Real 4D Lie algebras

J. Patera, P. Winternitz. Subalgebras of real three and four--dimensional Lie algebras. Journal of Mathematical Physics, 18, 1449--1455, 1977.

In[1]:=
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SetDirectory[NotebookDirectory[]];
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In[2]:=
```
<< "SymboLie.wl"
```

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SymboLie (v. 1.6)- A Package for determining Optimal Systems of Lie Subalgebras.
```

In[3]:=
```
CS = Table[0, {k, 1, 4}, {i, 1, 4}, {j, 1, 4}];
AllCS = Table[CS, {r, 1, 30}];
AllCS[[2, 2, 1, 2]] = 1; AllCS[[2, 2, 2, 1]] = -1;
AllCS[[3, 2, 1, 2]] = 1; AllCS[[3, 2, 2, 1]] = -1;
AllCS[[3, 4, 3, 4]] = 1; AllCS[[3, 4, 4, 3]] = -1;
AllCS[[4, 1, 2, 3]] = 1; AllCS[[4, 1, 3, 2]] = -1;
AllCS[[5, 1, 1, 3]] = 1; AllCS[[5, 1, 3, 1]] = -1;
AllCS[[5, 1, 2, 3]] = 1; AllCS[[5, 1, 3, 2]] = -1;
AllCS[[5, 2, 2, 3]] = 1; AllCS[[5, 2, 3, 2]] = -1;
AllCS[[6, 1, 1, 3]] = 1; AllCS[[6, 1, 3, 1]] = -1;
AllCS[[6, 2, 2, 3]] = 1; AllCS[[6, 2, 3, 2]] = -1;
AllCS[[7, 1, 1, 3]] = 1; AllCS[[7, 1, 3, 1]] = -1;
AllCS[[7, 2, 2, 3]] = -1; AllCS[[7, 2, 3, 2]] = 1;
AllCS[[8, 1, 1, 3]] = 1; AllCS[[8, 1, 3, 1]] = -1;
AllCS[[8, 2, 2, 3]] = a;
AllCS[[8, 2, 3, 2]] = -a;
(*con abs(a) tra 0 e 1*)AllCS[[9, 2, 1, 3]] = -1;
AllCS[[9, 2, 3, 1]] = 1;
AllCS[[9, 1, 2, 3]] = 1; AllCS[[9, 1, 3, 2]] = -1;
AllCS[[10, 1, 1, 3]] = a;
AllCS[[10, 1, 3, 1]] = -a;
(*a>0*)AllCS[[10, 1, 2, 3]] = 1;
AllCS[[10, 1, 3, 2]] = -1;
AllCS[[10, 2, 1, 3]] = -1; AllCS[[10, 2, 3, 1]] = 1;
AllCS[[10, 2, 2, 3]] = a;
AllCS[[10, 2, 3, 2]] = -a;
(*a>0*)AllCS[[11, 2, 3, 1]] = 2;
AllCS[[11, 2, 1, 3]] = -2;
AllCS[[11, 1, 1, 2]] = 1; AllCS[[11, 1, 2, 1]] = -1;
AllCS[[11, 3, 2, 3]] = 1; AllCS[[11, 3, 3, 2]] = -1;
AllCS[[12, 3, 1, 2]] = 1; AllCS[[12, 3, 2, 1]] = -1;
AllCS[[12, 1, 2, 3]] = 1; AllCS[[12, 1, 3, 2]] = -1;
AllCS[[12, 2, 3, 1]] = 1; AllCS[[12, 2, 1, 3]] = -1;
AllCS[[13, 1, 2, 4]] = 1; AllCS[[13, 1, 4, 2]] = -1;
AllCS[[13, 2, 3, 4]] = 1; AllCS[[13, 2, 4, 3]] = -1;
AllCS[[14, 1, 1, 4]] = a;
AllCS[[14, 1, 4, 1]] = -a;
(*a diverso da 0,1*)AllCS[[14, 2, 2, 4]] = 1;
AllCS[[14, 2, 4, 2]] = -1;
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AllCS[[14, 2, 3, 4]] = 1; AllCS[[14, 2, 4, 3]] = -1;
AllCS[[14, 3, 3, 4]] = 1; AllCS[[14, 3, 4, 3]] = -1;
AllCS[[15, 1, 1, 4]] = 1; AllCS[[15, 1, 4, 1]] = -1;
AllCS[[15, 2, 2, 4]] = 1; AllCS[[15, 2, 4, 2]] = -1;
AllCS[[15, 2, 3, 4]] = 1; AllCS[[15, 2, 4, 3]] = -1;
AllCS[[15, 3, 3, 4]] = 1; AllCS[[15, 3, 4, 3]] = -1;
AllCS[[16, 1, 1, 4]] = 1; AllCS[[16, 1, 4, 1]] = -1;
AllCS[[16, 2, 3, 4]] = 1; AllCS[[16, 2, 4, 3]] = -1;
AllCS[[17, 1, 1, 4]] = 1; AllCS[[17, 1, 4, 1]] = -1;
AllCS[[17, 1, 2, 4]] = 1; AllCS[[17, 1, 4, 2]] = -1;
AllCS[[17, 2, 2, 4]] = 1; AllCS[[17, 2, 4, 2]] = -1;
AllCS[[17, 2, 3, 4]] = 1; AllCS[[17, 2, 4, 3]] = -1;
AllCS[[17, 3, 3, 4]] = 1; AllCS[[17, 3, 4, 3]] = -1;
AllCS[[18, 1, 1, 4]] = 1; AllCS[[18, 1, 4, 1]] = -1;
AllCS[[18, 2, 2, 4]] = a; AllCS[[18, 2, 4, 2]] = -a;
AllCS[[18, 3, 3, 4]] = b;
AllCS[[18, 3, 4, 3]] = -b;
(*-1≤a,b<1,ab≠0*)AllCS[[19, 1, 1, 4]] = 1;
AllCS[[19, 1, 4, 1]] = -1;
AllCS[[19, 2, 2, 4]] = a; AllCS[[19, 2, 4, 2]] = -a;
AllCS[[19, 3, 3, 4]] = a;
AllCS[[19, 3, 4, 3]] = -a;
(*-1≤a<1,a≠0*)AllCS[[20, 1, 1, 4]] = 1;
AllCS[[20, 1, 4, 1]] = -1;
AllCS[[20, 2, 2, 4]] = a;
AllCS[[20, 2, 4, 2]] = -a;
(*-1≤a<1,a≠0*)AllCS[[20, 3, 3, 4]] = 1;
AllCS[[20, 3, 4, 3]] = -1;
AllCS[[21, 1, 1, 4]] = 1; AllCS[[21, 1, 4, 1]] = -1;
AllCS[[21, 2, 2, 4]] = 1; AllCS[[21, 2, 4, 2]] = -1;
AllCS[[21, 3, 3, 4]] = 1; AllCS[[21, 3, 4, 3]] = -1;
AllCS[[22, 1, 1, 4]] = a;
AllCS[[22, 1, 4, 1]] = -a;
(*a≠0,b≥0*)AllCS[[22, 2, 2, 4]] = b;
AllCS[[22, 2, 4, 2]] = -b;
AllCS[[22, 2, 3, 4]] = 1; AllCS[[22, 2, 4, 3]] = -1;
AllCS[[22, 3, 2, 4]] = -1; AllCS[[22, 3, 4, 2]] = 1;
AllCS[[22, 3, 3, 4]] = b; AllCS[[22, 3, 4, 3]] = -b;
AllCS[[23, 1, 1, 4]] = 2; AllCS[[23, 1, 4, 1]] = -2;
AllCS[[23, 2, 2, 4]] = 1; AllCS[[23, 2, 4, 2]] = -1;
AllCS[[23, 2, 3, 4]] = 1; AllCS[[23, 2, 4, 3]] = -1;
AllCS[[23, 3, 3, 4]] = 1; AllCS[[23, 3, 4, 3]] = -1;
AllCS[[23, 1, 2, 3]] = 1; AllCS[[23, 1, 3, 2]] = -1;
AllCS[[24, 1, 2, 3]] = 1; AllCS[[24, 1, 3, 2]] = -1;
AllCS[[24, 2, 2, 4]] = 1; AllCS[[24, 2, 4, 2]] = -1;
AllCS[[24, 3, 3, 4]] = -1; AllCS[[24, 3, 4, 3]] = 1;
AllCS[[25, 1, 2, 3]] = 1;
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AllCS[[25, 1, 3, 2]] = -1;
(*0<abs(b)<1*)AllCS[[25, 1, 1, 4]] = 1 + b;
AllCS[[25, 1, 4, 1]] = -1 - b;
AllCS[[25, 2, 2, 4]] = 1; AllCS[[25, 2, 4, 2]] = -1;
AllCS[[25, 3, 3, 4]] = b; AllCS[[25, 3, 4, 3]] = -b;
AllCS[[26, 1, 2, 3]] = 1; AllCS[[26, 1, 3, 2]] = -1;
AllCS[[26, 1, 1, 4]] = 2; AllCS[[26, 1, 4, 1]] = -2;
AllCS[[26, 2, 2, 4]] = 1; AllCS[[26, 2, 4, 2]] = -1;
AllCS[[26, 3, 3, 4]] = 1; AllCS[[26, 3, 4, 3]] = -1;
AllCS[[27, 1, 2, 3]] = 1; AllCS[[27, 1, 3, 2]] = -1;
AllCS[[27, 1, 1, 4]] = 1; AllCS[[27, 1, 4, 1]] = -1;
AllCS[[27, 2, 2, 4]] = 1; AllCS[[27, 2, 4, 2]] = -1;
AllCS[[28, 1, 2, 3]] = 1; AllCS[[28, 1, 3, 2]] = -1;
AllCS[[28, 3, 2, 4]] = -1; AllCS[[28, 3, 4, 2]] = 1;
AllCS[[28, 2, 3, 4]] = 1; AllCS[[28, 2, 4, 3]] = -1;
AllCS[[29, 1, 2, 3]] = 1;
AllCS[[29, 1, 3, 2]] = -1;
(*a>0*)AllCS[[29, 1, 1, 4]] = 2 a;
AllCS[[29, 1, 4, 1]] = -2 a;
AllCS[[29, 2, 2, 4]] = a; AllCS[[29, 2, 4, 2]] = -a;
AllCS[[29, 3, 2, 4]] = -1; AllCS[[29, 3, 4, 2]] = 1;
AllCS[[29, 2, 3, 4]] = 1; AllCS[[29, 2, 4, 3]] = -1;
AllCS[[29, 3, 3, 4]] = a; AllCS[[29, 3, 4, 3]] = -a;
AllCS[[30, 1, 1, 3]] = 1; AllCS[[30, 1, 3, 1]] = -1;
AllCS[[30, 2, 2, 3]] = 1; AllCS[[30, 2, 3, 2]] = -1;
AllCS[[30, 2, 1, 4]] = -1; AllCS[[30, 2, 4, 1]] = 1;
AllCS[[30, 1, 2, 4]] = 1;
AllCS[[30, 1, 4, 2]] = -1;
```

In[91]:=
```
pars = Table[{{}, {}}, {k1, 1, 30}];
pars[[8]] = {{a}, {-1 < a, a < 1, a ≠ 0}};(*pars[[8]]={a==1/2};*)
pars[[10]] = {{a}, {a > 0}};
pars[[14]] = {{a}, {a ≠ 0, a ≠ 1}};
pars[[18]] = {{a, b}, {-1 ≤ a, a < b, b < 1, a b ≠ 0}};
pars[[19]] = {{a}, {-1 ≤ a, a < 1, a ≠ 0}};
pars[[20]] = {{a}, {-1 ≤ a, a < 1, a ≠ 0}};
pars[[22]] = {{a, b}, {a ≠ 0, b ≥ 0}};
pars[[25]] = {{b}, {-1 < b, b < 1, b ≠ 0}};
pars[[29]] = {{a}, {a > 0}};
```

In[93]:=
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allalg1 = {}; allalg2 = {}; allalg3 = {};
```

In[94]:=
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For[k1 = 1, k1 ≤ 30, k1++,
  Print["=========================================="];
  Print["Algebra n. ", k1];
  t1 = SessionTime[];
  Print[CommutatorTable[AllCS[[k1]]] // MatrixForm];
  alg1 = SubAlgebra[AllCS[[k1]], pars[[k1]], 1];
  AppendTo[allalg1, alg1];
  optalg1 = PrintOptimal[alg1];
  Print[optalg1];
  G = PrintGraph[alg1];
  Print[G];
  alg2 = SubAlgebra[AllCS[[k1]], pars[[k1]], 2];
  AppendTo[allalg2, alg2];
  optalg2 = PrintOptimal[alg2];
  Print[optalg2];
  G = PrintGraph[alg2];
  Print[G];
  alg3 = SubAlgebra[AllCS[[k1]], pars[[k1]], 3];
  AppendTo[allalg3, alg3];
  optalg3 = PrintOptimal[alg3];
  Print[optalg3];
  G = PrintGraph[alg3];
  Print[G];
  t1 = SessionTime[] - t1;
  Print["Time of computation: ", t1]
 ];
Print["=========================================="];
```

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==========================================
Algebra n. 1
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
There are 15 1-D families of subalgebras to be analyzed.

{{1}}

Done.

There are 15 optimal families of 1-dimensional Lie subalgebras.
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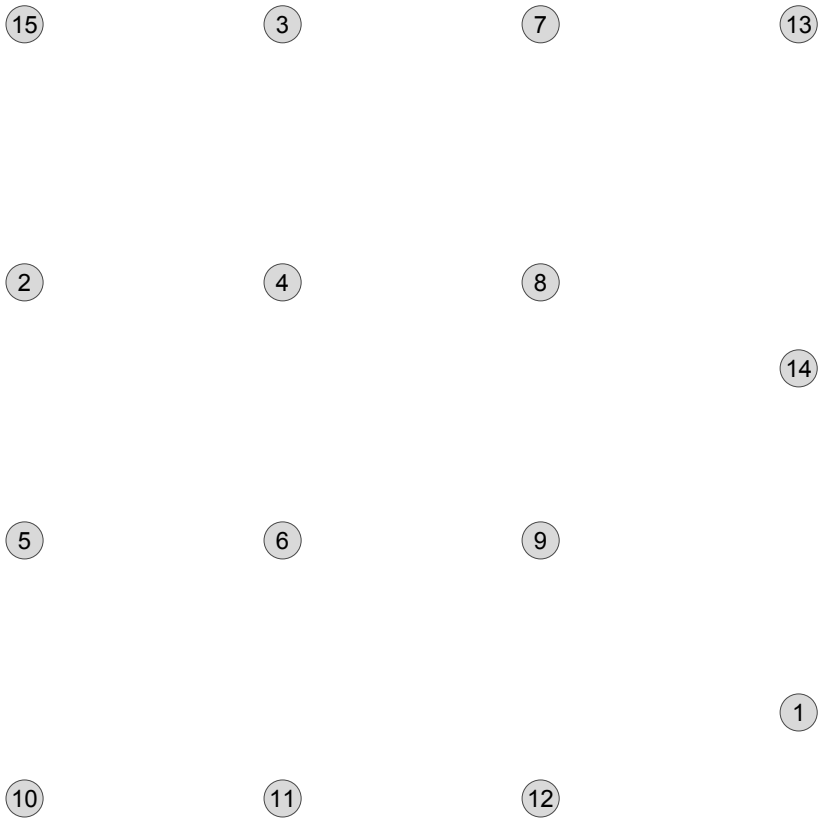
$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4\}, \{\Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + a_1 \Xi_2\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}, 7 \to \{\Xi_2 + a_1 \Xi_3\}, 8 \to \{\Xi_1 + a_1 \Xi_4\}, 9 \to \{\Xi_2 + a_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 11 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3\}, 12 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4\}, 13 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4\}, 14 \to \{\Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, 15 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\}\}$$



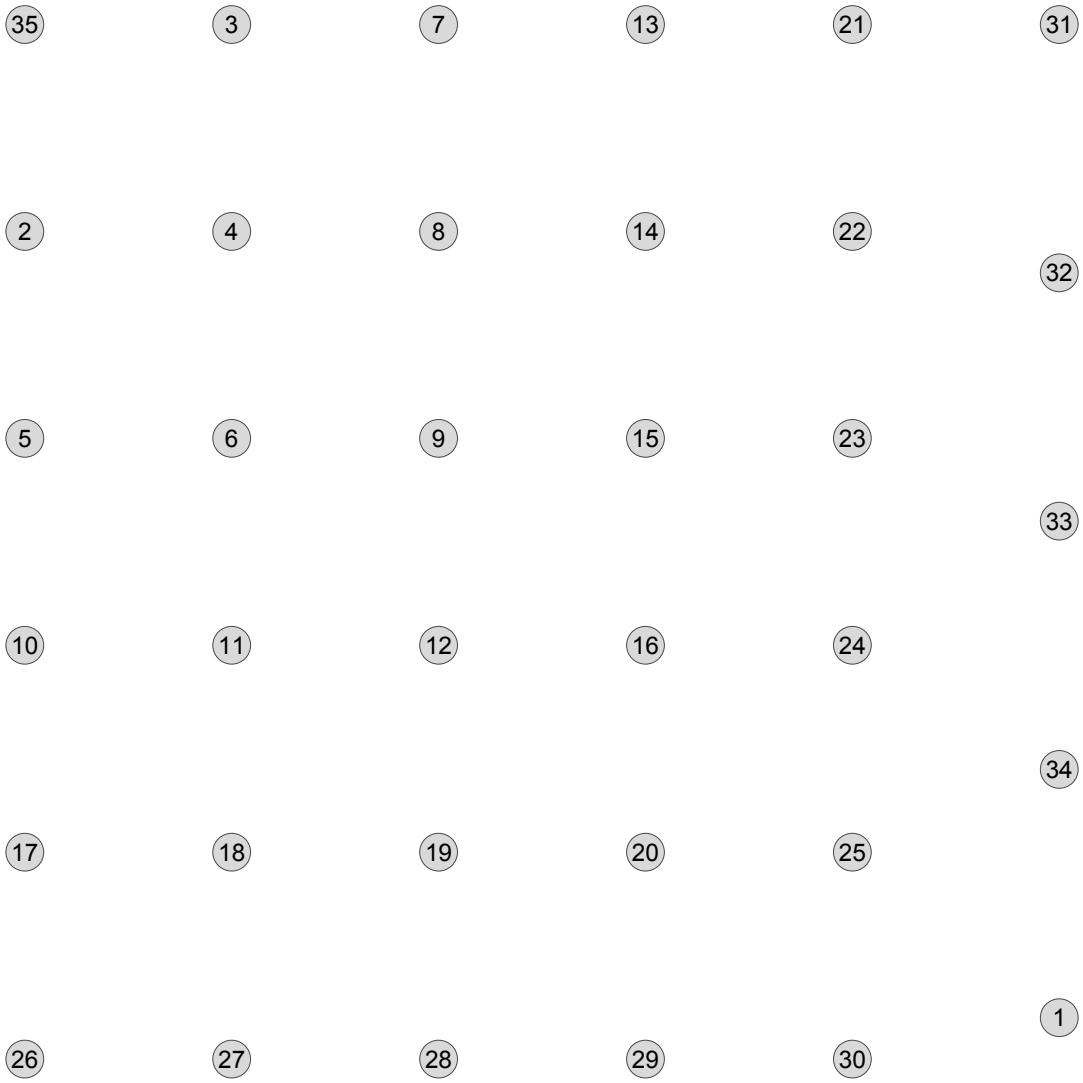
There are 35 2-D families of subalgebras to be analyzed.

Done.

There are 35 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3\},$$
$$\{\Xi_1, \Xi_2 + a_1 \Xi_4\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2\}, \{\Xi_1 + a_1 \Xi_4, \Xi_3\},$$
$$\{\Xi_2 + a_1 \Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, \{\Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4, \Xi_3\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4, \Xi_3 + a_3 \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_3\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_3 + a_4 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\},$$
$$6 \to \{\Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\}, 8 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4\}, 9 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\},$$
$$10 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, 11 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 12 \to \{\Xi_1 + a_1 \Xi_2, \Xi_4\}, 13 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\},$$
$$14 \to \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, 15 \to \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, 16 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2\}, 17 \to \{\Xi_1 + a_1 \Xi_4, \Xi_3\},$$
$$18 \to \{\Xi_2 + a_1 \Xi_4, \Xi_3\}, 19 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, 20 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3 + a_2 \Xi_4\},$$
$$21 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\}, 22 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_4\}, 23 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3, \Xi_4\},$$
$$24 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3\}, 25 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4\}, 26 \to \{\Xi_1 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\},$$
$$27 \to \{\Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, 28 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4, \Xi_3\}, 29 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2\},$$
$$30 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\}, 31 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3 + a_3 \Xi_4\},$$
$$32 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_4, \Xi_3 + a_3 \Xi_4\}, 33 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_3\},$$
$$34 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_4\}, 35 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2 + a_3 \Xi_3 + a_4 \Xi_4\}\}$$



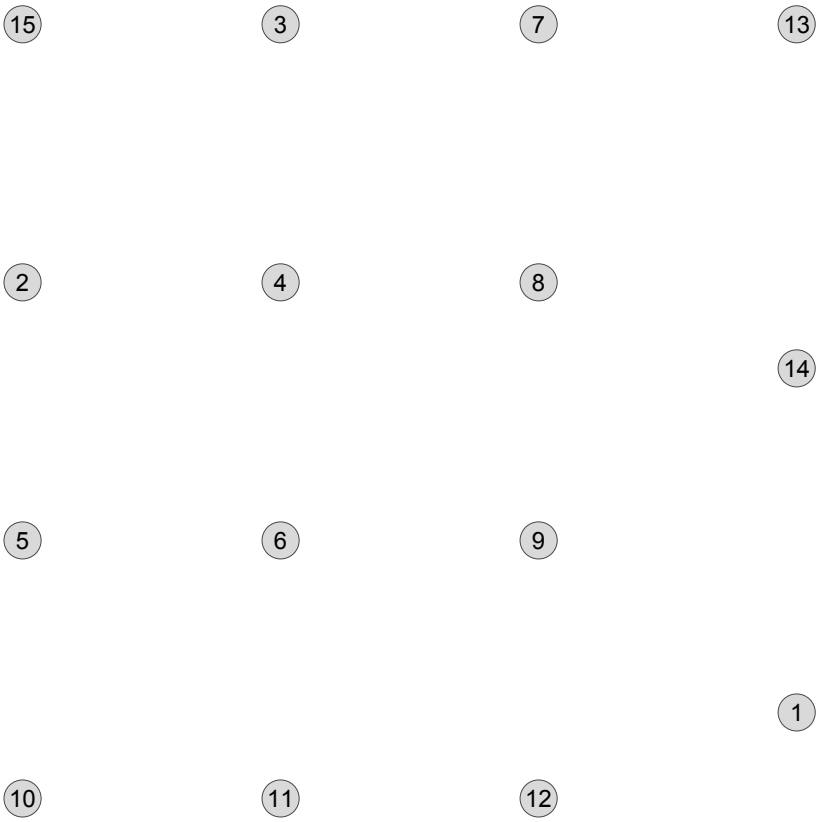
There are 15 3-D families of subalgebras to be analyzed.

