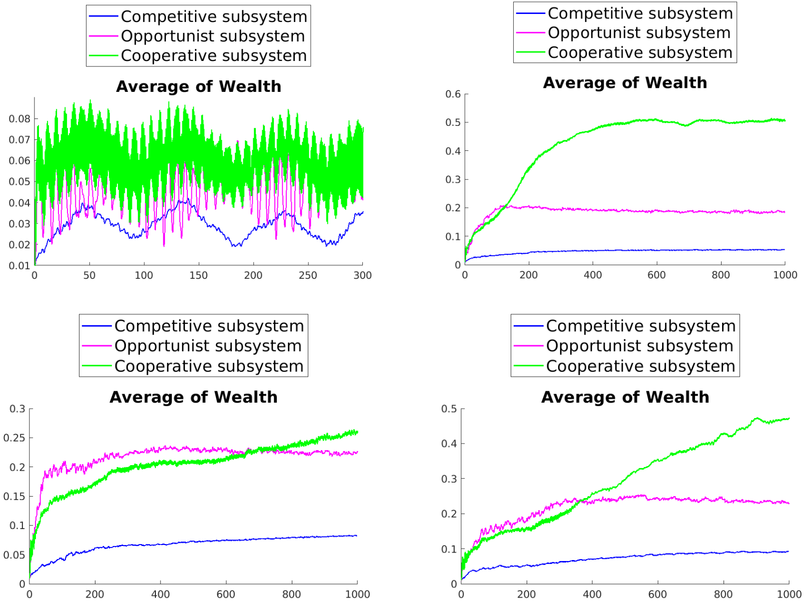
$$\begin{aligned}
\delta_j^{(k)} &= n_j(k\tau) - n_j((k-1)\tau), \quad k = 1,\dots,N,\\
\delta^{(k)} &= \max\left\{\left|\delta_j^{(k)}\right|,\ j = 1,\dots,N\right\};
\end{aligned}$$

then, at times $k\tau$ the inertia parameters change according to the rule:

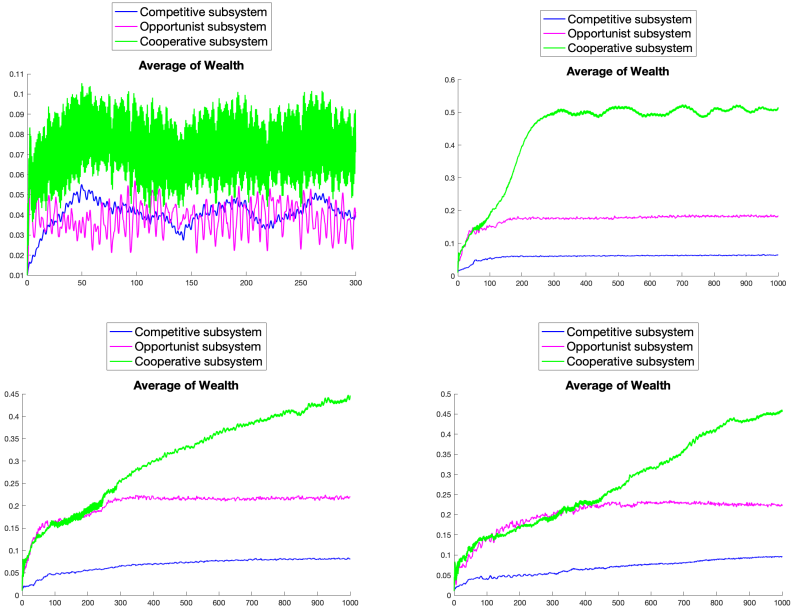
$$\omega_j \mapsto \omega_j\left(1 + \frac{\delta_j^{(k)}}{\delta^{(k)}}\right), \qquad j = 1,\dots,N.$$



