Real 3 D Lie algebras

J. Patera, P. Winternitz. Subalgebras of real three and four--dimensional Lie algebras.

Journal of Mathematical Physics, 18, 1449--1455, 1977.

In[1]:=
```
SetDirectory[NotebookDirectory[]];
```

In[2]:=
```
<< "SymboLie.wl"
```

SymboLie (v. 1.6)– A Package for determining Optimal Systems of Lie Subalgebras.

In[3]:=
```
CS = Table[0, {k, 1, 3}, {i, 1, 3}, {j, 1, 3}];
AllCS = Table[CS, {k, 1, 11}];
AllCS⟦2, 2, 1, 2⟧ = 1; AllCS⟦2, 2, 2, 1⟧ = -1;

AllCS⟦3, 1, 2, 3⟧ = 1; AllCS⟦3, 1, 3, 2⟧ = -1;

AllCS⟦4, 1, 1, 3⟧ = 1; AllCS⟦4, 1, 3, 1⟧ = -1;
AllCS⟦4, 1, 2, 3⟧ = 1; AllCS⟦4, 1, 3, 2⟧ = -1;
AllCS⟦4, 2, 2, 3⟧ = 1; AllCS⟦4, 2, 3, 2⟧ = -1;

AllCS⟦5, 1, 1, 3⟧ = 1; AllCS⟦5, 1, 3, 1⟧ = -1;
AllCS⟦5, 2, 2, 3⟧ = 1; AllCS⟦5, 2, 3, 2⟧ = -1;

AllCS⟦6, 1, 1, 3⟧ = 1; AllCS⟦6, 1, 3, 1⟧ = -1;
AllCS⟦6, 2, 2, 3⟧ = -1; AllCS⟦6, 2, 3, 2⟧ = 1;

AllCS⟦7, 1, 1, 3⟧ = 1; AllCS⟦7, 1, 3, 1⟧ = -1;
AllCS⟦7, 2, 2, 3⟧ = a; AllCS⟦7, 2, 3, 2⟧ = -a; (* 0<|a|<1 *)

AllCS⟦8, 2, 1, 3⟧ = -1; AllCS⟦8, 2, 3, 1⟧ = 1;
AllCS⟦8, 1, 2, 3⟧ = 1; AllCS⟦8, 1, 3, 2⟧ = -1;

AllCS⟦9, 1, 1, 3⟧ = a; AllCS⟦9, 1, 3, 1⟧ = -a; (* a>0 *)
AllCS⟦9, 2, 1, 3⟧ = -1; AllCS⟦9, 2, 3, 1⟧ = 1;
AllCS⟦9, 1, 2, 3⟧ = 1; AllCS⟦9, 1, 3, 2⟧ = -1;
AllCS⟦9, 2, 2, 3⟧ = a; AllCS⟦9, 2, 3, 2⟧ = -a;

AllCS⟦10, 1, 1, 2⟧ = 1; AllCS⟦10, 1, 2, 1⟧ = -1;
AllCS⟦10, 3, 2, 3⟧ = 1; AllCS⟦10, 3, 3, 2⟧ = -1;
AllCS⟦10, 2, 3, 1⟧ = 2; AllCS⟦10, 2, 1, 3⟧ = -2;

AllCS⟦11, 3, 1, 2⟧ = 1; AllCS⟦11, 3, 2, 1⟧ = -1;
AllCS⟦11, 2, 3, 1⟧ = 1; AllCS⟦11, 2, 1, 3⟧ = -1;
AllCS⟦11, 1, 2, 3⟧ = 1; AllCS⟦11, 1, 3, 2⟧ = -1;
```

In[28]:=
```
pars = Table[{{}, {}}, {k, 1, 11}];
pars⟦7⟧ = {{a}, {Element[a, Reals], 0 < Abs[a] < 1}};
pars⟦9⟧ = {{a}, {a > 0}};
```

In[31]:=
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PrintDebug = 0;
```

In[32]:=
```
allalg1 = {}; allalg2 = {};
```

In[33]:=
```
results = {{"Id Algebra", "# 1D OS", "# 2D OS"}};
```

In[34]:=
```
For[cont = 1, cont ≤ 11, cont++,
  Print["=========================================="];
  Print["Algebra n. ", cont];
  Print[CommutatorTable[AllCS⟦cont⟧] // MatrixForm];
  alg1 = SubAlgebra[AllCS⟦cont⟧, pars⟦cont⟧, 1];
  AppendTo[allalg1, alg1];
  opt1 = PrintOptimal[alg1];
  Print[opt1];
  PrintClasses[alg1];
  G = PrintGraph[alg1, 1];
  Print[G];
  alg2 = SubAlgebra[AllCS⟦cont⟧, pars⟦cont⟧, 2];
  AppendTo[allalg2, alg2];
  opt2 = PrintOptimal[alg2];
  Print[opt2];
  PrintClasses[alg2];
  G = PrintGraph[alg2, 1];
  Print[G];
 ];
Print["=========================================="];
```