Done.

There are 15 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3, \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4, \Xi_3\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4, \Xi_3 + a_3 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\},$$
$$5 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3\},$$
$$8 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}, 9 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}, 10 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3\},$$
$$11 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3, \Xi_4\}, 13 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3 + a_2 \Xi_4\},$$
$$14 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4, \Xi_3\}, 15 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_4, \Xi_3 + a_3 \Xi_4\}\}$$



Time of computation: 0.269663

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Algebra n. 2

$$\begin{pmatrix} 0 & \Xi_2 & 0 & 0 \\ -\Xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

$\{\{1\}\}$

Done.

There are 11 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_4\},$
$\{\Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 8 \to \{\Xi_1 + a_1 \Xi_4\},$
$9 \to \{\Xi_2 + \alpha_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 13 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4\}, 14 \to \{\Xi_2 + \alpha_1 \Xi_3 + a_1 \Xi_4\}\}$

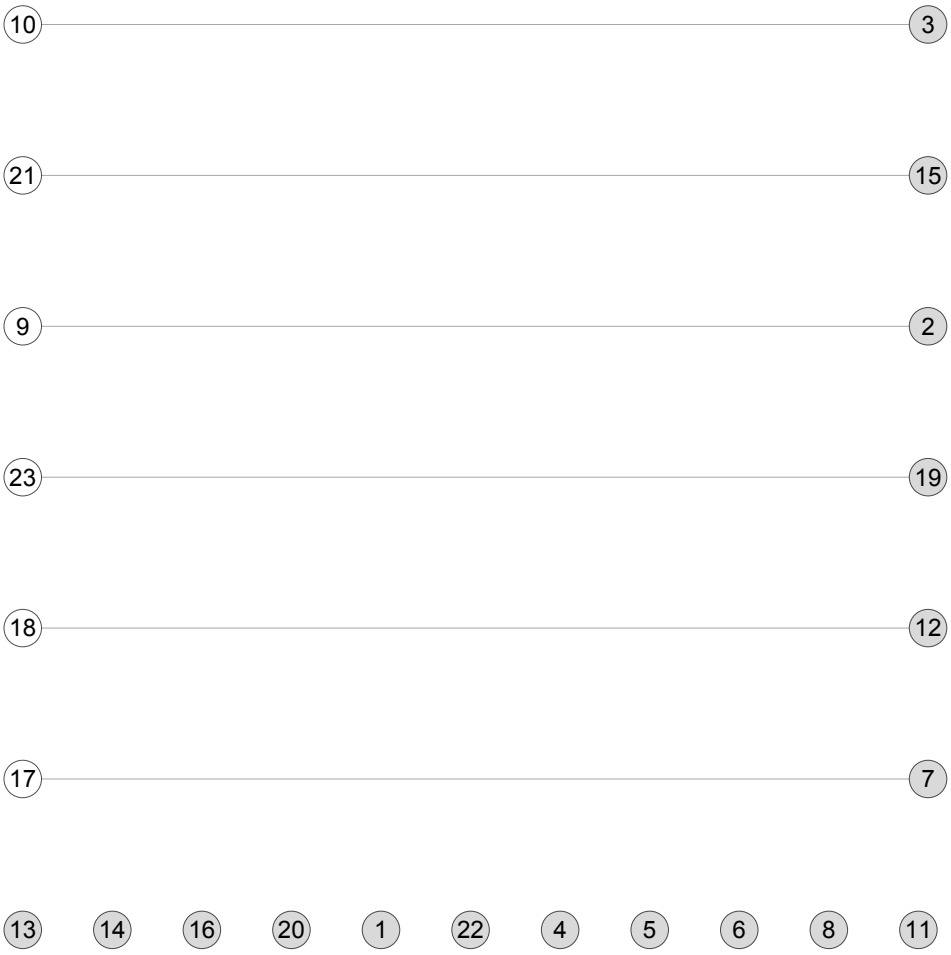
There are 23 2-D families of subalgebras to be analyzed.

Done.

There are 17 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2\}, \{\Xi_1 + a_1 \Xi_4, \Xi_3\},$$
$$\{\Xi_2 + \alpha_1 \Xi_4, \Xi_3\}, \{\Xi_1 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\},$$
$$7 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 8 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, 11 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, 12 \to \{\Xi_1 + a_1 \Xi_3, \Xi_4\},$$
$$13 \to \{\Xi_2 + \alpha_1 \Xi_3, \Xi_4\}, 14 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2\}, 15 \to \{\Xi_1 + a_1 \Xi_4, \Xi_3\}, 16 \to \{\Xi_2 + \alpha_1 \Xi_4, \Xi_3\},$$
$$19 \to \{\Xi_1 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}, 20 \to \{\Xi_2 + \alpha_1 \Xi_4, \Xi_3 + a_1 \Xi_4\}, 22 \to \{\Xi_1 + a_1 \Xi_3 + a_2 \Xi_4, \Xi_2\}\}$$



There are 9 3-D families of subalgebras to be analyzed.

Done.

There are 8 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3 + a_2 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$7 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}, 8 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3\}, 9 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2, \Xi_3 + a_2 \Xi_4\}\}$$

Time of computation: 0.685922

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Algebra n. 3

$$\begin{pmatrix} 0 & \Xi_2 & 0 & 0 \\ -\Xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_4 \\ 0 & 0 & -\Xi_4 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

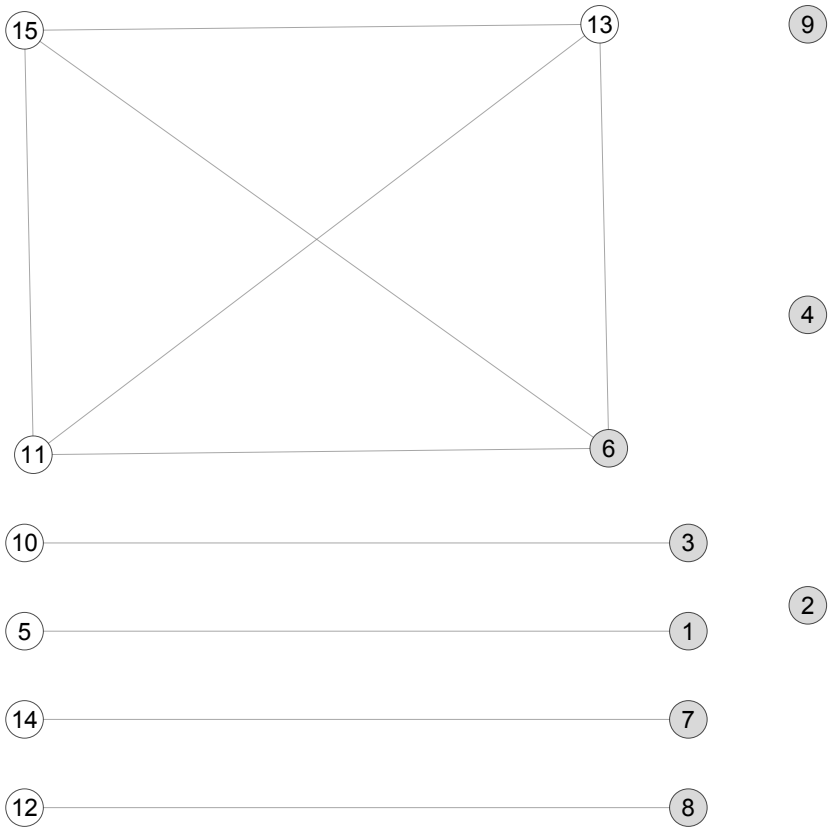
{{1, 2}}

Done.

There are 8 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\},$
$6 \to \{\Xi_1 + a_1 \Xi_3\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + \alpha_1 \Xi_4\}\}$



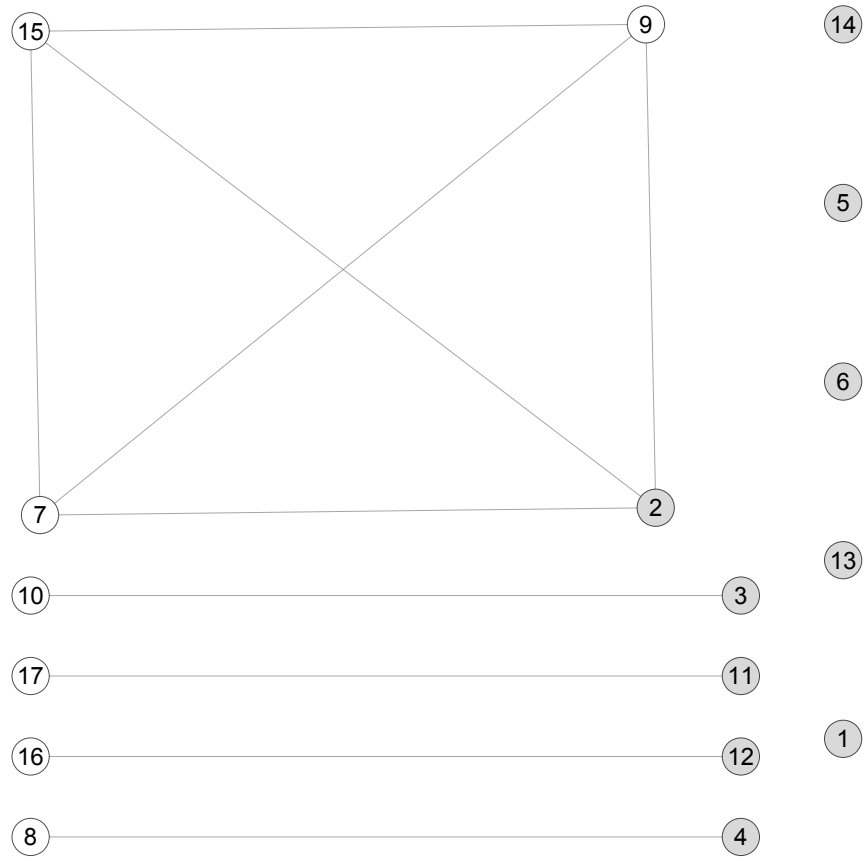
There are 17 2-D families of subalgebras to be analyzed.

Done.

There are 10 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\},\ \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_4\}, \{\Xi_2 + \underset{\cdot}{\alpha}_1 \Xi_3, \Xi_4\}, \{\Xi_1 + \underset{\cdot}{\alpha}_1 \Xi_4, \Xi_2\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\},\ 11 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}, 12 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_4\}, 13 \to \{\Xi_2 + \underset{\cdot}{\alpha}_1 \Xi_3, \Xi_4\}, 14 \to \{\Xi_1 + \underset{\cdot}{\alpha}_1 \Xi_4, \Xi_2\}\}$$



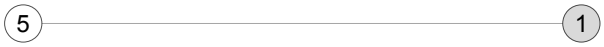
There are 7 3-D families of subalgebras to be analyzed.

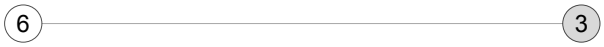
Done.

There are 5 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 7 \to \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2, \Xi_4\}\}$





Time of computation: 0.809559

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Algebra n. 4

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Xi_1 & 0 \\ 0 & -\Xi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

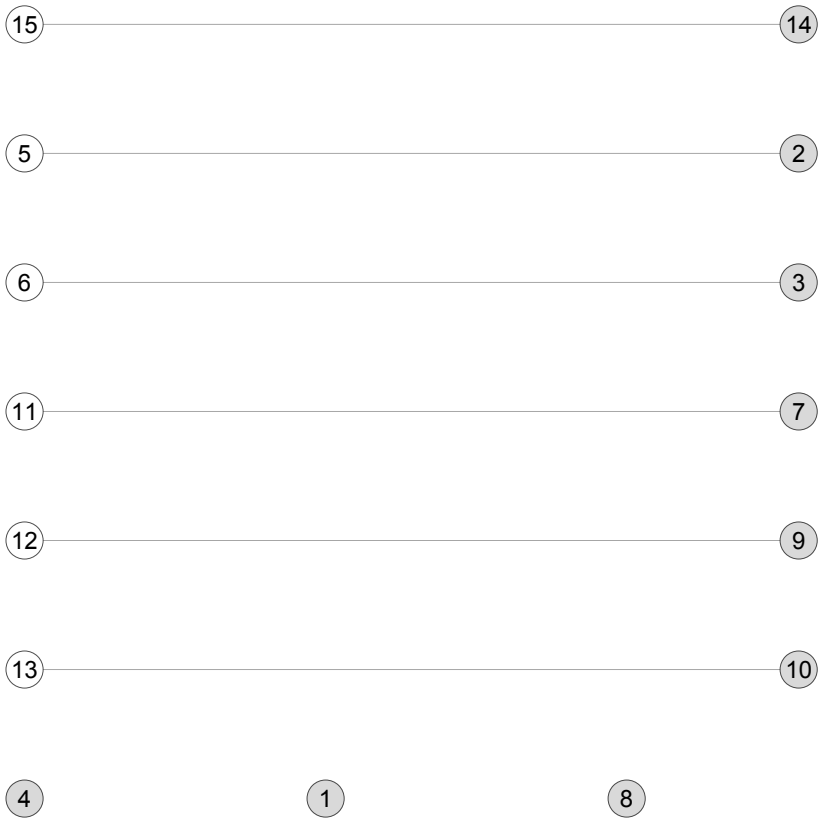
$\{\{1, 2\}\}$

Done.

There are 9 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 7 \to \{\Xi_2 + a_1 \Xi_3\},$
$8 \to \{\Xi_1 + a_1 \Xi_4\}, 9 \to \{\Xi_2 + a_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 14 \to \{\Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}\}$



There are 19 2-D families of subalgebras to be analyzed.

Done.

There are 13 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3\},$
$\{\Xi_1, \Xi_2 + a_1 \Xi_4\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2\},$
$\{\Xi_1 + a_1 \Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\}, 5 \to \{\Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\},$
$7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4\}, 8 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 11 \to \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2\},$
$13 \to \{\Xi_1 + a_1 \Xi_4, \Xi_3\}, 14 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3 + a_2 \Xi_4\}, 16 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2 + a_2 \Xi_3\}\}$

There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 7 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$5 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3\}, 7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4, \Xi_3 + a_2 \Xi_4\}\}$$

1

3

7

2

4

5

6

Time of computation: 1.002533

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Algebra n. 5

$$\begin{pmatrix} 0 & 0 & \Xi_1 & 0 \\ 0 & 0 & \Xi_1 + \Xi_2 & 0 \\ -\Xi_1 & -\Xi_1 - \Xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

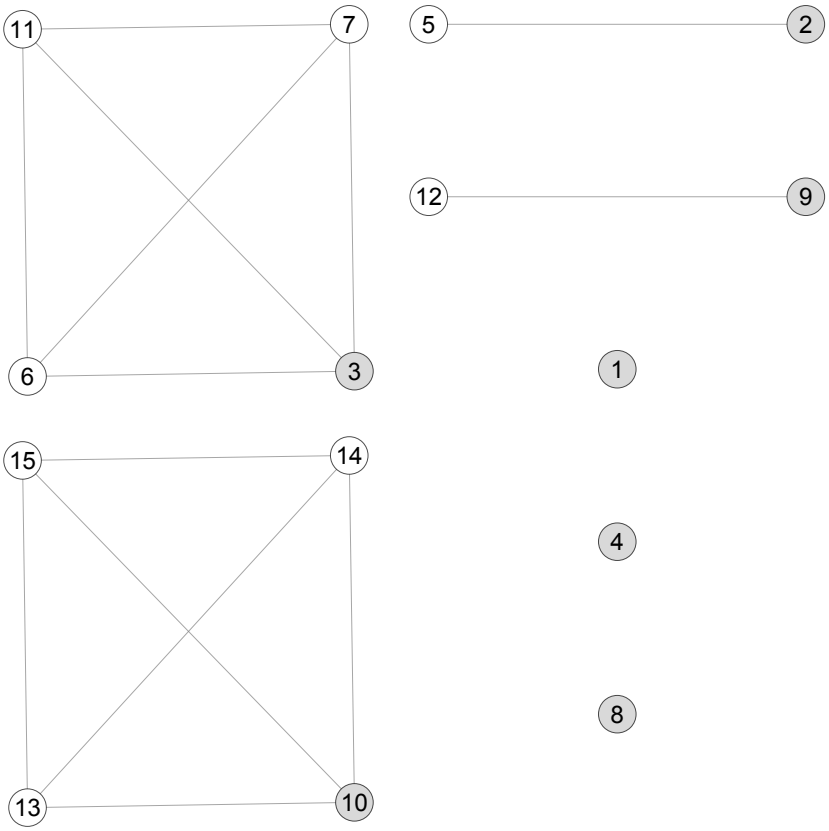
{{1, 2, 3}, {1, 3, 2}, {2, 3, 1}, {3, 1, 2}}

Done.

There are 7 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + a_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}\}$



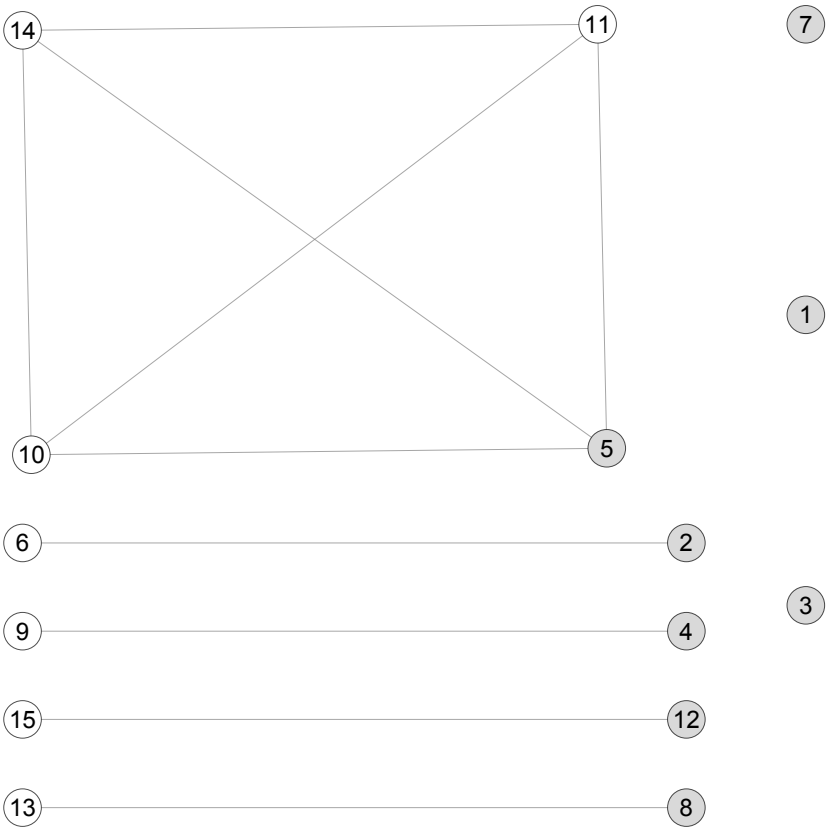
There are 15 2-D families of subalgebras to be analyzed.

Done.

There are 8 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\},\ \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_4, \Xi_2\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\},\ 5 \to \{\Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, 8 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_4, \Xi_2\}\}$$



There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

Time of computation: 2.928295

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Algebra n. 6

$$\begin{pmatrix} 0 & 0 & \Xi_1 & 0 \\ 0 & 0 & \Xi_2 & 0 \\ -\Xi_1 & -\Xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

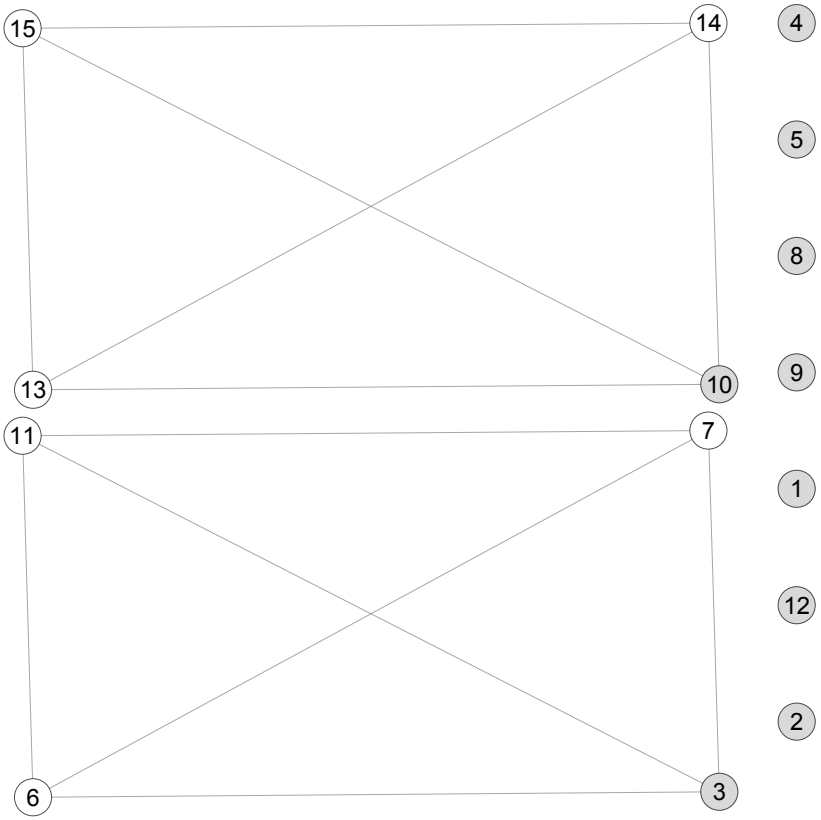
```
{{1, 2}}
```

Done.