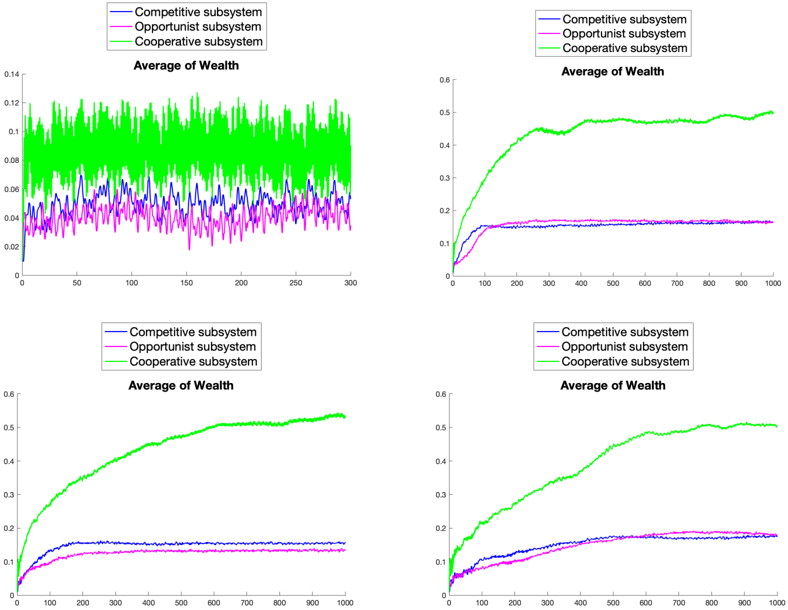
Cooperative and Competitive subgroups contain 45 agents and the Opportunist subgroup 10 agents; no rule, $(\mathcal{H}, \rho)$–induced dynamics with $\tau = 1$, $\tau = 2$ and $\tau = 4$.



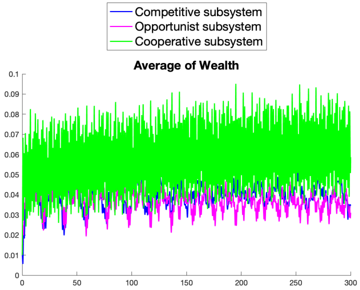
Cooperative and Competitive subgroups contain 42 agents and the Opportunist subgroup 16 agents; no rule, $(\mathcal{H}, \rho)$–induced dynamics with $\tau = 1$, $\tau = 2$ and $\tau = 4$.

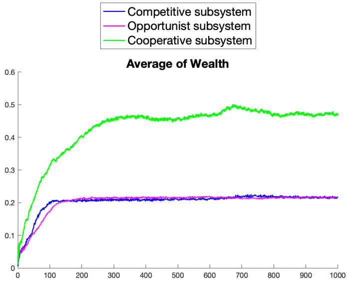


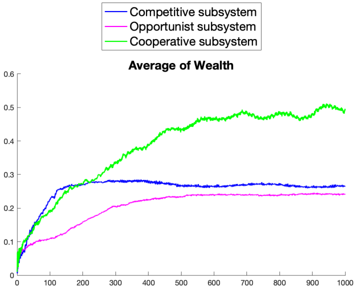
Cooperative and Competitive subgroups contain 40 agents and the Opportunist subgroup 20 agents; no rule, $(\mathcal{H}, \rho)$–induced dynamics with $\tau = 1$, $\tau = 2$ and $\tau = 4$.

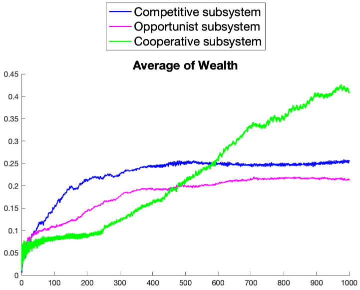


Cooperative and Competitive subgroups contain 33 agents and the Opportunist subgroup 34 agents; no rule, $(\mathcal{H}, \rho)$–induced dynamics with $\tau = 1$, $\tau = 2$ and $\tau = 4$.









Cooperative and Competitive subgroups contain 25 agents and the Opportunist subgroup 50 agents; no rule, $(\mathcal{H}, \rho)$–induced dynamics with $\tau = 1$, $\tau = 2$ and $\tau = 4$.

Comments

- Without the rule, the evolution is trivial and a never ending oscillatory outcome is obtained.
- With the $(\mathcal{H}, \rho)$–induced dynamics approach, cooperation gives better results in terms of wealth:
  - Cooperative subgroup, as time increases, obtains an amount of average wealth always greater than that of the purely competitive subgroup and often greater than that of the opportunist subgroup.
  - For a low number of opportunist agents, the time needed for having an average wealth of the cooperative subsystem greater than that of the opportunist subsystem increases with τ: a lower frequency of adjusting the attitudes of the agents makes a better performance for the opportunist agents for longer times.
  - When the number of opportunist agents increases, their average wealth definitely becomes less than that of the cooperative subsystem, regardless the choice of τ; also, the average wealth of opportunist subsystem is similar to that of competitive subsystem.