There are 9 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2 + \alpha_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + a_1 \Xi_2\},$$
$$8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + \alpha_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_2 + \alpha_1 \Xi_4\}\}$$



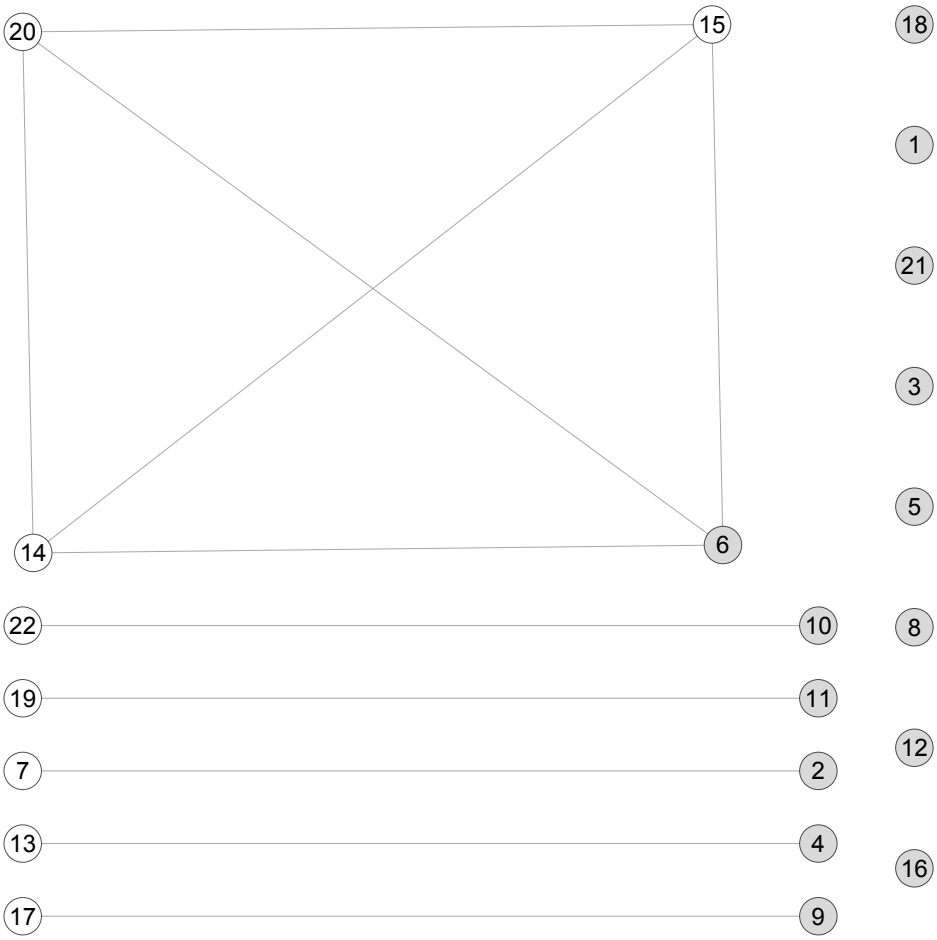
There are 22 2-D families of subalgebras to be analyzed.

Done.

There are 14 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\},$$
$$\{\Xi_1 + a_1 \Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3 + a_2 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\},$$
$$5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\}, 8 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\},$$
$$10 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, 11 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 12 \to \{\Xi_1 + a_1 \Xi_2, \Xi_4\},$$
$$16 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, 18 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3 + a_2 \Xi_4\}, 21 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$$



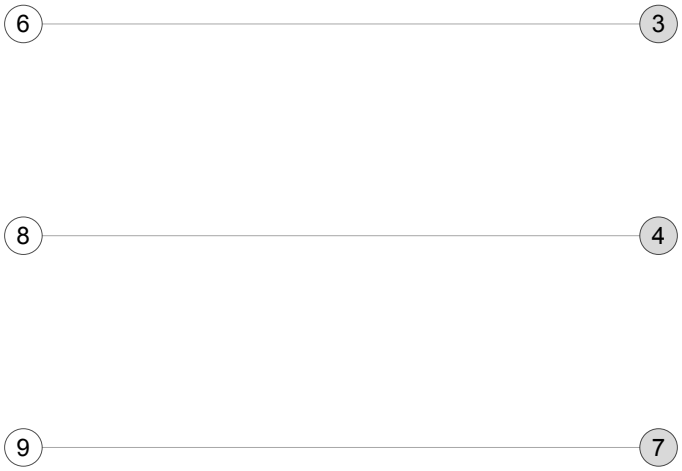
There are 9 3-D families of subalgebras to be analyzed.

Done.

There are 6 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\},$
$4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}, 7 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}\}$



Time of computation: 1.091080

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Algebra n. 7

$$\begin{pmatrix} 0 & 0 & \Xi_1 & 0 \\ 0 & 0 & -\Xi_2 & 0 \\ -\Xi_1 & \Xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

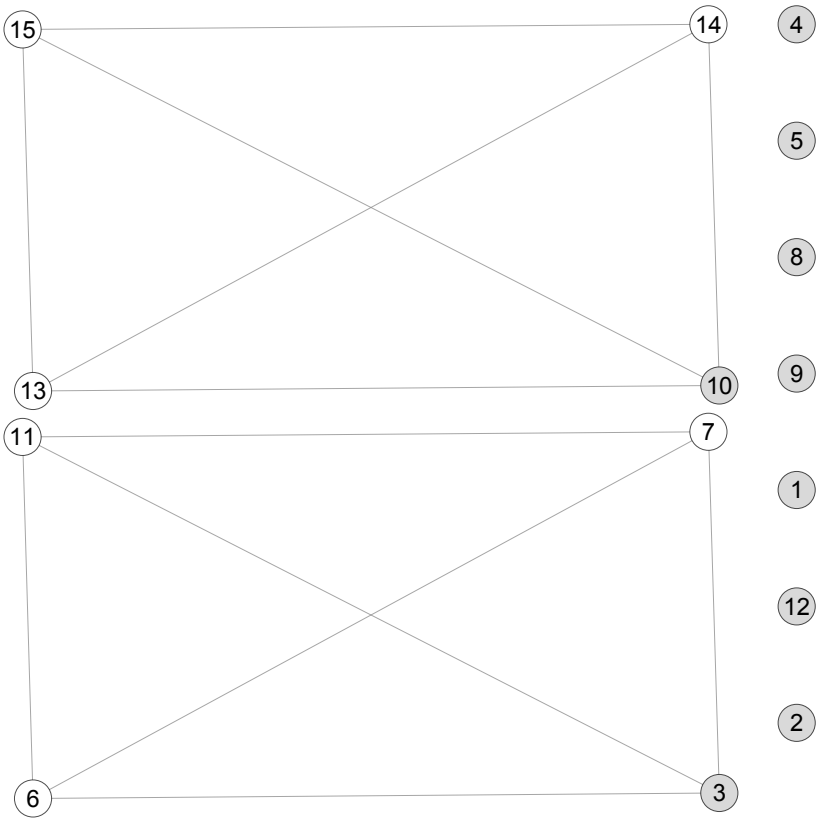
{{1, 2}}

Done.

There are 9 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\},$$
$$8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + \alpha_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 12 \to \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_4\}\}$$



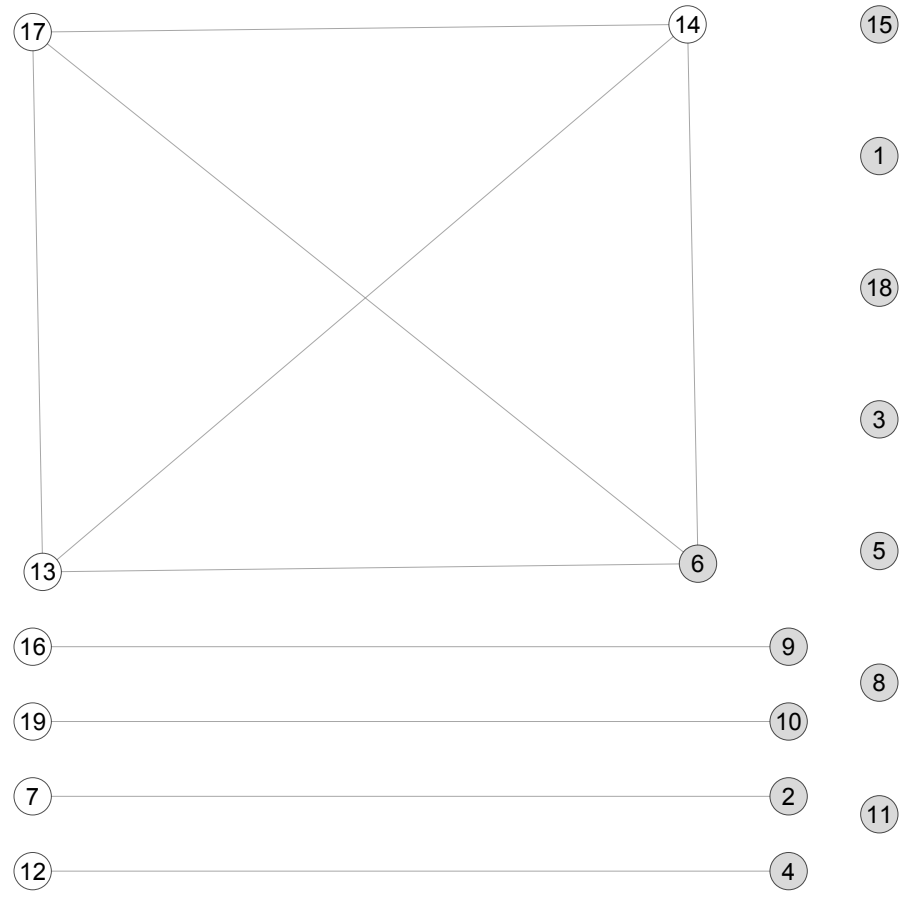
There are 19 2-D families of subalgebras to be analyzed.

Done.

There are 12 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\},$$
$$\{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\},$$
$$6 \to \{\Xi_3, \Xi_4\}, 8 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 10 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\},$$
$$11 \to \{\Xi_1 + \alpha_1 \Xi_2, \Xi_4\}, 15 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, 18 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$$



There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 5 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

Time of computation: 0.924666

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Algebra n. 8

$$\begin{pmatrix} 0 & 0 & \Xi_1 & 0 \\ 0 & 0 & a\,\Xi_2 & 0 \\ -\Xi_1 & -a\,\Xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

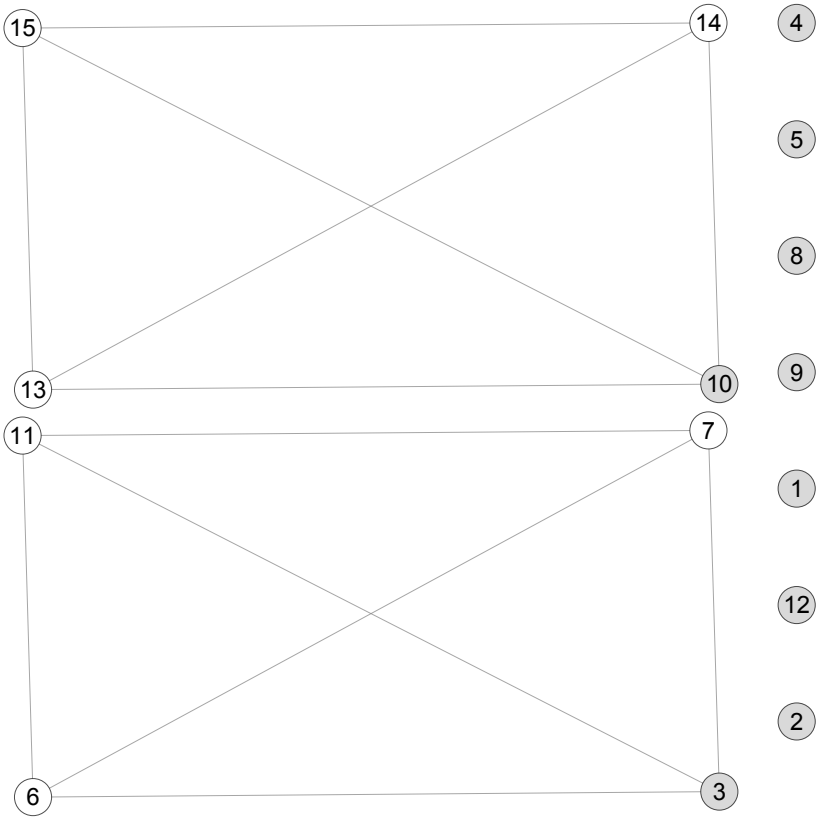
{{1, 2}}

Done.

There are 9 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + \alpha_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}, 12 \to \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_4\}\}$



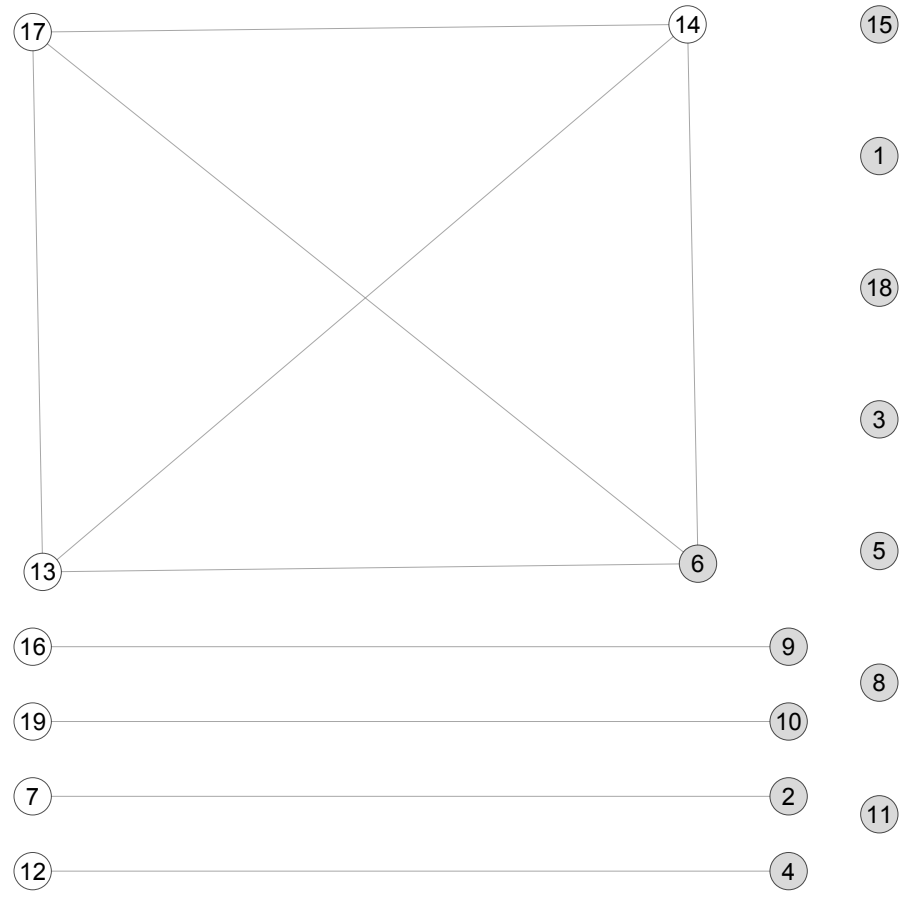
There are 19 2-D families of subalgebras to be analyzed.

Done.

There are 12 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\},$
$\{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\},$
$6 \to \{\Xi_3, \Xi_4\}, 8 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 10 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\},$
$11 \to \{\Xi_1 + \alpha_1 \Xi_2, \Xi_4\}, 15 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, 18 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2 + a_1 \Xi_4\}\}$



There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 5 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$

Time of computation: 0.930177

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Algebra n. 9

$$\begin{pmatrix} 0 & 0 & -\Xi_2 & 0 \\ 0 & 0 & \Xi_1 & 0 \\ \Xi_2 & -\Xi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

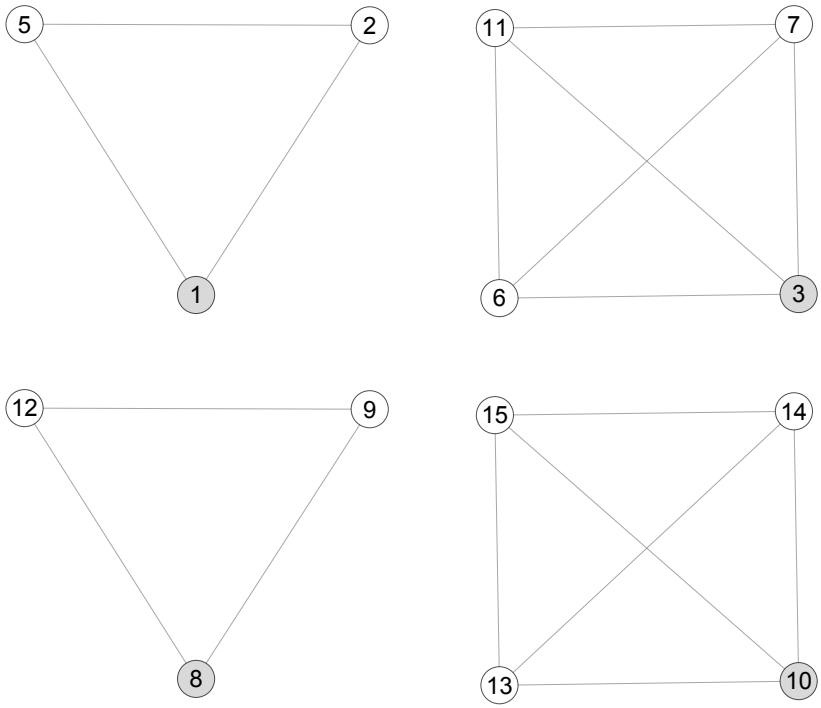
{{1, 2, 3}, {1, 3, 2}, {2, 3, 1}, {3, 1, 2}}

Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \dot{a}_1\, \Xi_4\}, \{\Xi_3 + \dot{a}_1\, \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 8 \to \{\Xi_1 + \dot{a}_1\, \Xi_4\}, 10 \to \{\Xi_3 + \dot{a}_1\, \Xi_4\}\}$



4

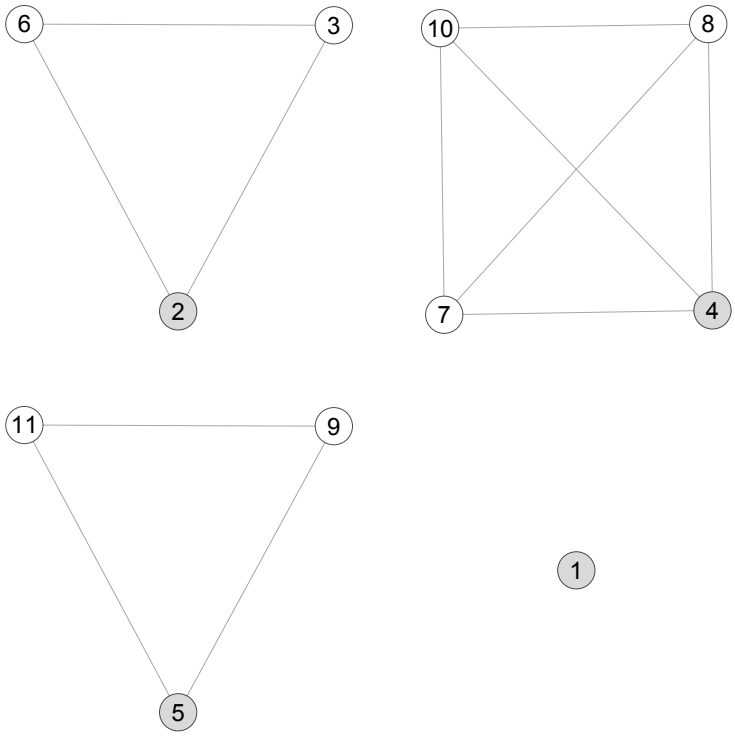
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \dot{a}_1\, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2 + \dot{a}_1\, \Xi_4\}\}$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \mathring{a}_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_2, \Xi_3 + \mathring{a}_1 \Xi_4\}\}$$

Time of computation: 2.732099

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Algebra n. 10

$$\begin{pmatrix} 0 & 0 & a\,\Xi_1 - \Xi_2 & 0 \\ 0 & 0 & \Xi_1 + a\,\Xi_2 & 0 \\ -a\,\Xi_1 + \Xi_2 & -\Xi_1 - a\,\Xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

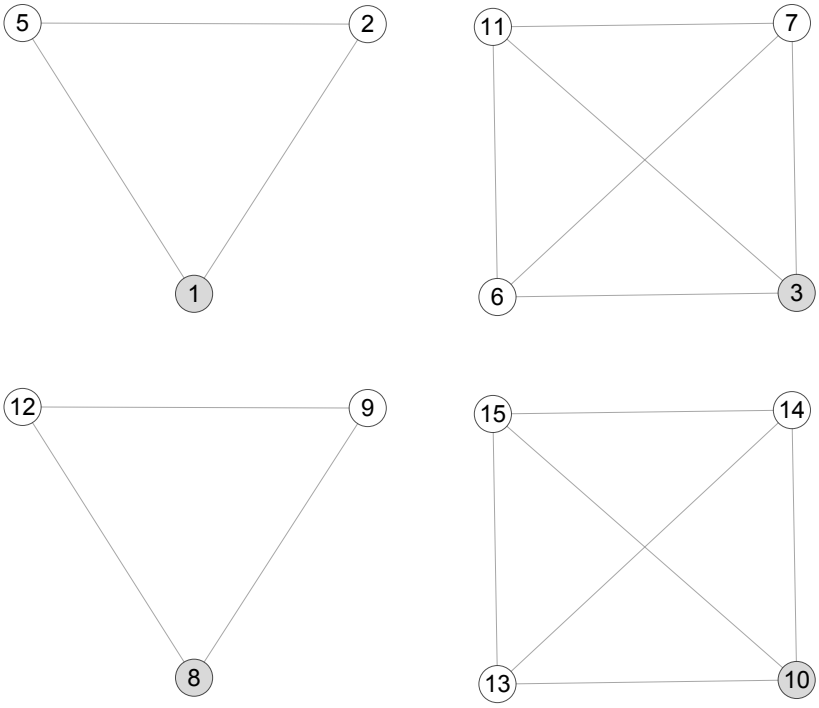
$$\{\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{3, 1, 2\}\}$$

Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 8 \to \{\Xi_1 + a_1 \Xi_4\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}\}$$



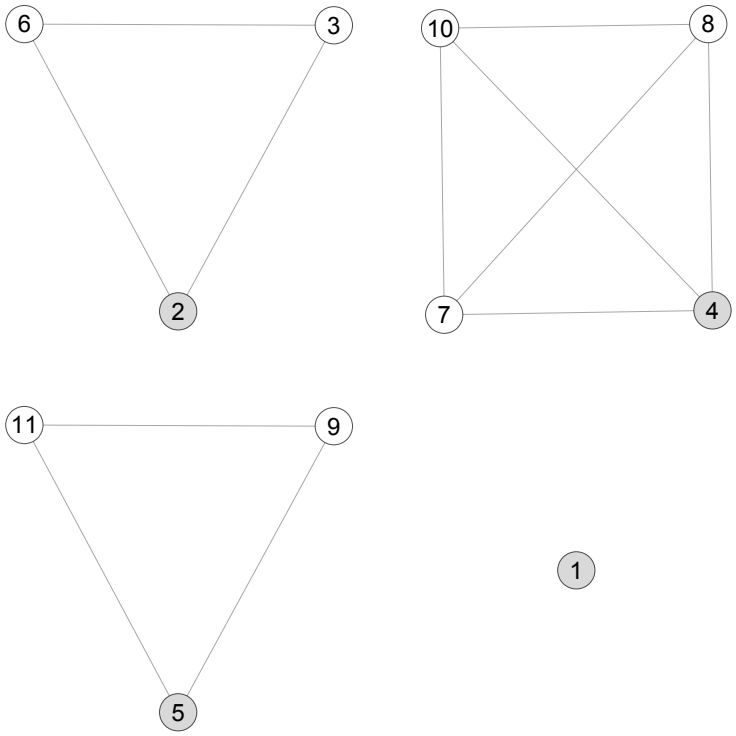
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4\}\}$$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \mathring{a}_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_2, \Xi_3 + \mathring{a}_1 \Xi_4\}\}$$

Time of computation: 18.757360

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Algebra n. 11

$$\begin{pmatrix} 0 & \Xi_1 & -2\,\Xi_2 & 0 \\ -\Xi_1 & 0 & \Xi_3 & 0 \\ 2\,\Xi_2 & -\Xi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

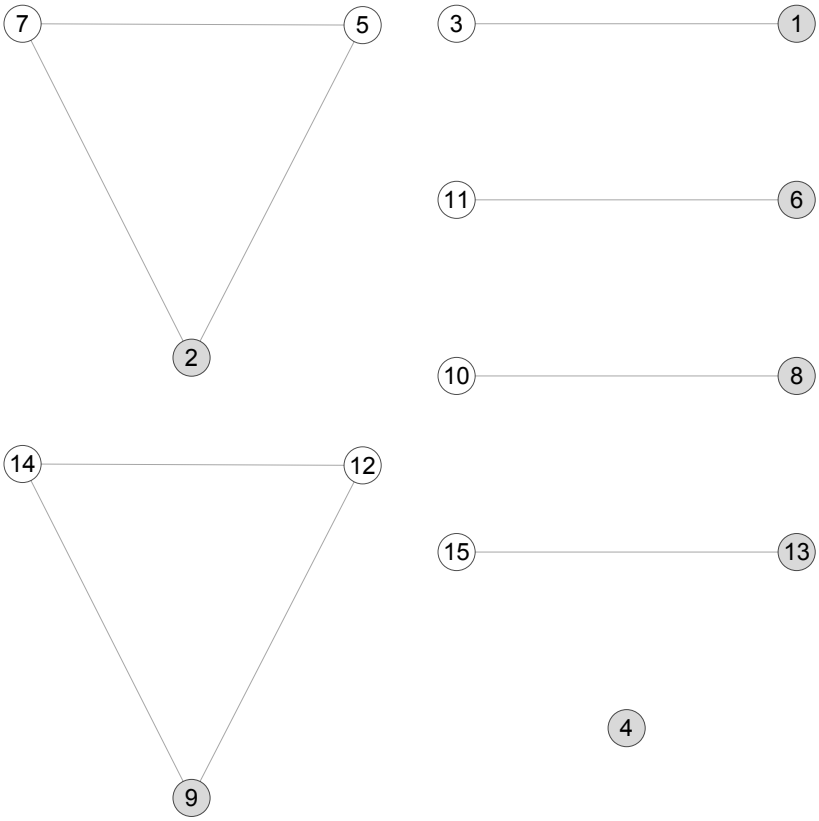
$$\{\{1, 2\}, \{2, 1\}\}$$

Done.

There are 7 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_2 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3 + a_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 4 \to \{\Xi_4\}, 6 \to \{\Xi_1 + \alpha_1 \Xi_3\},$$
$$8 \to \{\Xi_1 + \alpha_1 \Xi_4\}, 9 \to \{\Xi_2 + a_1 \Xi_4\}, 13 \to \{\Xi_1 + \alpha_1 \Xi_3 + a_1 \Xi_4\}\}$$



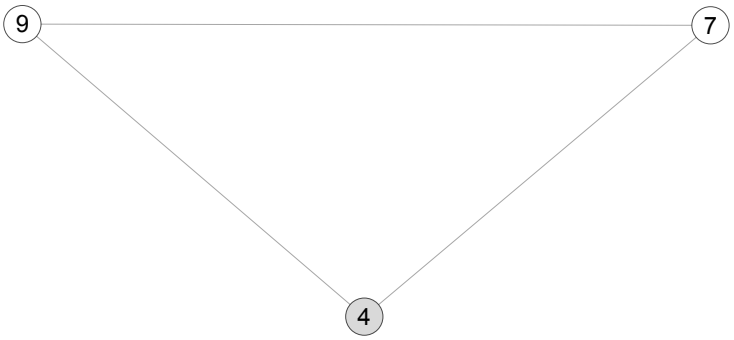
There are 11 2-D families of subalgebras to be analyzed.

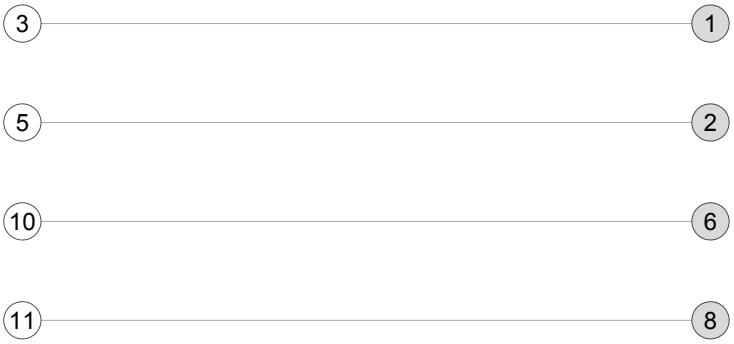
Done.

There are 5 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_4\}, 8 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_4\}\}$$





There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 2 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}\}$

Time of computation: 1.447425

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Algebra n. 12

$$\begin{pmatrix} 0 & \Xi_3 & -\Xi_2 & 0 \\ -\Xi_3 & 0 & \Xi_1 & 0 \\ \Xi_2 & -\Xi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

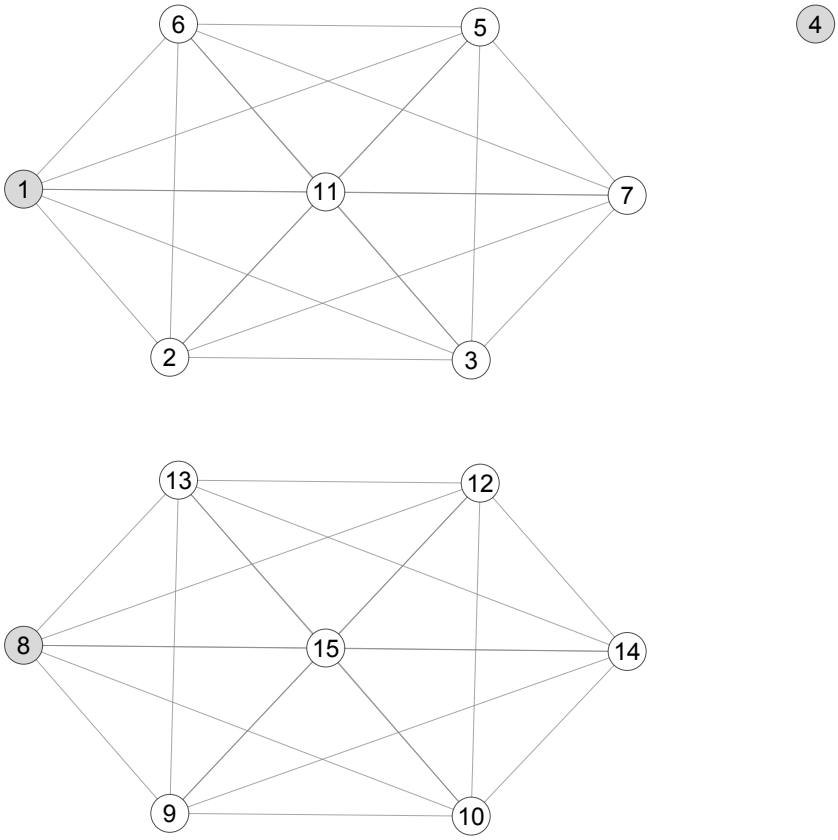
$\{\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}\}$

Done.

There are 3 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 4 \to \{\Xi_4\}, 8 \to \{\Xi_1 + a_1 \Xi_4\}\}$



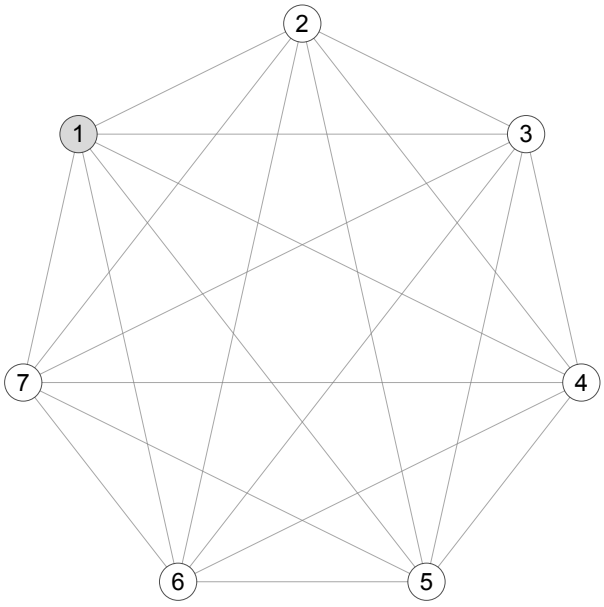
There are 7 2-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_4\}\}$



There are 1 3-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}\}$

Time of computation: 17.269050

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Algebra n. 13

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & \Xi_2 \\ 0 & -\Xi_1 & -\Xi_2 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

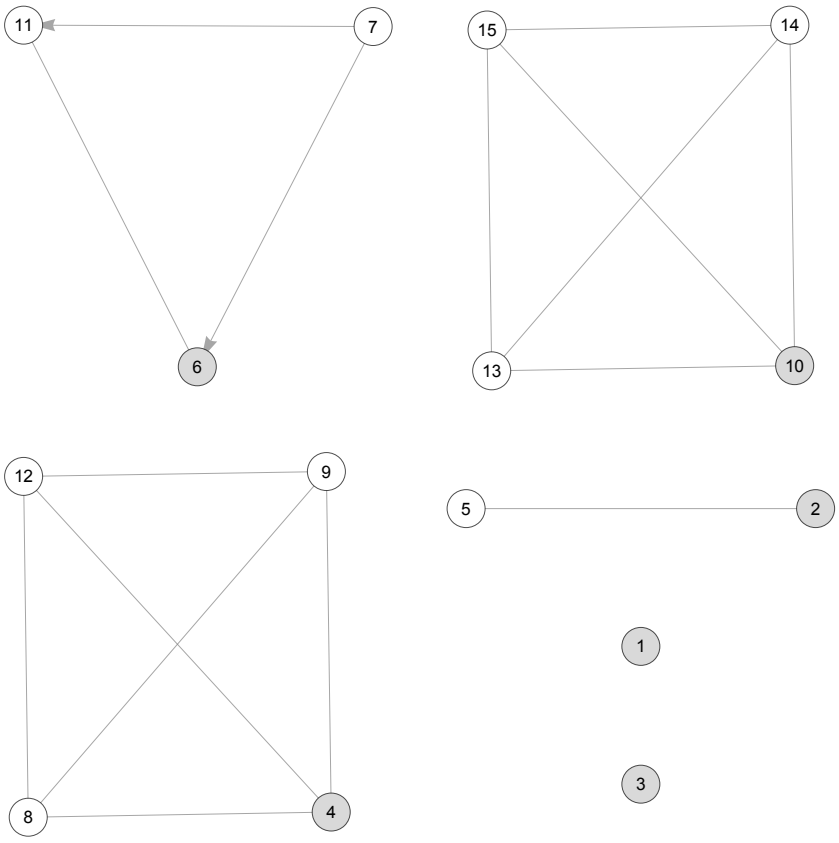
$\{\{1, 2, 3\}, \{1, 3, 2\}\}$

Done.

There are 6 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}\}$



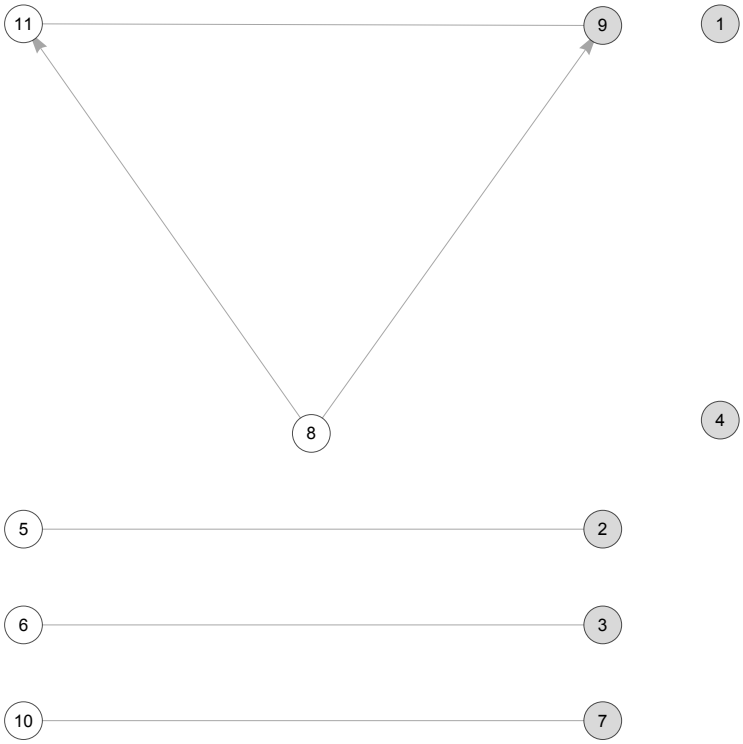
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 6 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_1, \Xi_3 + \dot{a}_1 \Xi_4\}, \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 7 \to \{\Xi_1, \Xi_3 + \dot{a}_1 \Xi_4\}, 9 \to \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2\}\}$$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$$

Time of computation: 1.113251

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Algebra n. 14

$$\begin{pmatrix} 0 & 0 & 0 & a\,\Xi_1 \\ 0 & 0 & 0 & \Xi_2 \\ 0 & 0 & 0 & \Xi_2 + \Xi_3 \\ -a\,\Xi_1 & -\Xi_2 & -\Xi_2 - \Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

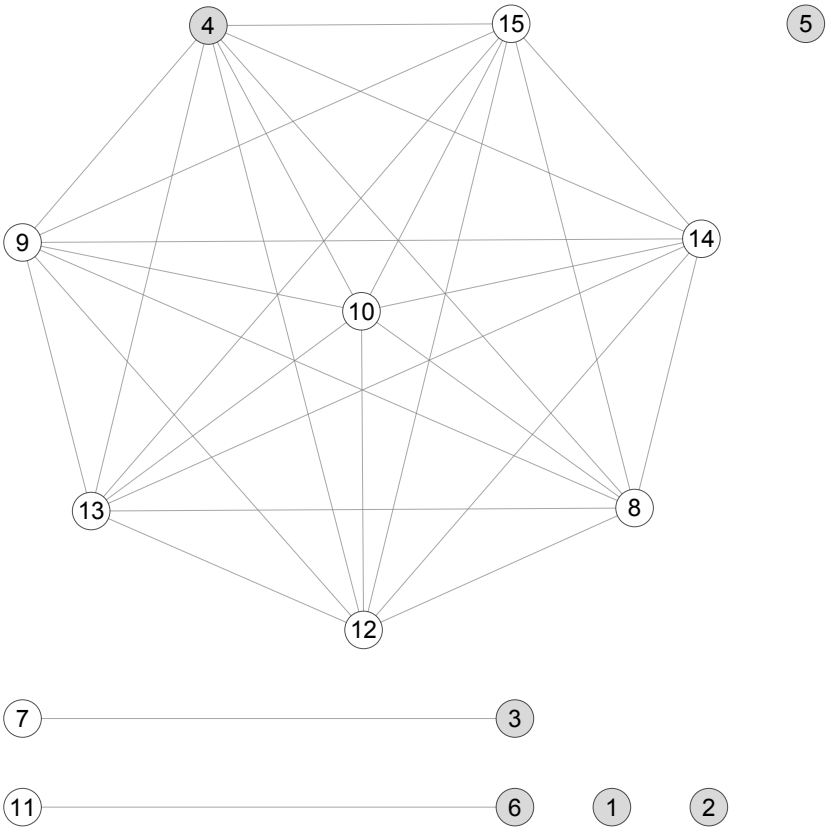
$$\{\{1, 2, 3, 4\}, \{1, 2, 4, 3\}, \{1, 3, 4, 2\}, \{1, 4, 2, 3\},$$
$$\{2, 3, 4, 1\}, \{2, 4, 1, 3\}, \{3, 4, 1, 2\}, \{4, 1, 2, 3\}\}$$

Done.

There are 6 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}\}$



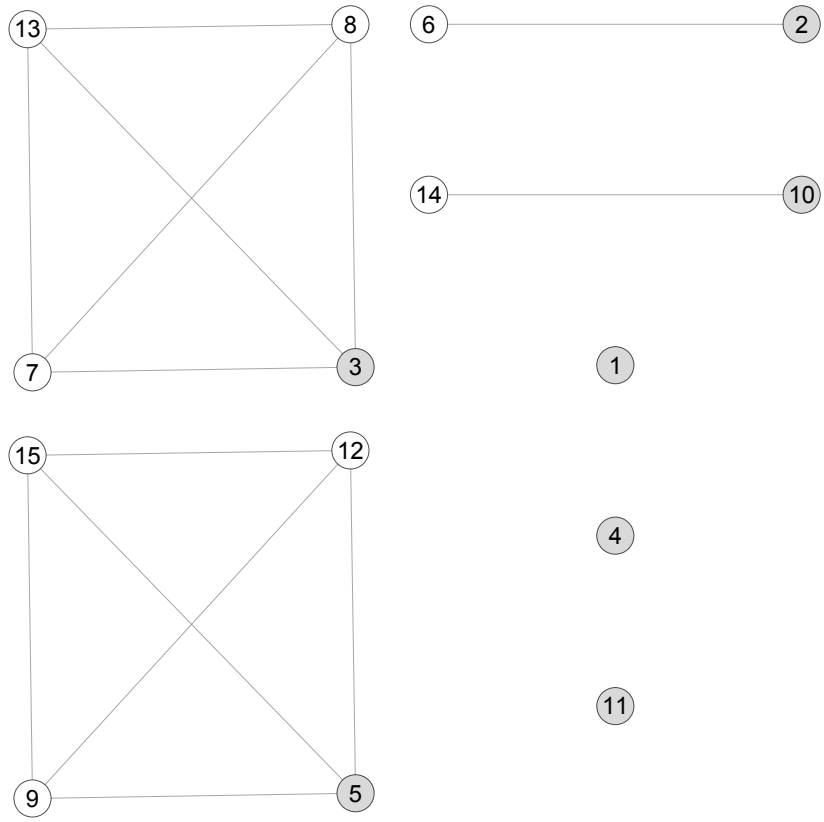
There are 15 2-D families of subalgebras to be analyzed.

Done.