Comments

Opportunist behavior may be for some time successful if the number of opportunist agents is not too high, but as time increases their success vanishes.

In the long run opportunist agents are losers, at least in this model.

In real life, often not!

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In the long run opportunist agents are losers, at least in this model.

In real life, often not!

"You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time."
Abraham Lincoln.

Rules modifying inertia and nature of interactions

Let us consider a system made by N agents; each agent is located in a cell of a one–dimensional torus partitioned in N cells.The maximum distance between the cells is $d_{\mathsf{max}} = \lfloor N/2 \rfloor$.
The agents are partitioned in three subgroups:

- N_1 agents with inertia parameters randomly chosen in the interval $[0.2, 0.4]$;
- N_2 agents with inertia parameters randomly chosen in the interval $[0.4, 0.6]$;
- N_3 agents with inertia parameters randomly chosen in the interval $[0.6, 0.8]$.

As far as the interactions are concerned:

- N_{comp} couples of agents (randomly chosen) interact with a competition mechanism;
- N_{coop} couples of agents (randomly chosen) interact with a cooperation mechanism.

$$\lambda_{j,k} = \lambda \left(1 - \tanh\left(d(j,k) - d_{\mathsf{max}}/2\right)\right),$$
$$\mu_{j,k} = \mu \left(1 + \tanh\left(d(j,k) - d_{\mathsf{max}}/2\right)\right),$$

where $\lambda = \mu = 0.1$ (so that $\lambda_{j,k}, \mu_{j,k} \in [0, 0.2]$.

Rules modifying inertia and nature of the interactions

- Fix τ for updating the parameters.
- Compute in each subinterval of length τ

$$\delta_i^{(k)} = n_i(k\tau) - n_i((k-1)\tau), \qquad i = 1, \dots, N.$$

- At times $k\tau$, update the inertia parameters according to

$$\omega_i \mapsto \omega_i(1 + \delta_i^{(k)}),$$

and update interaction parameters (**weak attitude to cooperate**) according to:

$$\mu_{i,j} \mapsto \begin{cases} \text{min}\left(\mu_{i,j} + \delta_i^{(k)} + \delta_j^{(k)}, \mu_{max}\right) & \text{if} \quad \delta_i^{(k)}\delta_j^{(k)} > 0 \text{ and } \delta_i^{(k)} + \delta_j^{(k)} > 0 \\ \text{max}\left(\mu_{i,j} + \delta_i^{(k)} + \delta_j^{(k)}, \mu_{min}\right) & \text{if} \quad \delta_i^{(k)} < 0 \text{ and } \delta_j^{(k)} < 0 \end{cases}$$

$$\lambda_{i,j} \mapsto \begin{cases} \text{max}\left(\lambda_{i,j} - \delta_i^{(k)} - \delta_j^{(k)}, \lambda_{min}\right) & \text{if} \quad \delta_i^{(k)} > 0 \text{ and } \delta_j^{(k)} > 0 \\ \text{min}\left(\lambda_{i,j} - \delta_i^{(k)} - \delta_j^{(k)}, \lambda_{max}\right) & \text{if} \quad \delta_i^{(k)} < 0 \text{ and } \delta_j^{(k)} < 0 \end{cases} .$$

- In the case of **strong attitude to cooperate**, besides previous rules, it is also:

$$\mu_{i,j} \mapsto \text{min}\left(\mu_{i,j} + (\delta_i^{(k)} + \delta_j^{(k)})/2, \mu_{max}\right) \text{ if } (\delta_i^{(k)} + \delta_j^{(k)} > 0).$$

Rules modifying inertia and nature of the interactions

The agents of the network can be classified as competitive, cooperative or neutral. Compute

$$t_i = \sum_{j=1}^{N} (\lambda_{i,j} - \mu_{i,j})\,;$$

then

- the i–th agent is **cooperative** if $t_i < 0$;
- the i–th agent is **neutral** if $t_i = 0$;
- the i–th agent is **competitive** if $t_i > 0$.