There are 7 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\},$
$5 \to \{\Xi_2, \Xi_4\}, 10 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$



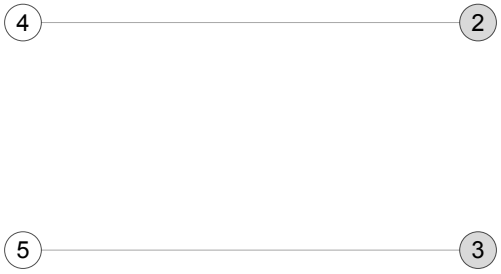
There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_2, \Xi_3, \Xi_4\}\}$



Time of computation: 9.147339

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Algebra n. 15

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & \Xi_2 \\ 0 & 0 & 0 & \Xi_2 + \Xi_3 \\ -\Xi_1 & -\Xi_2 & -\Xi_2 - \Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

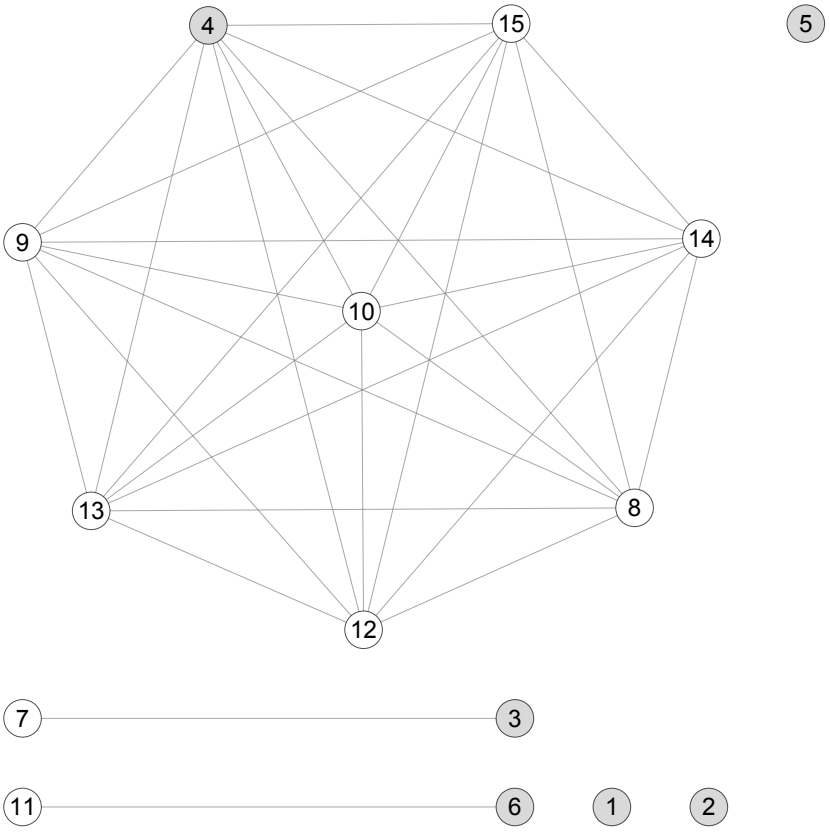
$$\{\{1, 2, 3, 4\}, \{1, 2, 4, 3\}, \{1, 3, 4, 2\}, \{1, 4, 2, 3\}, \{2, 3, 4, 1\}, \{2, 4, 1, 3\}, \{3, 4, 1, 2\}, \{4, 1, 2, 3\}\}$$

Done.

There are 6 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + a_1 \Xi_2\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}\}$$



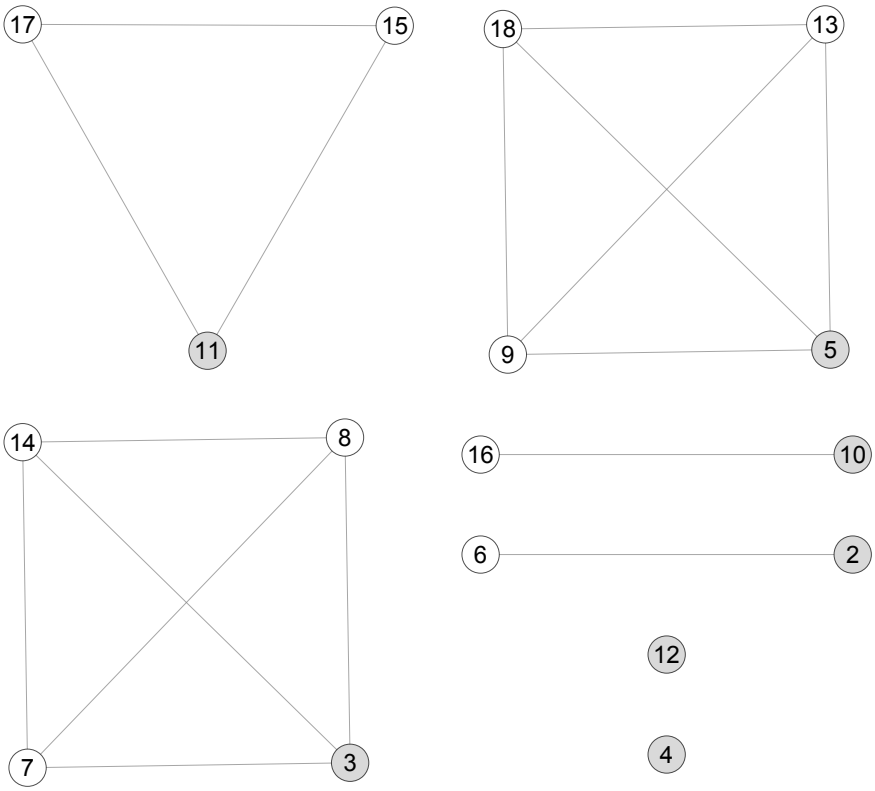
There are 18 2-D families of subalgebras to be analyzed.

Done.

There are 8 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\}, 10 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 11 \to \{\Xi_1 + a_1 \Xi_2, \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\}\}$$



There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_2, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}\}$

Time of computation: 41.039728

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Algebra n. 16

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_2 \\ -\Xi_1 & 0 & -\Xi_2 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

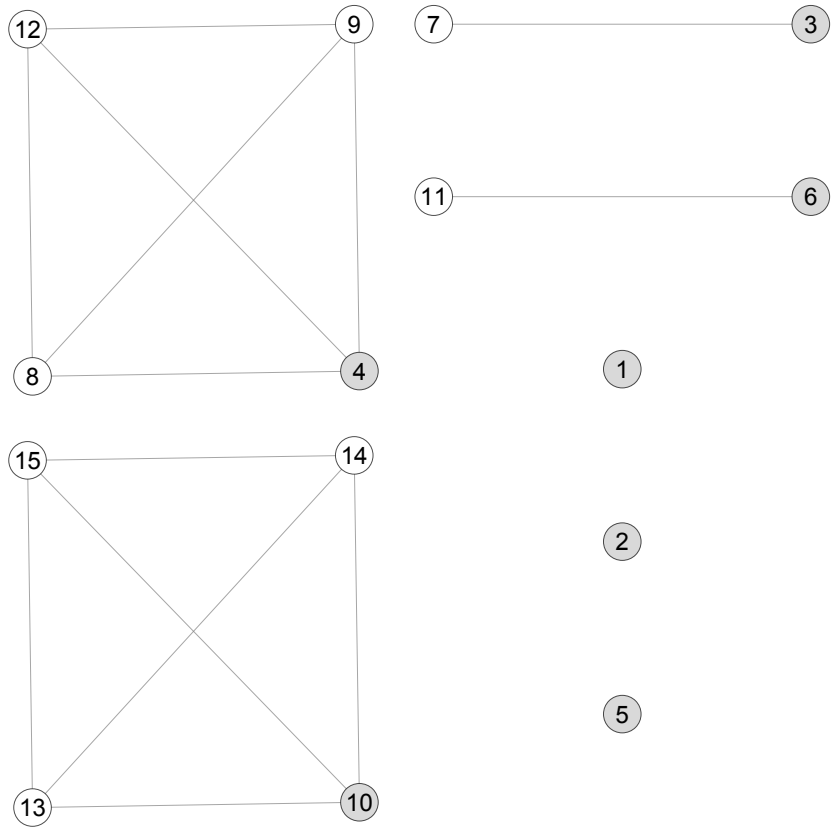
{{1, 2, 3}, {2, 3, 1}}

Done.

There are 7 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\}, 6 \to \{\Xi_1 + a_1 \Xi_3\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}\}$



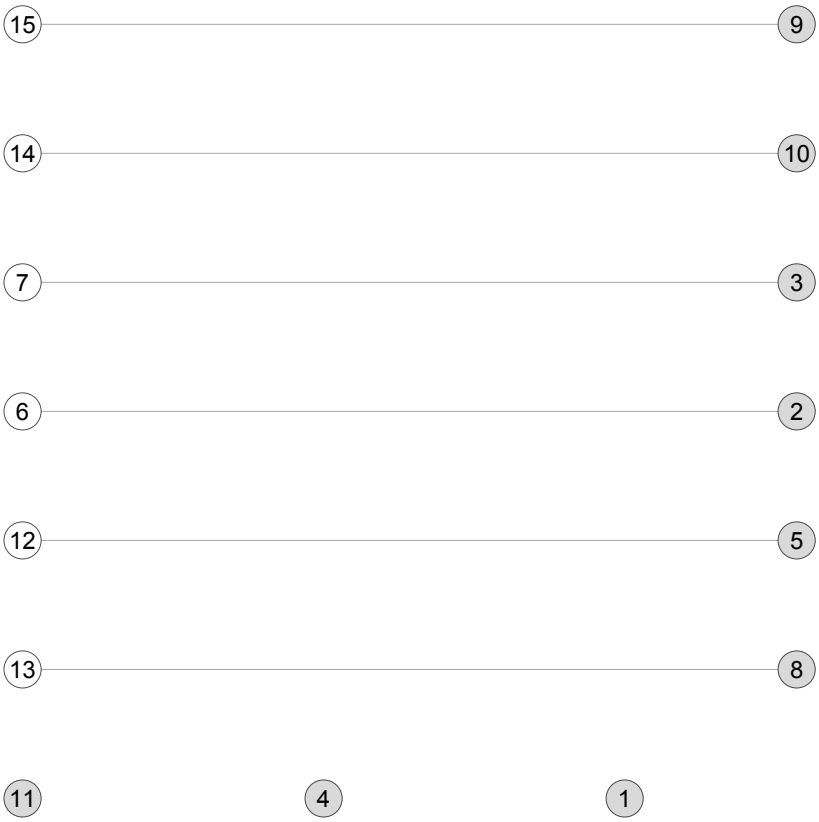
There are 15 2-D families of subalgebras to be analyzed.

Done.

There are 9 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\},$
$\{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\},$
$8 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, 9 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\}, 10 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$



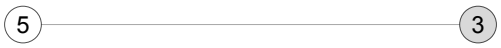
There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_2, \Xi_3, \Xi_4\}, 4 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$



Time of computation: 2.286267

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Algebra n. 17

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & \Xi_1 + \Xi_2 \\ 0 & 0 & 0 & \Xi_2 + \Xi_3 \\ -\Xi_1 & -\Xi_1 - \Xi_2 & -\Xi_2 - \Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

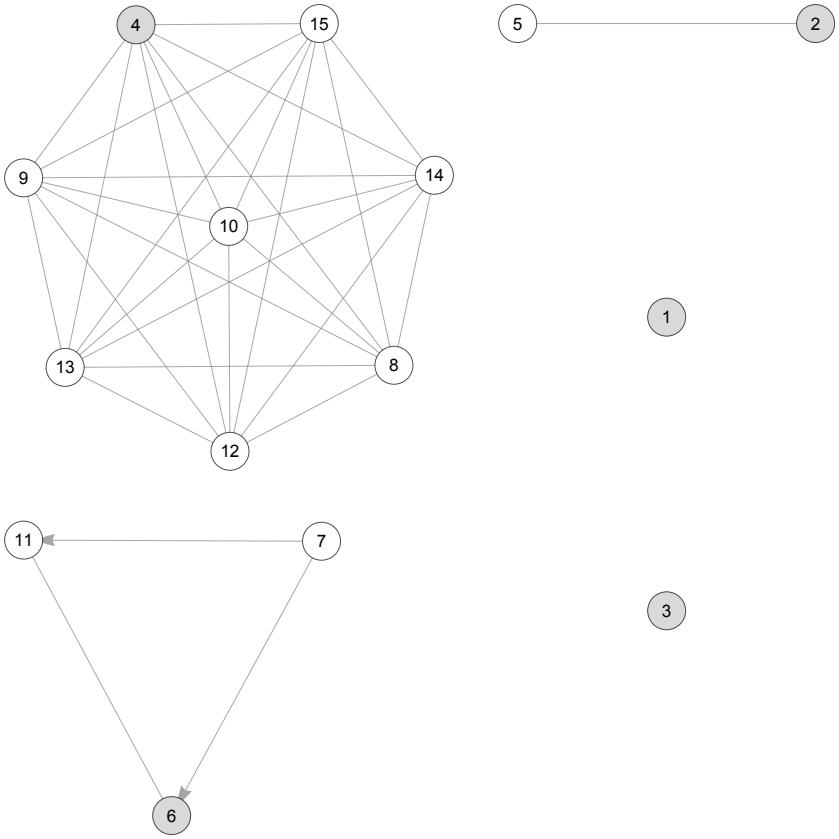
{{1, 2, 3, 4}, {1, 2, 4, 3}, {1, 3, 4, 2}, {1, 4, 2, 3},
 {2, 3, 4, 1}, {2, 4, 1, 3}, {3, 4, 1, 2}, {4, 1, 2, 3}}

Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 6 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3\}\}$



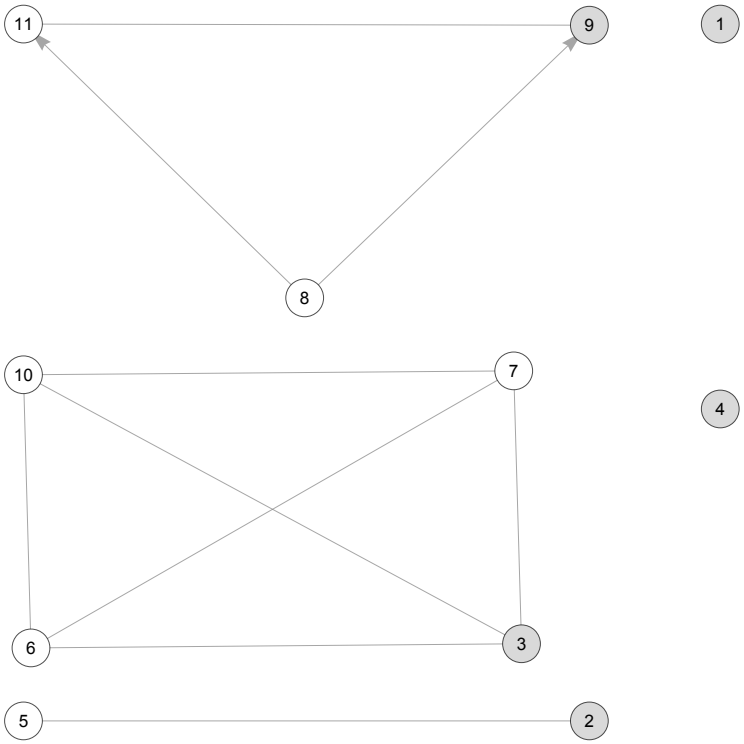
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 5 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 9 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}\}$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 2 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}\}$

Time of computation: 11.840944

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Algebra n. 18

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & a\,\Xi_2 \\ 0 & 0 & 0 & b\,\Xi_3 \\ -\Xi_1 & -a\,\Xi_2 & -b\,\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

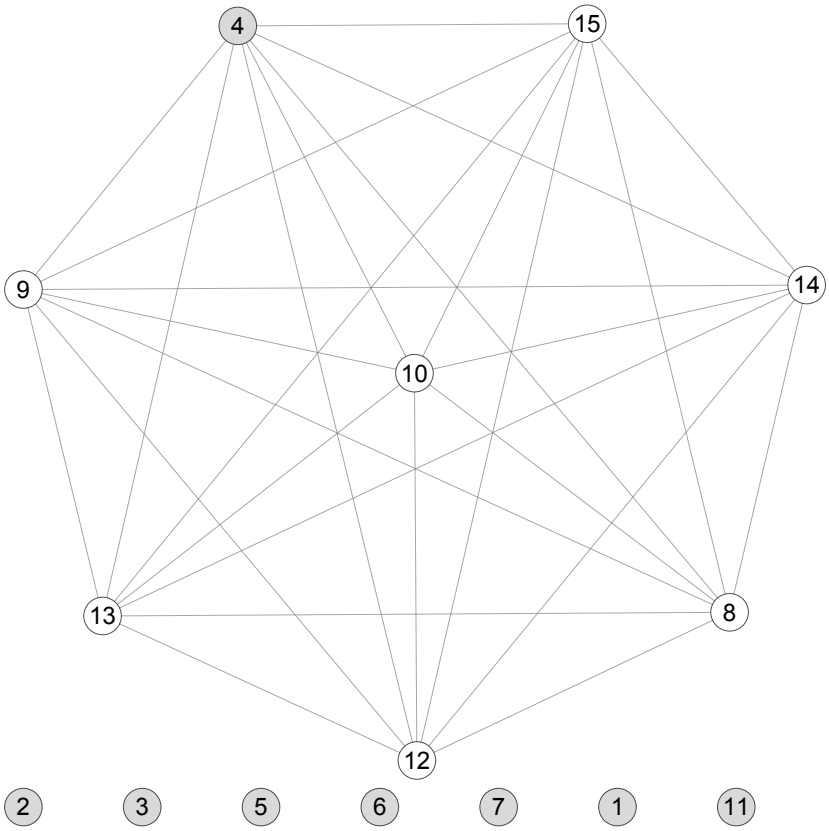
$\{\{1, 2, 3\}\}$

Done.

There are 8 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_3\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\},$$
$$6 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$$



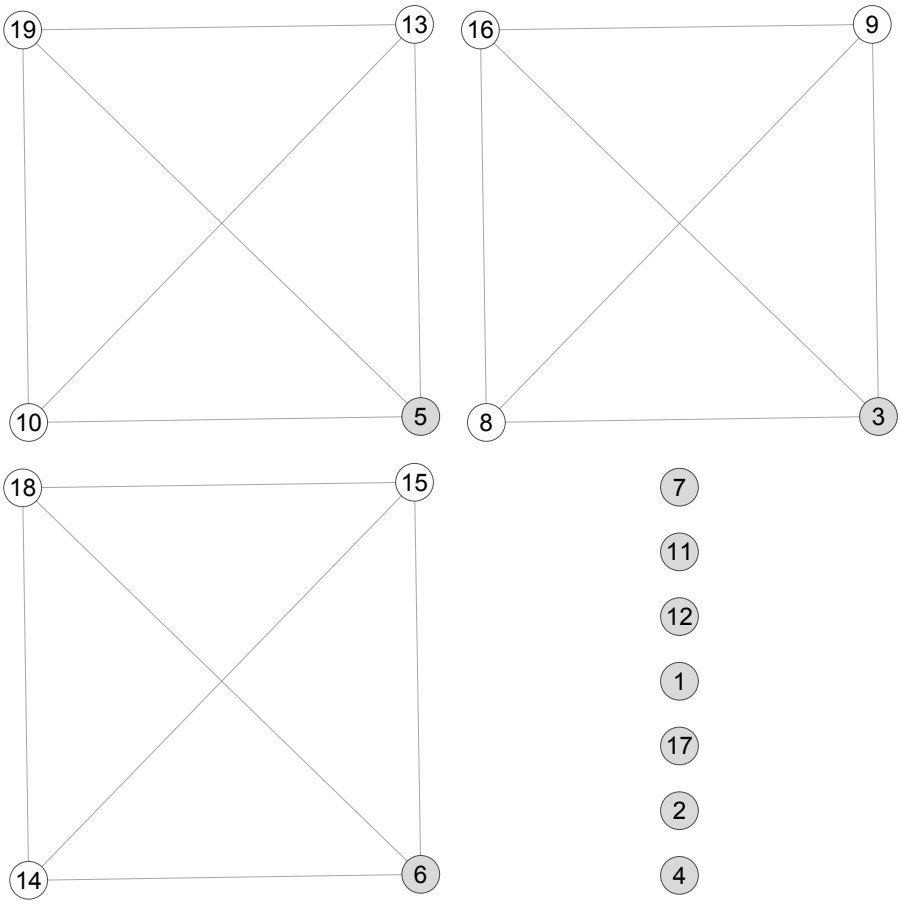
There are 19 2-D families of subalgebras to be analyzed.

Done.

There are 10 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + a_1 \Xi_3\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\},$$
$$7 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\}, 12 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, 17 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + a_1 \Xi_3\}\}$$



There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}\}$

Time of computation: 1.604672

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Algebra n. 19

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & a\,\Xi_2 \\ 0 & 0 & 0 & a\,\Xi_3 \\ -\Xi_1 & -a\,\Xi_2 & -a\,\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

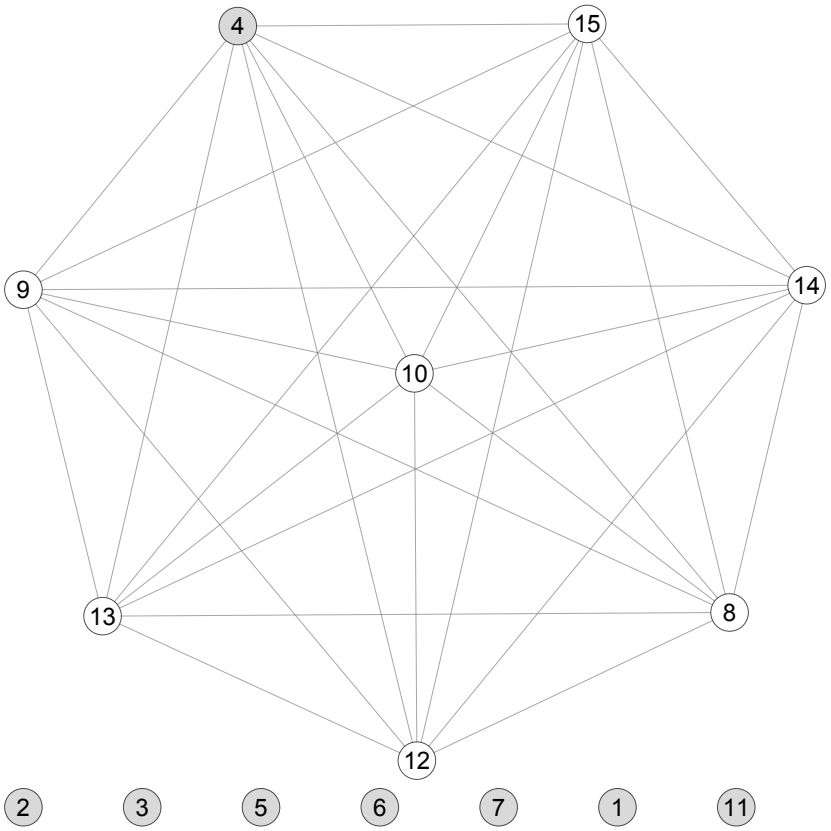
{{1, 2, 3}}

Done.

There are 8 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_3\}, \{\Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$6 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_2 + a_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$



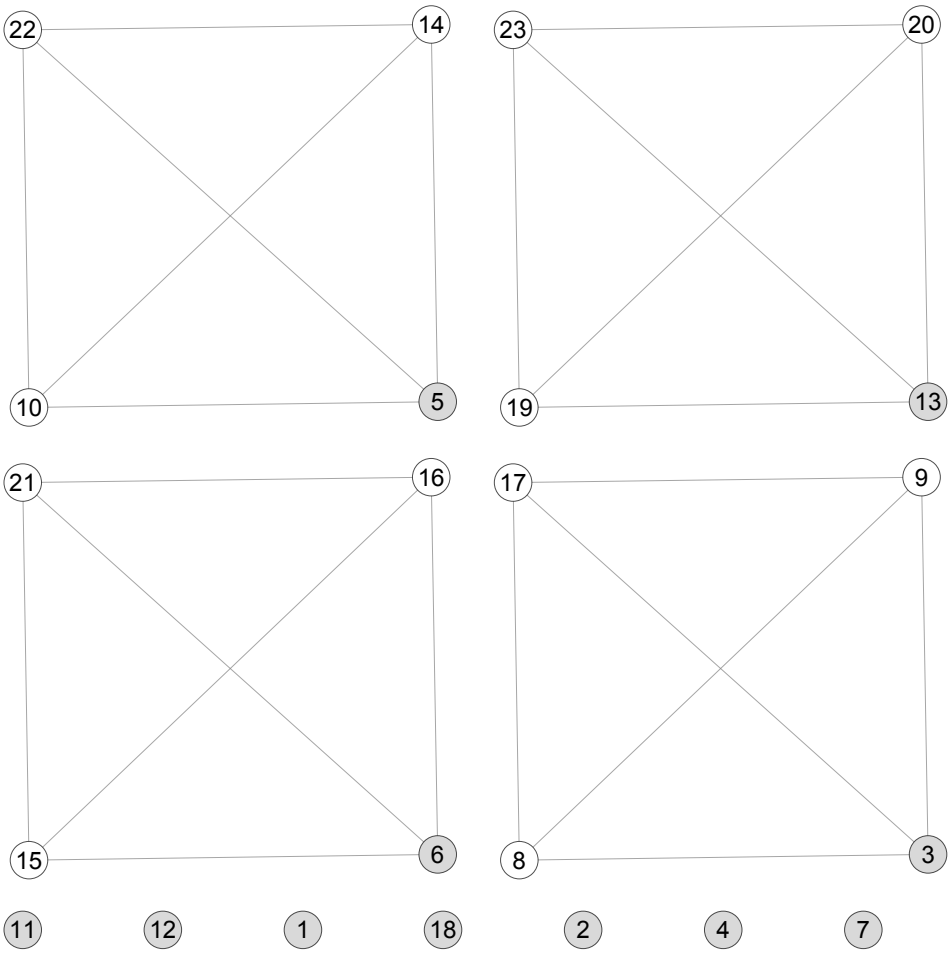
There are 23 2-D families of subalgebras to be analyzed.

Done.

There are 11 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3\},$
$\{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + a_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\},$
$5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\},$
$12 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, 13 \to \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, 18 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + a_1 \Xi_3\}\}$



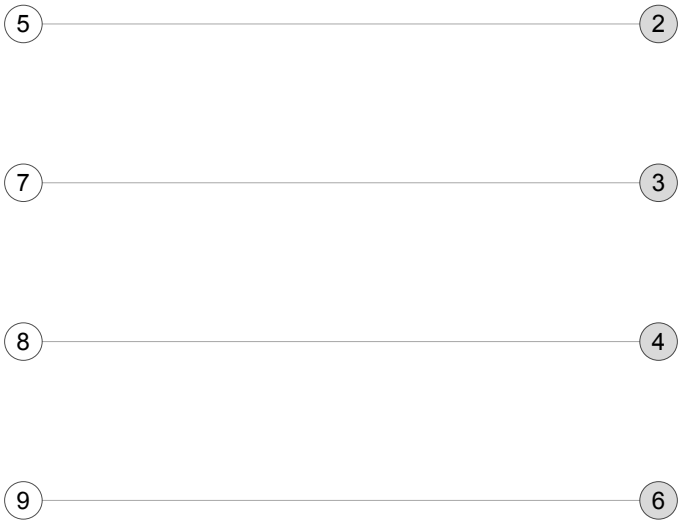
There are 9 3-D families of subalgebras to be analyzed.

Done.

There are 5 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}\}$



1

Time of computation: 2.298426

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Algebra n. 20

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & a\,\Xi_2 \\ 0 & 0 & 0 & \Xi_3 \\ -\Xi_1 & -a\,\Xi_2 & -\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

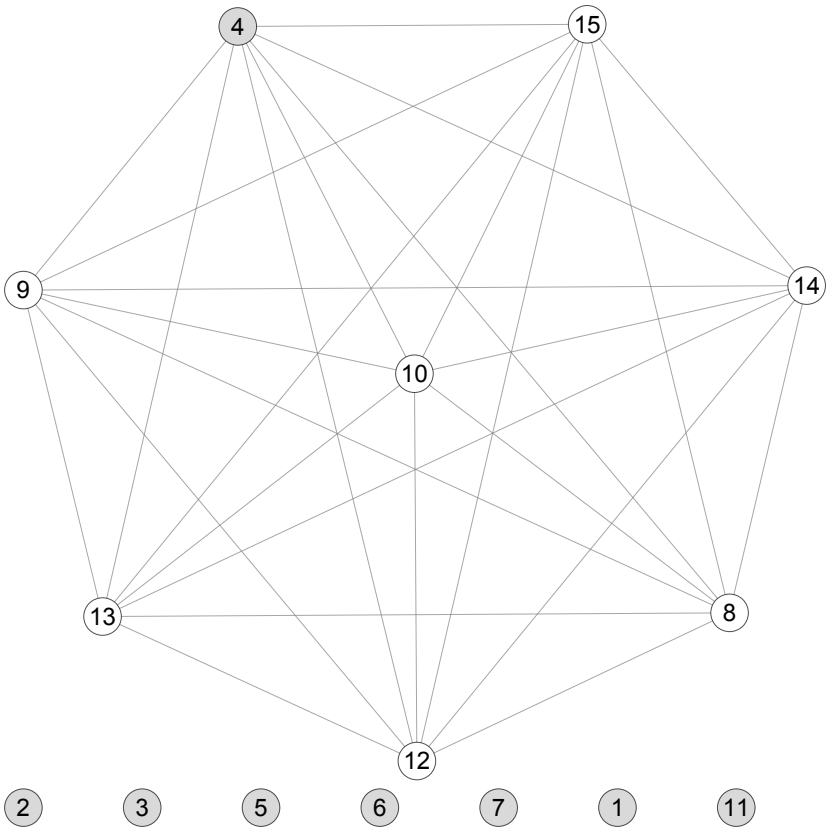
{{1, 2, 3}}

Done.

There are 8 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$6 \to \{\Xi_1 + a_1 \Xi_3\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_3\}\}$



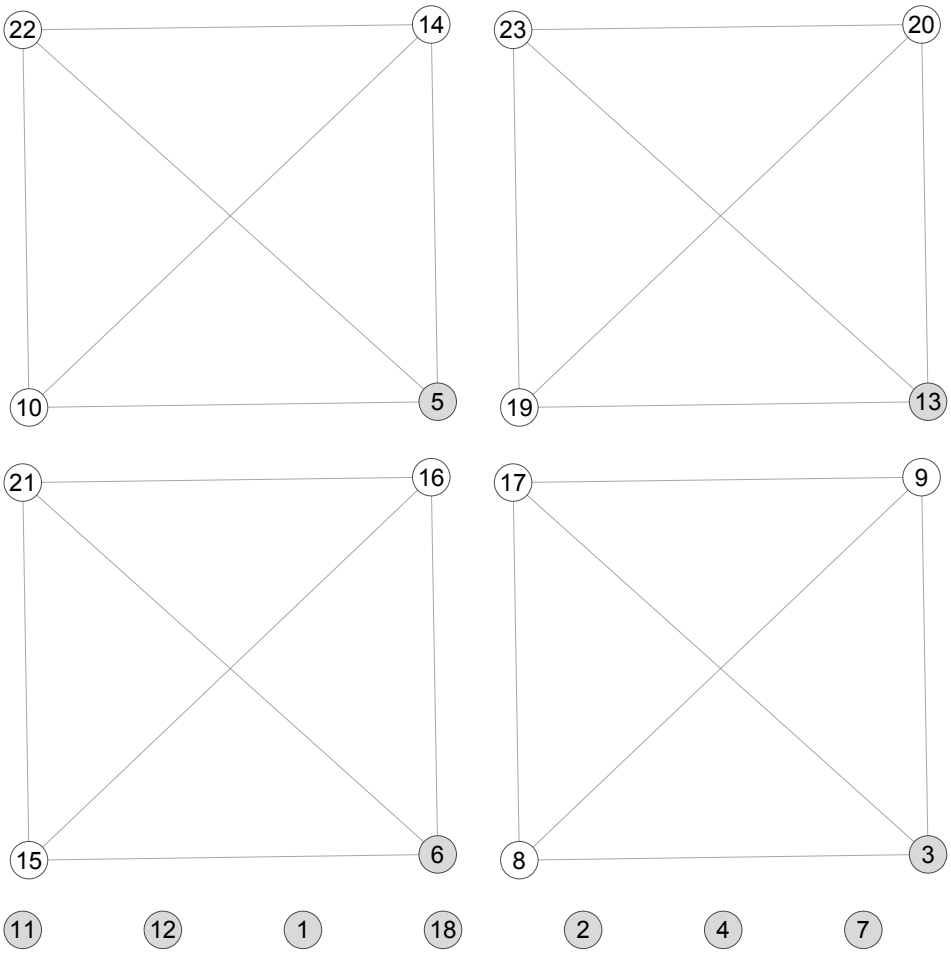
There are 23 2-D families of subalgebras to be analyzed.

Done.

There are 11 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\},$$
$$\{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + \alpha_1 \Xi_3\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\},$$
$$5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, 11 \to \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\},$$
$$12 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, 13 \to \{\Xi_1 + a_1 \Xi_3, \Xi_4\}, 18 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + \alpha_1 \Xi_3\}\}$$



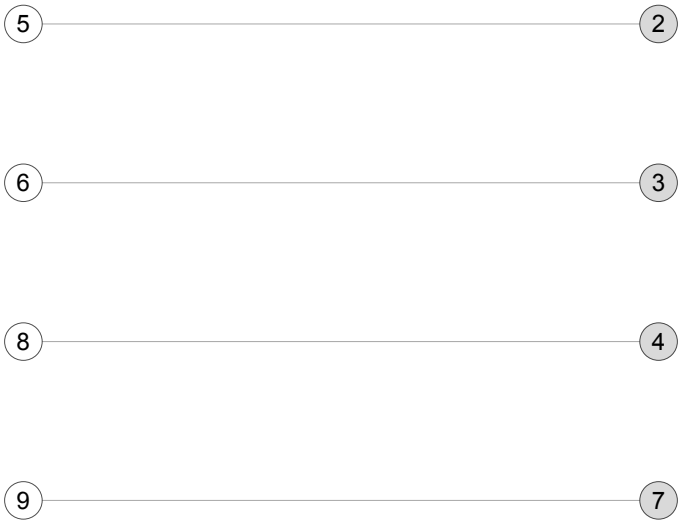
There are 9 3-D families of subalgebras to be analyzed.

Done.

There are 5 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 7 \to \{\Xi_1 + \dot{a}_1 \Xi_3, \Xi_2, \Xi_4\}\}$



1

Time of computation: 1.926168

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Algebra n. 21

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & \Xi_2 \\ 0 & 0 & 0 & \Xi_3 \\ -\Xi_1 & -\Xi_2 & -\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

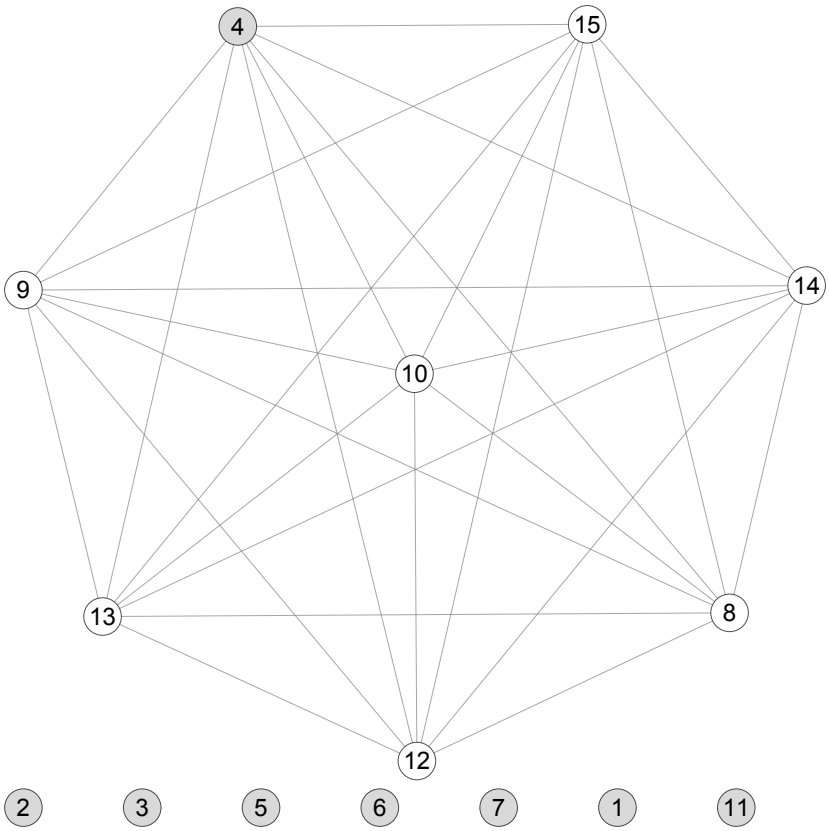
$\{\{1, 2, 3\}\}$

Done.

There are 8 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1 \Xi_2\}, \{\Xi_1 + a_1 \Xi_3\}, \{\Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + a_1 \Xi_2\},$
$6 \to \{\Xi_1 + a_1 \Xi_3\}, 7 \to \{\Xi_2 + a_1 \Xi_3\}, 11 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3\}\}$



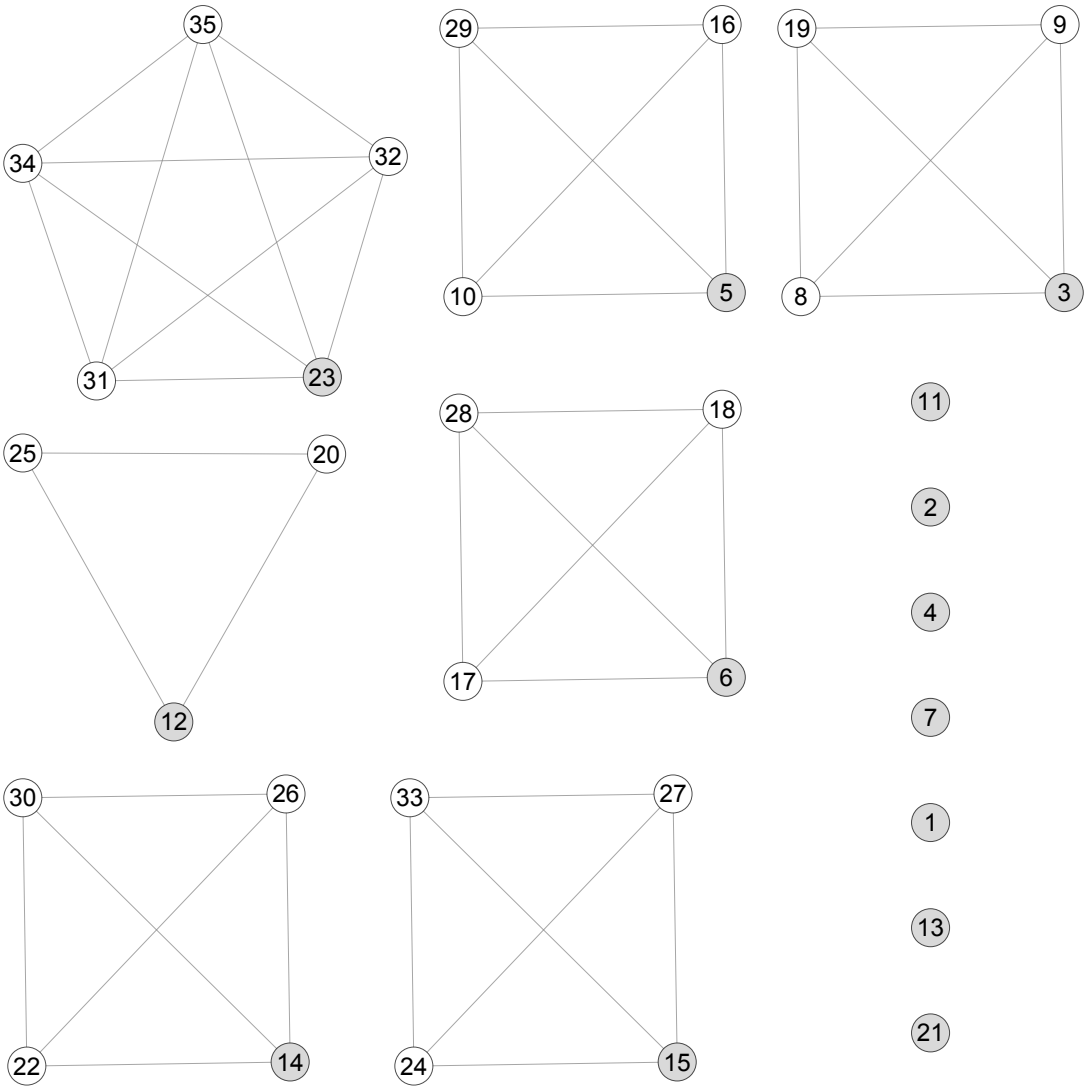
There are 35 2-D families of subalgebras to be analyzed.

Done.

There are 14 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\},$$
$$\{\Xi_1, \Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2\},$$
$$\{\Xi_1 + a_1 \Xi_3, \Xi_4\}, \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\}, \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3, \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\},$$
$$5 \to \{\Xi_2, \Xi_4\}, 6 \to \{\Xi_3, \Xi_4\}, 7 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\}, 11 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\},$$
$$12 \to \{\Xi_1 + a_1 \Xi_2, \Xi_4\}, 13 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, 14 \to \{\Xi_1 + a_1 \Xi_3, \Xi_4\},$$
$$15 \to \{\Xi_2 + a_1 \Xi_3, \Xi_4\}, 21 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\}, 23 \to \{\Xi_1 + a_1 \Xi_2 + a_2 \Xi_3, \Xi_4\}\}$$



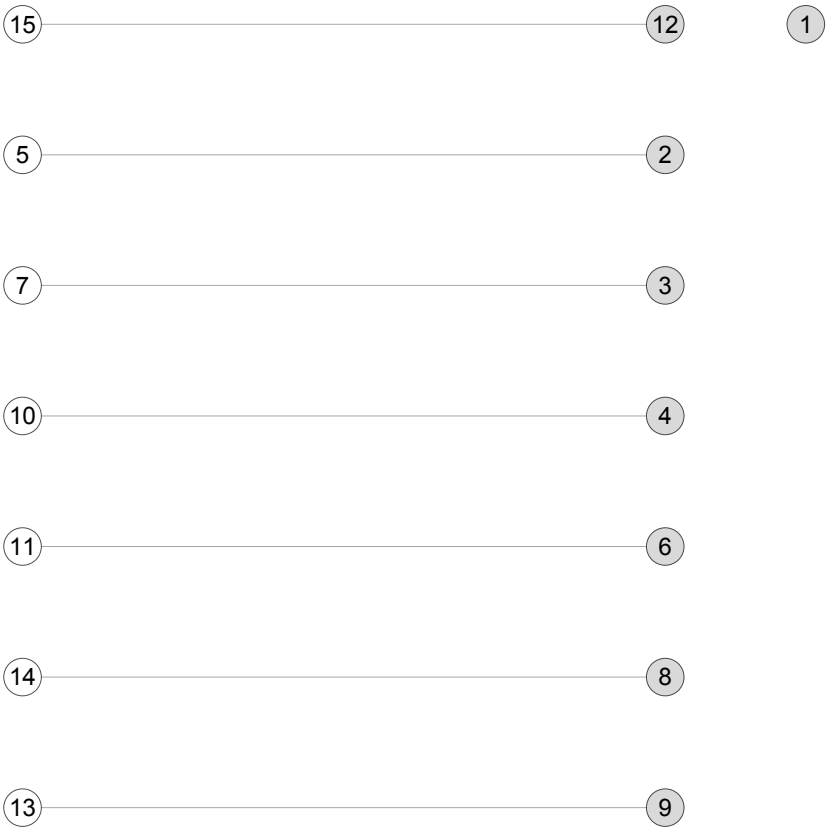
There are 15 3-D families of subalgebras to be analyzed.