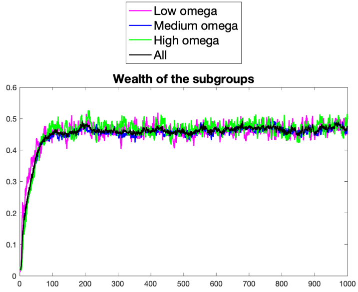
The distribution of wealth $n_i(t)$ among the agents could be analyzed by means of **Gini Index**:

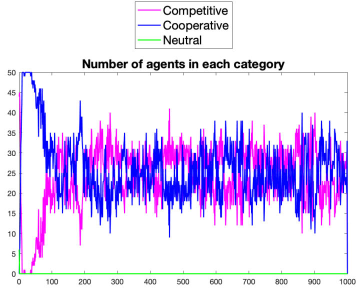
$$G(t) = \frac{\sum_{i,j=1}^{N} |n_i(t) - n_j(t)|}{2N \sum_{i=1}^{N} n_i(t)}.$$

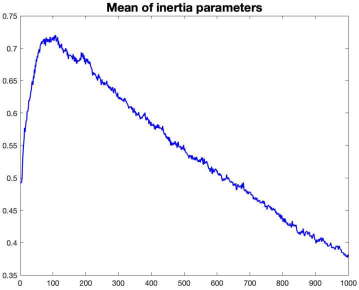
belonging to the interval $[0, 1]$. A value close to 0 corresponds to an almost uniform wealth distribution, whereas a value close to 1 to a wealth distribution with strong inequalities.

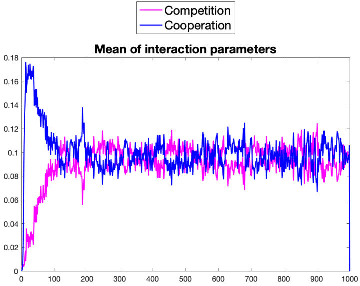
Rules modifying inertia and nature of the interactions

Only competing agents initially: weak attitude to cooperate.



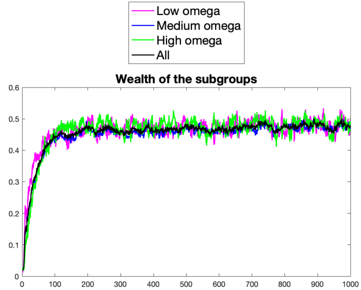


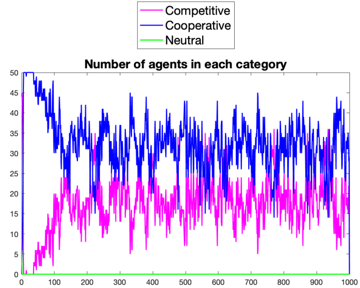


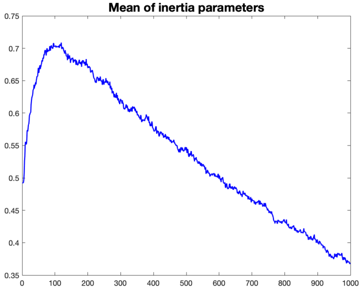


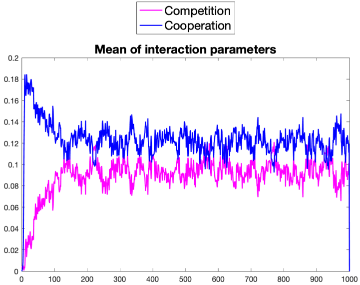
Rules modifying inertia and nature of the interactions

Only competing agents initially: strong attitude to cooperate.