Done.

There are 8 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\},$
$\{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\},$
$8 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3, \Xi_4\}, 9 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2, \Xi_4\}, 12 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3, \Xi_4\}\}$



Time of computation: 6.223046

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Algebra n. 22

$$\begin{pmatrix} 0 & 0 & 0 & a\,\Xi_1 \\ 0 & 0 & 0 & b\,\Xi_2 - \Xi_3 \\ 0 & 0 & 0 & \Xi_2 + b\,\Xi_3 \\ -a\,\Xi_1 & -b\,\Xi_2 + \Xi_3 & -\Xi_2 - b\,\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

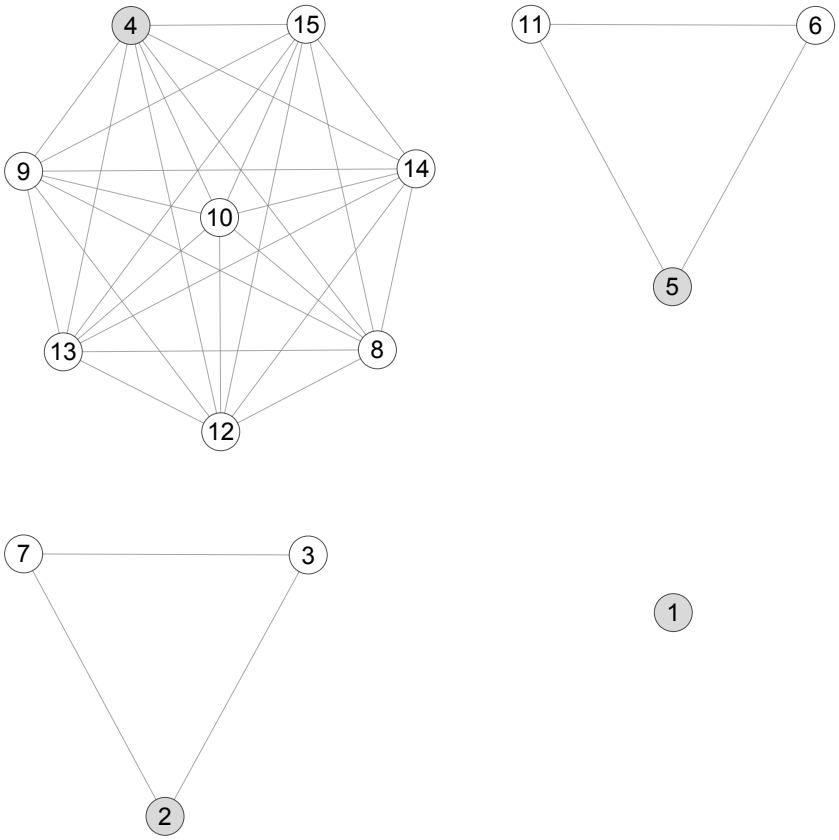
{{1, 2, 3, 4}, {1, 2, 4, 3}, {1, 3, 4, 2}, {1, 4, 2, 3},
 {2, 3, 4, 1}, {2, 4, 1, 3}, {3, 4, 1, 2}, {4, 1, 2, 3}}

Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1 + \dot{a}_1\,\Xi_2\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 4 \to \{\Xi_4\}, 5 \to \{\Xi_1 + \dot{a}_1\,\Xi_2\}\}$



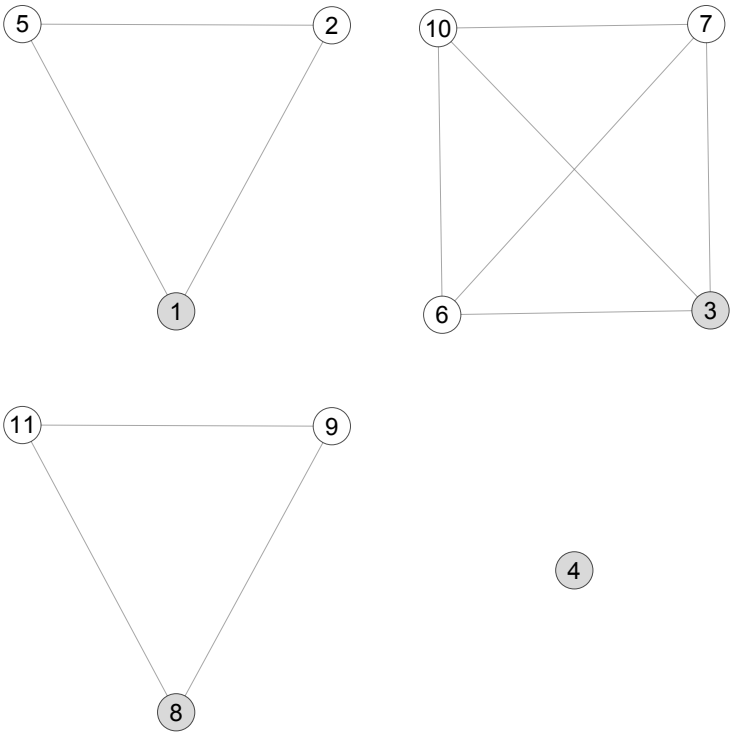
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_3\}, 8 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}\}$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 2 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_2, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_2, \Xi_3, \Xi_4\}\}$

Time of computation: 43.781596

==========

Algebra n. 23

$$\begin{pmatrix} 0 & 0 & 0 & 2\,\Xi_1 \\ 0 & 0 & \Xi_1 & \Xi_2 \\ 0 & -\Xi_1 & 0 & \Xi_2 + \Xi_3 \\ -2\,\Xi_1 & -\Xi_2 & -\Xi_2 - \Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

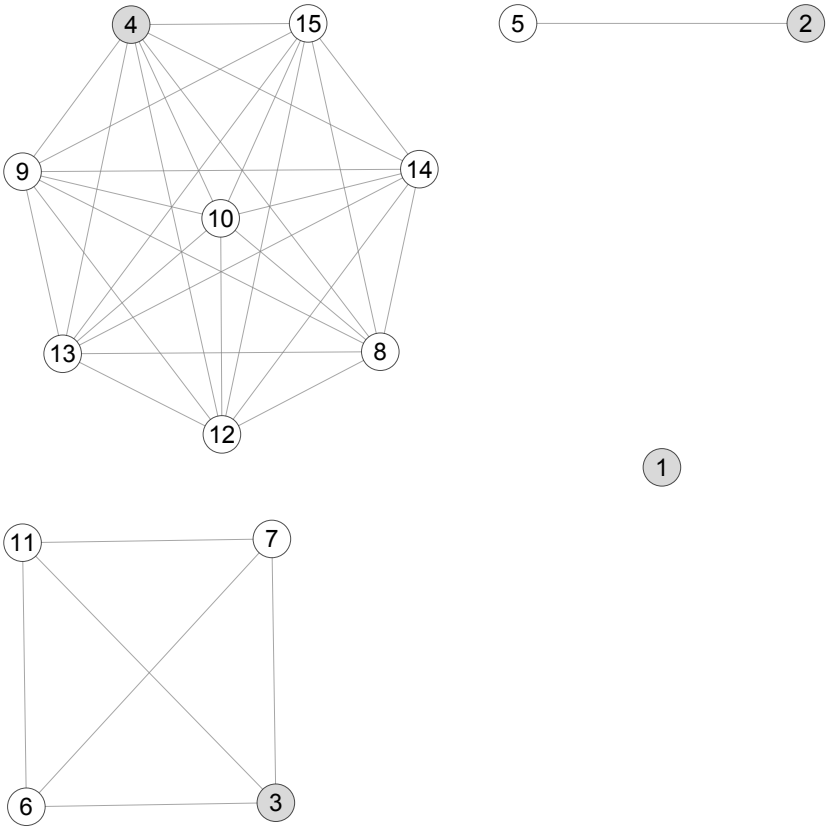
$\{\{1, 2, 3, 4\}, \{1, 2, 4, 3\}, \{1, 3, 2, 4\}, \{1, 3, 4, 2\}, \{1, 4, 2, 3\}, \{1, 4, 3, 2\},$
$\{2, 3, 4, 1\}, \{2, 4, 1, 3\}, \{3, 2, 4, 1\}, \{3, 4, 1, 2\}, \{4, 1, 2, 3\}, \{4, 1, 3, 2\}\}$

Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}\}$



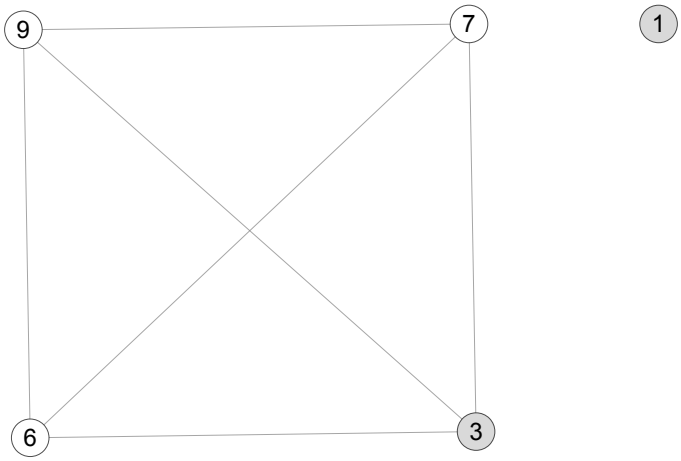
There are 9 2-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\}\}$



There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 2 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}\}$

Time of computation: 25.010848

==========

Algebra n. 24

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Xi_1 & \Xi_2 \\ 0 & -\Xi_1 & 0 & -\Xi_3 \\ 0 & -\Xi_2 & \Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

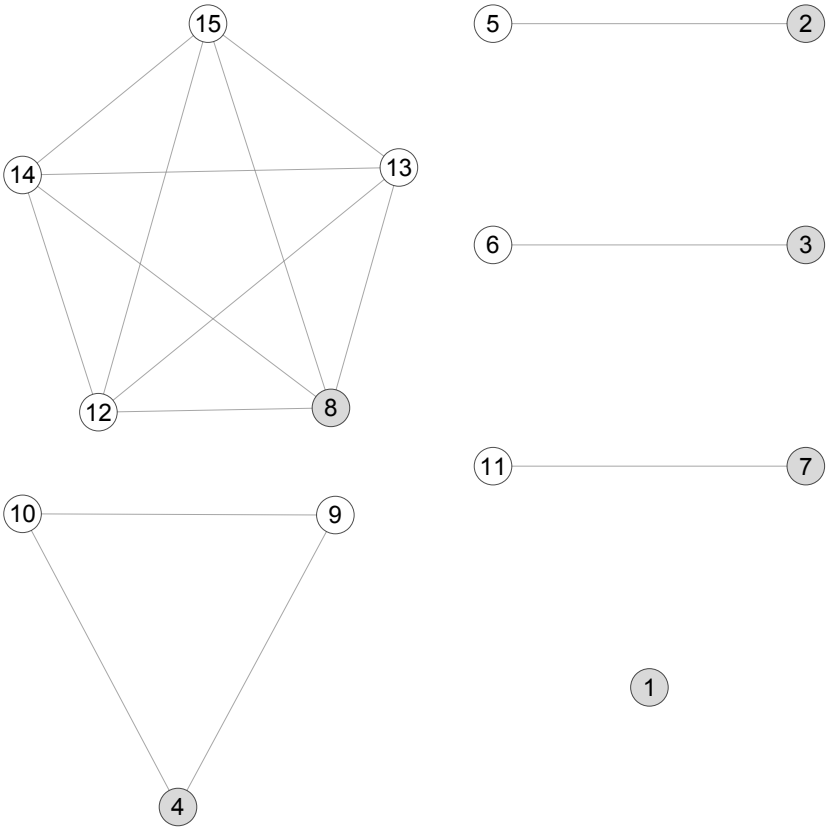
$\{\{1, 2\}\}$

Done.

There are 6 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + \underset{\cdot}{\alpha}_1 \Xi_3\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_4\}\}$

$$\{1 \to \{\Xi_1\},\ 2 \to \{\Xi_2\},\ 3 \to \{\Xi_3\},\ 4 \to \{\Xi_4\},\ 7 \to \{\Xi_2 + \alpha_{\circ 1}\, \Xi_3\},\ 8 \to \{\Xi_1 + a_{\circ 1}\, \Xi_4\}\}$$



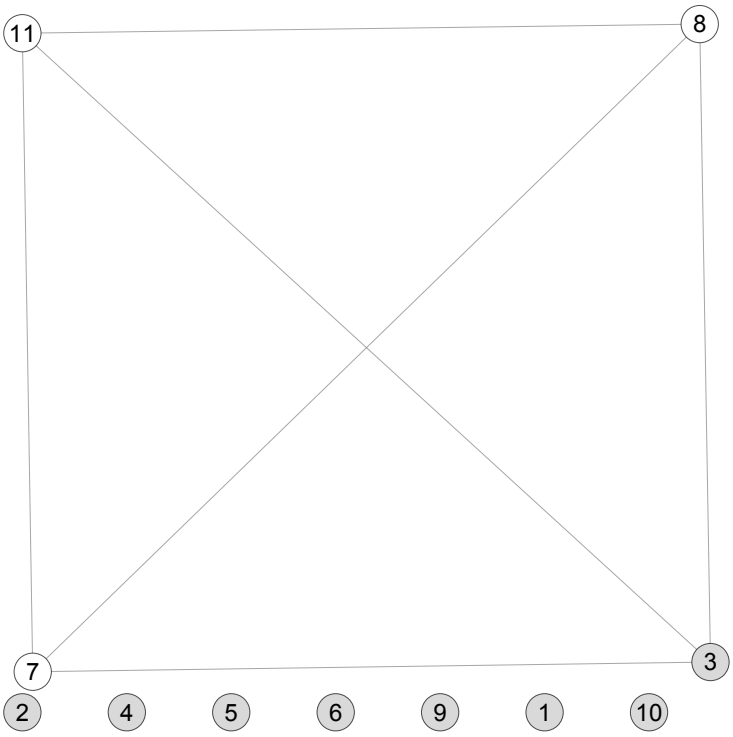
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 8 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\},\ \{\Xi_1, \Xi_3\},\ \{\Xi_1, \Xi_4\},\ \{\Xi_2, \Xi_4\},$$
$$\{\Xi_3, \Xi_4\},\ \{\Xi_1, \Xi_2 + \alpha_{\circ 1}\, \Xi_3\},\ \{\Xi_1 + a_{\circ 1}\, \Xi_4, \Xi_2\},\ \{\Xi_1 + a_{\circ 1}\, \Xi_4, \Xi_3\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\},\ 2 \to \{\Xi_1, \Xi_3\},\ 3 \to \{\Xi_1, \Xi_4\},\ 4 \to \{\Xi_2, \Xi_4\},$$
$$5 \to \{\Xi_3, \Xi_4\},\ 6 \to \{\Xi_1, \Xi_2 + \alpha_{\circ 1}\, \Xi_3\},\ 9 \to \{\Xi_1 + a_{\circ 1}\, \Xi_4, \Xi_2\},\ 10 \to \{\Xi_1 + a_{\circ 1}\, \Xi_4, \Xi_3\}\}$$



There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}\}$

Time of computation: 0.884514

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Algebra n. 25

$$\begin{pmatrix} 0 & 0 & 0 & (1+b)\,\Xi_1 \\ 0 & 0 & \Xi_1 & \Xi_2 \\ 0 & -\Xi_1 & 0 & b\,\Xi_3 \\ (-1-b)\,\Xi_1 & -\Xi_2 & -b\,\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

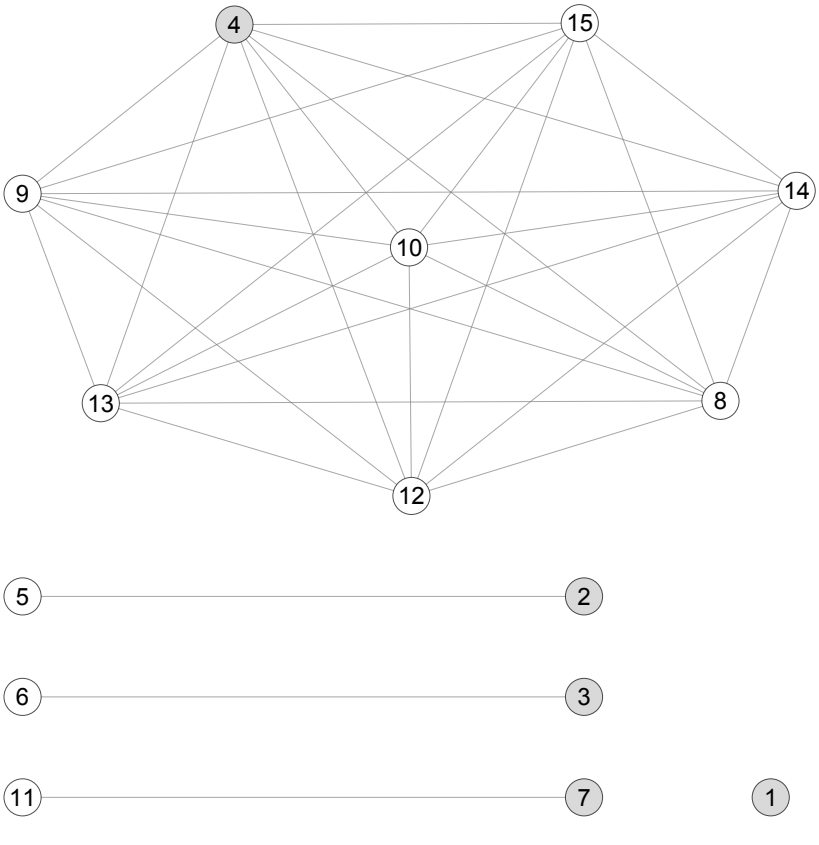
{{1, 2, 3}, {1, 3, 2}}

Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}\}$



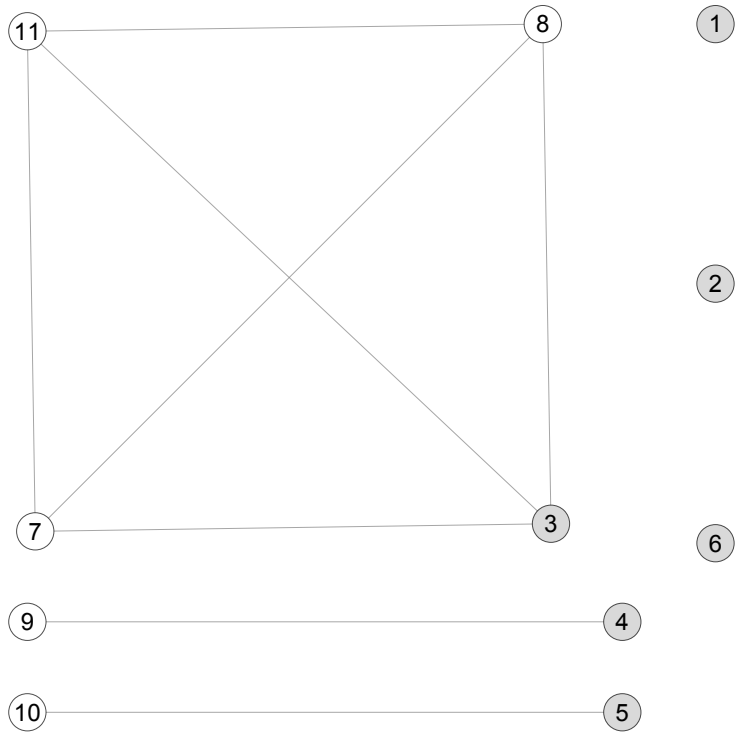
There are 11 2-D families of subalgebras to be analyzed.

Done.

There are 6 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\}, 4 \to \{\Xi_2, \Xi_4\}, 5 \to \{\Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}\}$



There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}\}$

Time of computation: 1.440139

==========

Algebra n. 26

$$\begin{pmatrix} 0 & 0 & 0 & 2\Xi_1 \\ 0 & 0 & \Xi_1 & \Xi_2 \\ 0 & -\Xi_1 & 0 & \Xi_3 \\ -2\Xi_1 & -\Xi_2 & -\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

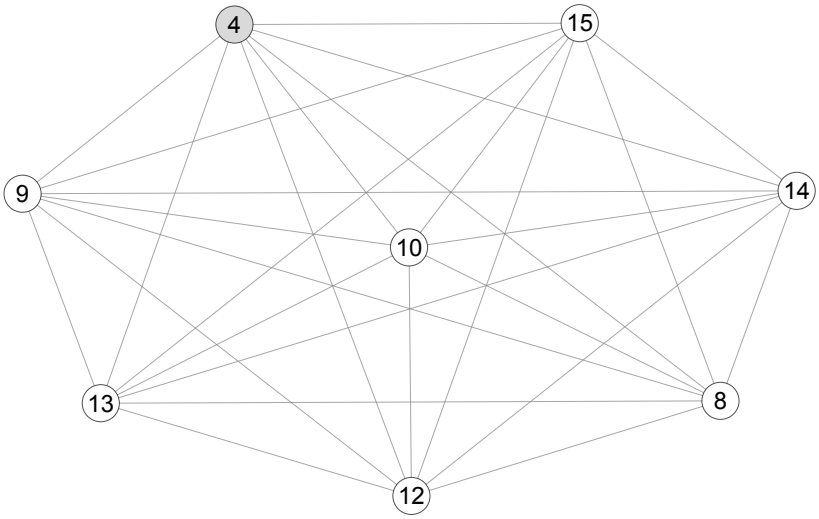
{{1, 2, 3}, {1, 3, 2}}

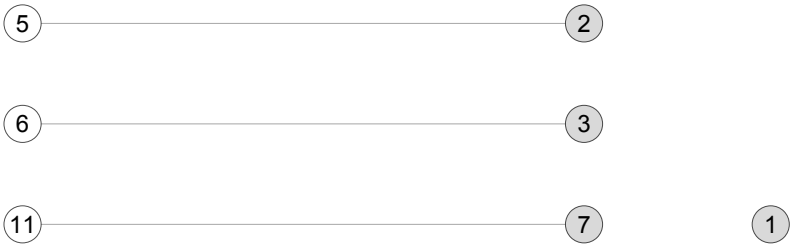
Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

{{Ξ_1}, {Ξ_2}, {Ξ_3}, {Ξ_4}, {$\Xi_2 + \dot{a}_1\, \Xi_3$}}

{1 → {Ξ_1}, 2 → {Ξ_2}, 3 → {Ξ_3}, 4 → {Ξ_4}, 7 → {$\Xi_2 + \dot{a}_1\, \Xi_3$}}





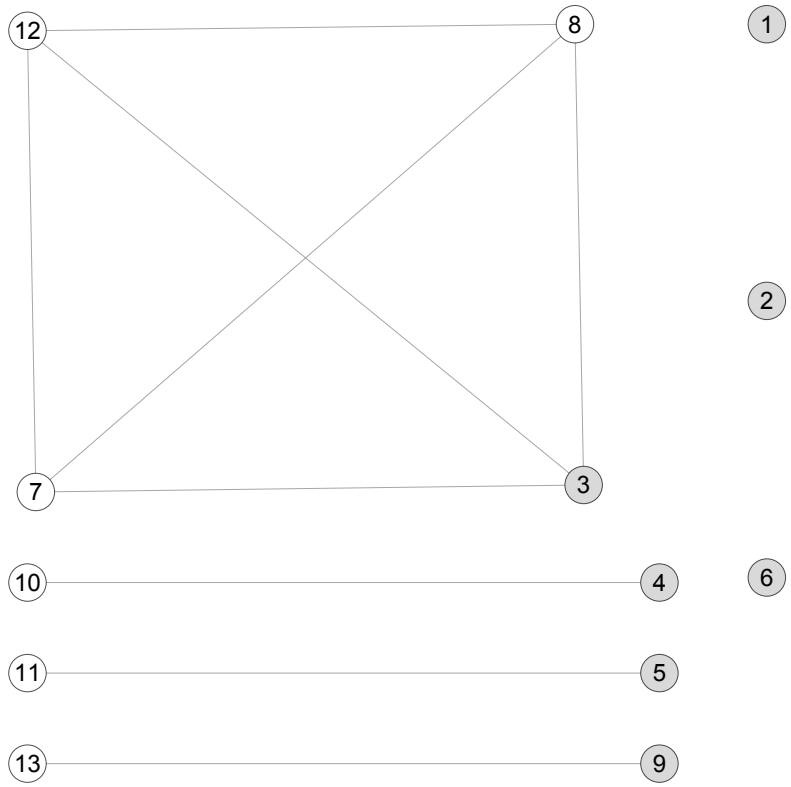
There are 13 2-D families of subalgebras to be analyzed.

Done.

There are 7 optimal families of 2-dimensional Lie subalgebras.

{{Ξ_1, Ξ_2}, {Ξ_1, Ξ_3}, {Ξ_1, Ξ_4}, {Ξ_2, Ξ_4}, {Ξ_3, Ξ_4}, {Ξ_1, $\Xi_2 + \dot{a}_1\, \Xi_3$}, {$\Xi_2 + \dot{a}_1\, \Xi_3$, Ξ_4}}

{1 → {Ξ_1, Ξ_2}, 2 → {Ξ_1, Ξ_3}, 3 → {Ξ_1, Ξ_4},
 4 → {Ξ_2, Ξ_4}, 5 → {Ξ_3, Ξ_4}, 6 → {Ξ_1, $\Xi_2 + \dot{a}_1\, \Xi_3$}, 9 → {$\Xi_2 + \dot{a}_1\, \Xi_3$, Ξ_4}}



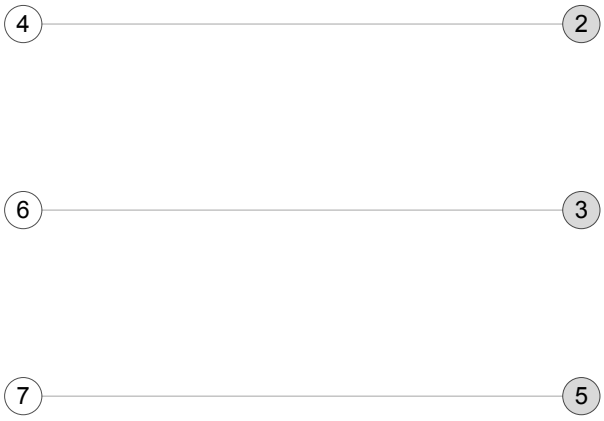
There are 7 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 5 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3, \Xi_4\}\}$



Time of computation: 2.035767

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Algebra n. 27

$$\begin{pmatrix} 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & \Xi_1 & \Xi_2 \\ 0 & -\Xi_1 & 0 & 0 \\ -\Xi_1 & -\Xi_2 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

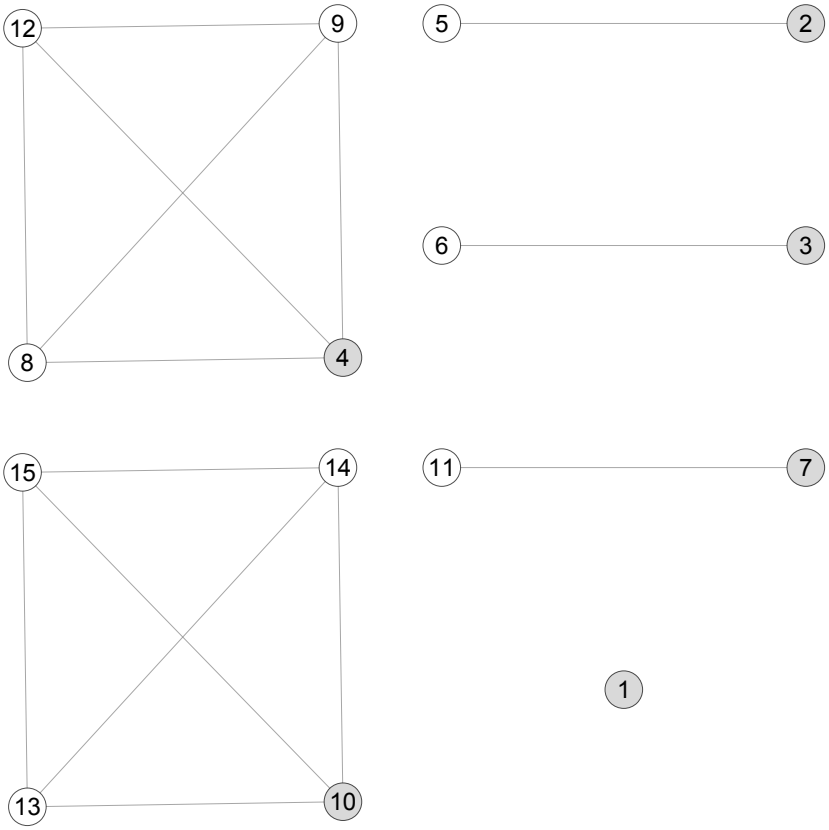
$\{\{1, 2, 3\}, \{1, 3, 2\}\}$

Done.

There are 6 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 7 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 10 \to \{\Xi_3 + a_1 \Xi_4\}\}$



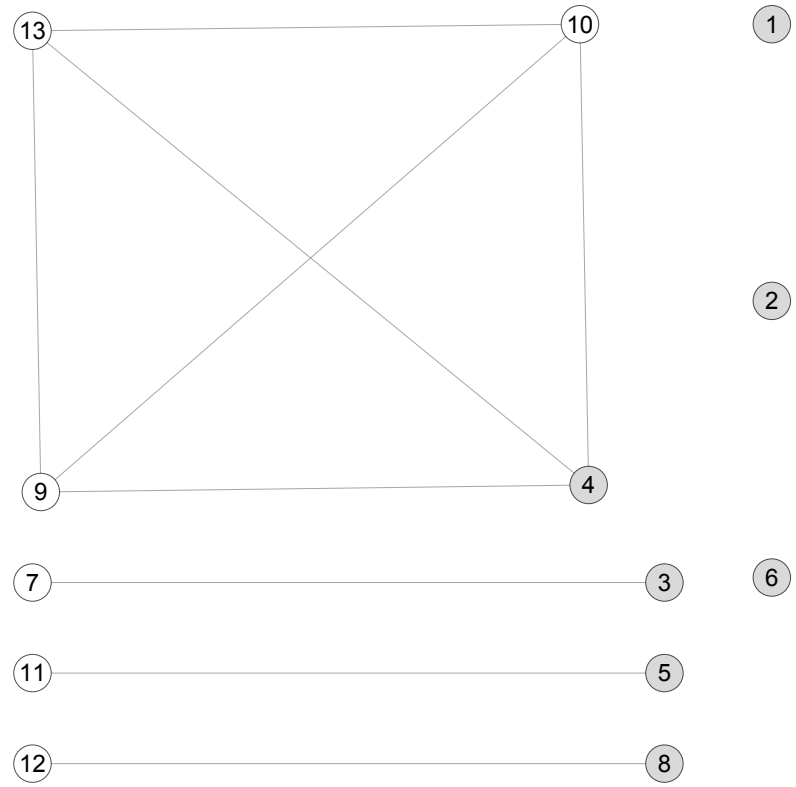
There are 13 2-D families of subalgebras to be analyzed.

Done.

There are 7 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_4\},$
$4 \to \{\Xi_2, \Xi_4\}, 5 \to \{\Xi_3, \Xi_4\}, 6 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, 8 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}\}$



There are 5 3-D families of subalgebras to be analyzed.

Done.

There are 4 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_3, \Xi_4\}, 4 \to \{\Xi_1, \Xi_2, \Xi_3 + \dot{a}_1 \Xi_4\}\}$

Time of computation: 1.282051

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Algebra n. 28

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Xi_1 & -\Xi_3 \\ 0 & -\Xi_1 & 0 & \Xi_2 \\ 0 & \Xi_3 & -\Xi_2 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

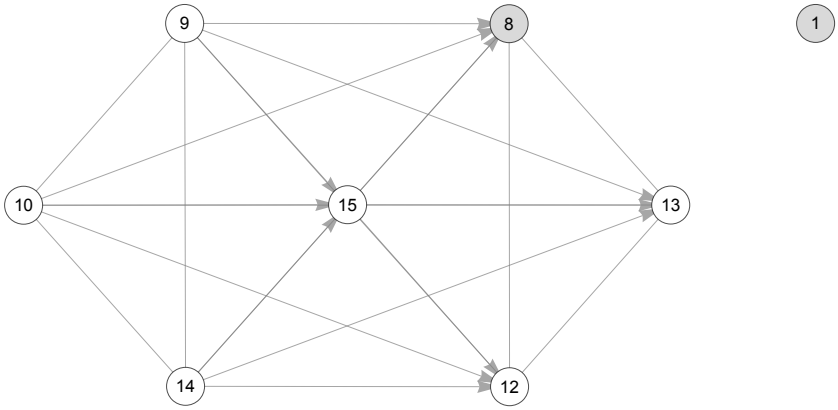
$\{\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{3, 1, 2\}\}$

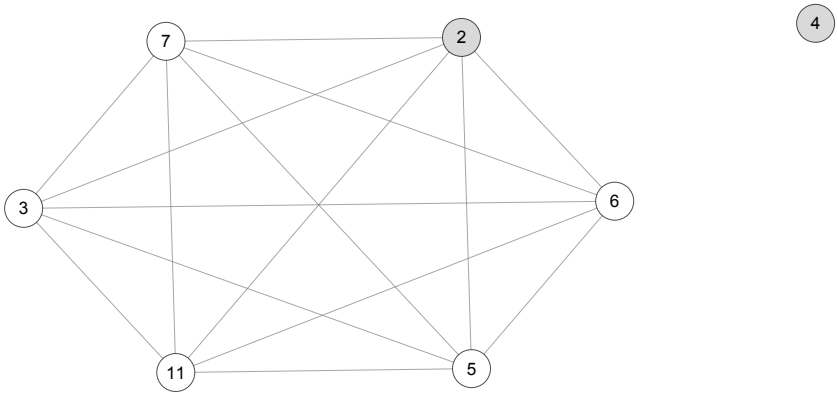
Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1 + \dot{a}_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 4 \to \{\Xi_4\}, 8 \to \{\Xi_1 + \dot{a}_1 \Xi_4\}\}$





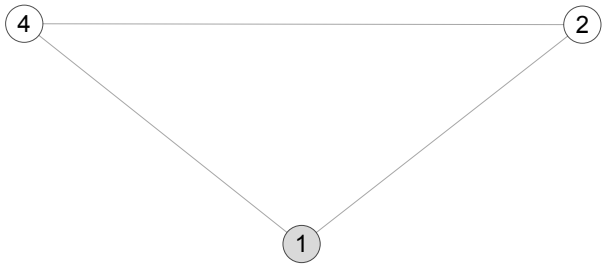
There are 7 2-D families of subalgebras to be analyzed.

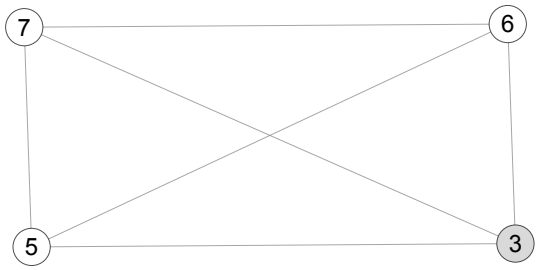
Done.

There are 2 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 3 \to \{\Xi_1, \Xi_4\}\}$





There are 1 3-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}\}$$

Time of computation: 12.613478

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Algebra n. 29

$$\begin{pmatrix} 0 & 0 & 0 & 2\,a\,\Xi_1 \\ 0 & 0 & \Xi_1 & a\,\Xi_2 - \Xi_3 \\ 0 & -\Xi_1 & 0 & \Xi_2 + a\,\Xi_3 \\ -2\,a\,\Xi_1 & -a\,\Xi_2 + \Xi_3 & -\Xi_2 - a\,\Xi_3 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

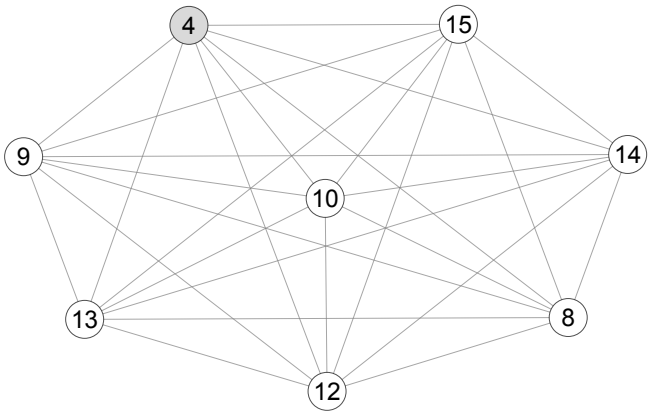
$$\{\{1, 2, 3, 4\}, \{1, 2, 4, 3\}, \{1, 3, 2, 4\}, \{1, 3, 4, 2\}, \{1, 4, 2, 3\}, \{1, 4, 3, 2\}, \{2, 3, 4, 1\}, \{2, 4, 1, 3\}, \{3, 2, 4, 1\}, \{3, 4, 1, 2\}, \{4, 1, 2, 3\}, \{4, 1, 3, 2\}\}$$

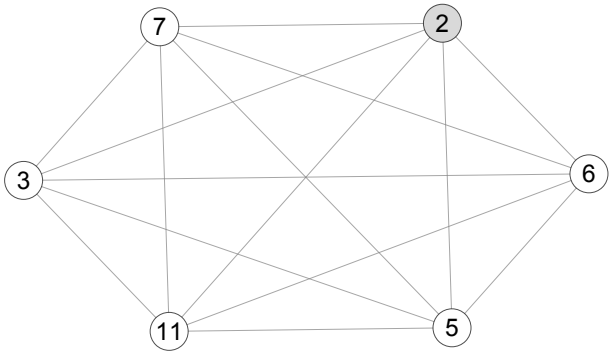
Done.

There are 3 optimal families of 1-dimensional Lie subalgebras.

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 4 \to \{\Xi_4\}\}$$





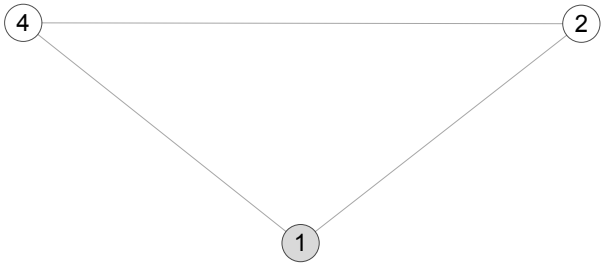
There are 7 2-D families of subalgebras to be analyzed.

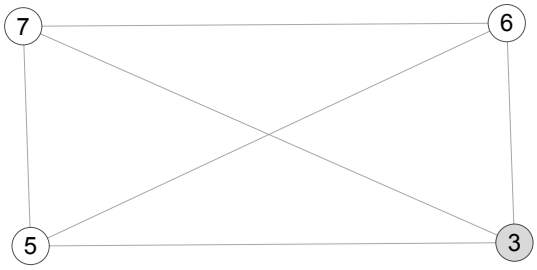
Done.

There are 2 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2\}, 3 \to \{\Xi_1, \Xi_4\}\}$





There are 1 3-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 3-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2, \Xi_3\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}\}$$

Time of computation: 174.899092

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Algebra n. 30

$$\begin{pmatrix} 0 & 0 & \Xi_1 & -\Xi_2 \\ 0 & 0 & \Xi_2 & \Xi_1 \\ -\Xi_1 & -\Xi_2 & 0 & 0 \\ \Xi_2 & -\Xi_1 & 0 & 0 \end{pmatrix}$$

There are 15 1-D families of subalgebras to be analyzed.

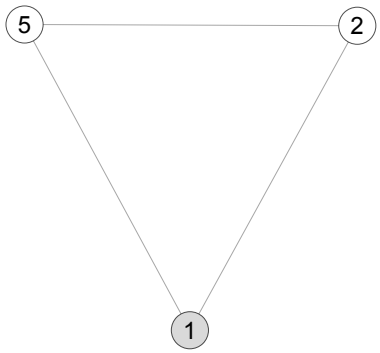
$$\{\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{3, 1, 2\}\}$$

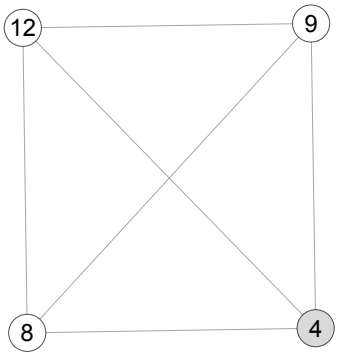
Done.

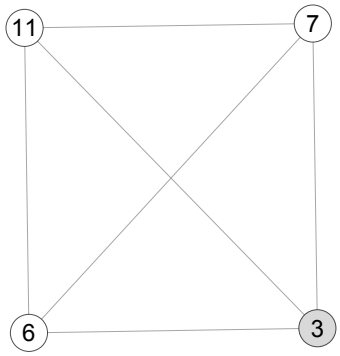
There are 4 optimal families of 1-dimensional Lie subalgebras.

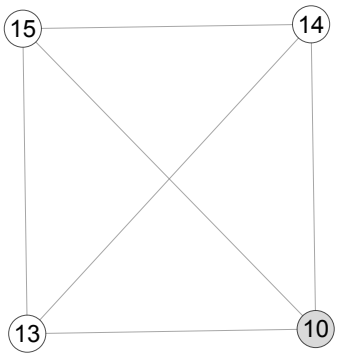
$$\{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + \mathring{a}_1\, \Xi_4\}\}$$

$$\{1 \to \{\Xi_1\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_4\}, 10 \to \{\Xi_3 + \mathring{a}_1\, \Xi_4\}\}$$









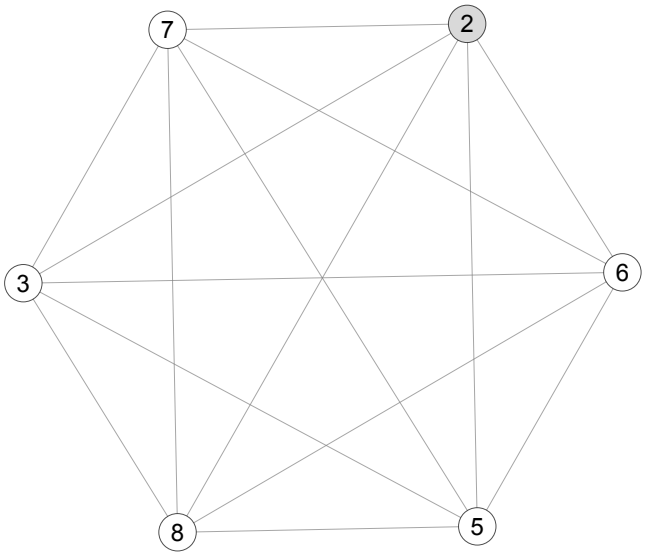
There are 8 2-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 2-dimensional Lie subalgebras.

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_3, \Xi_4\}\}$$

$$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 4 \to \{\Xi_3, \Xi_4\}\}$$



1 4

There are 3 3-D families of subalgebras to be analyzed.

Done.

There are 3 optimal families of 3-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$

$\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}\}$

1 3

2

Time of computation: 5.947459

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Print["Computation done in ", SessionTime[] / 60, " minutes."]

Computation done in 6.58565797 minutes.