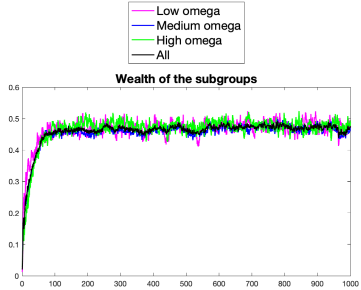


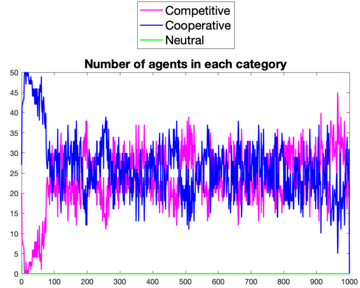


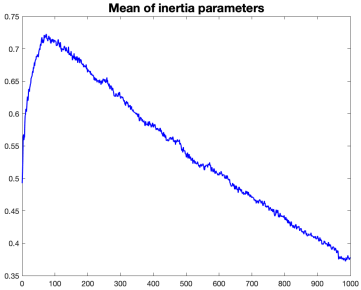


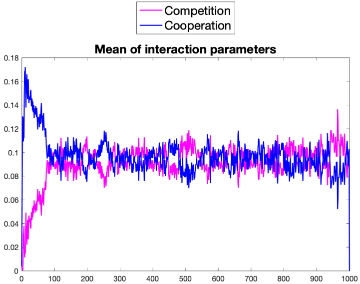
Rules modifying inertia and nature of the interactions

Same initial number of competing and cooperating couples: weak attitude to cooperate.



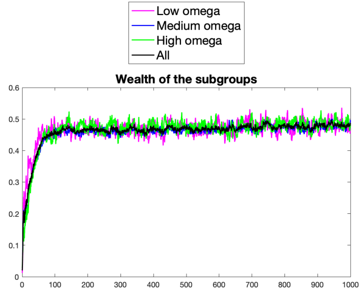


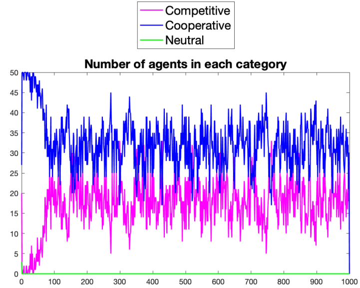


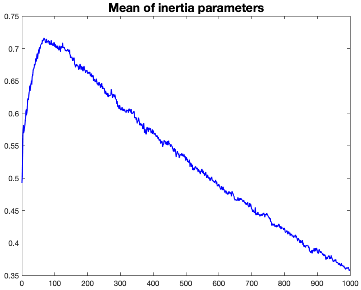


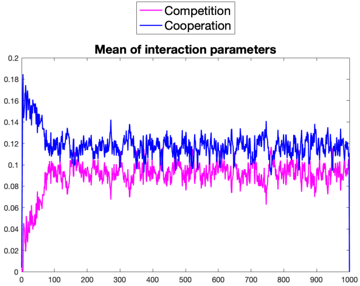
Rules modifying inertia and nature of the interactions

Same initial number of competing and cooperating couples: strong attitude to cooperate.

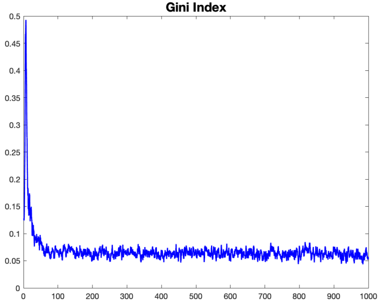


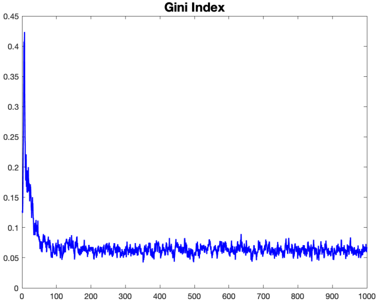


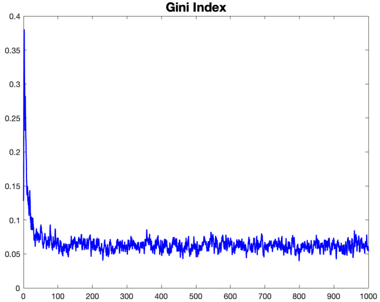


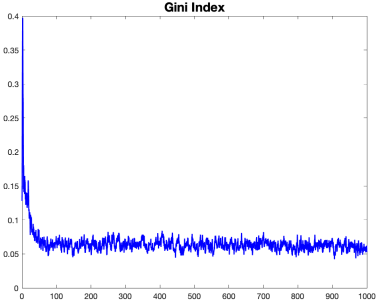


Rules modifying inertia and nature of the interactions









Thank you for the attention.