```
==========================================
Algebra n. 1
```

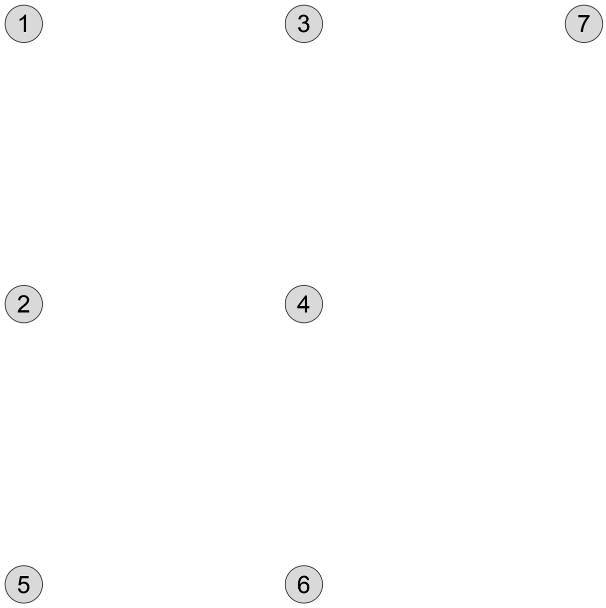
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
There are 7 1-D families of subalgebras to be analyzed.
Done.
There are 7 optimal families of 1-dimensional Lie subalgebras.
```

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \underset{.}{a}_1 \Xi_2\}, \{\Xi_1 + \underset{.}{a}_1 \Xi_3\}, \{\Xi_2 + \underset{.}{a}_1 \Xi_3\}, \{\Xi_1 + \underset{.}{a}_1 \Xi_2 + \underset{.}{a}_2 \Xi_3\}\}$

```
There are 7 optimal families of 1-dimensional Lie subalgebras.
```

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \underset{.}{a}_1 \Xi_2\},$
$5 \to \{\Xi_1 + \underset{.}{a}_1 \Xi_3\}, 6 \to \{\Xi_2 + \underset{.}{a}_1 \Xi_3\}, 7 \to \{\Xi_1 + \underset{.}{a}_1 \Xi_2 + \underset{.}{a}_2 \Xi_3\}\}$



There are 7 2–D families of subalgebras to be analyzed.

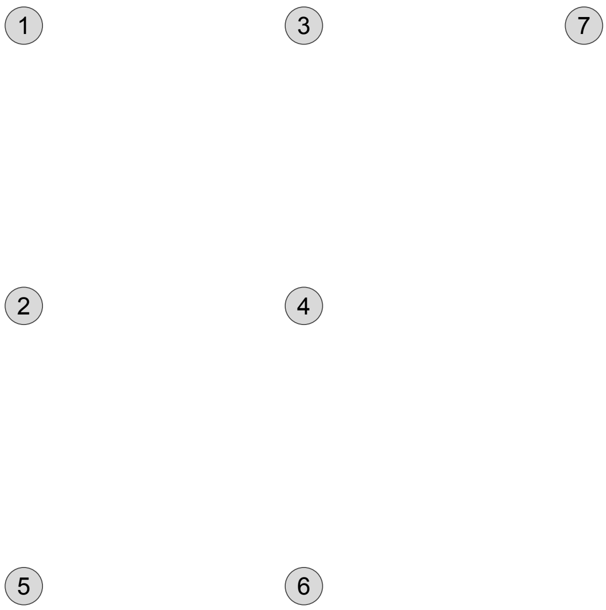
Done.

There are 7 optimal families of 2–dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\}\}$

There are 7 optimal families of 2–dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\}, 5 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 6 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2\}, 7 \to \{\Xi_1 + a_1 \Xi_3, \Xi_2 + a_2 \Xi_3\}\}$



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Algebra n. 2

$$\begin{pmatrix} 0 & \Xi_2 & 0 \\ -\Xi_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are 7 1–D families of subalgebras to be analyzed.

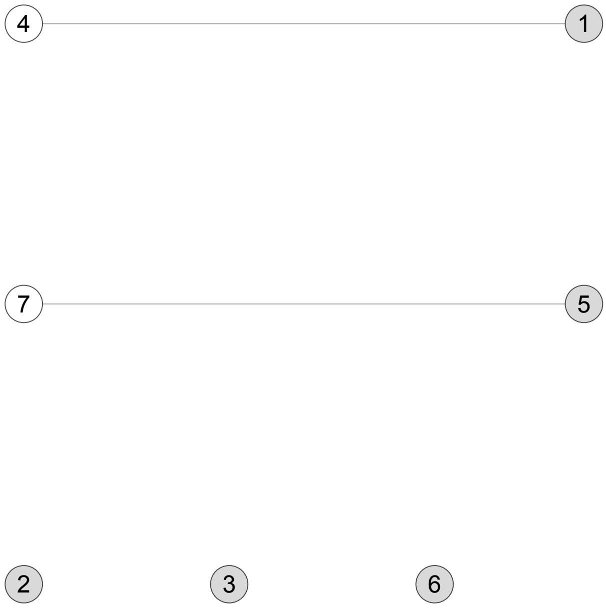
Done.

There are 5 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3\}, \{\Xi_2 + \underset{\cdot}{\alpha}_1 \Xi_3\}\}$

There are 5 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \underset{\cdot}{\alpha}_1 \Xi_2\},$
$5 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3\}, 6 \to \{\Xi_2 + \underset{\cdot}{\alpha}_1 \Xi_3\}, 7 \to \{\Xi_1 + \underset{\cdot}{\alpha}_1 \Xi_2 + \underset{\cdot}{a}_1 \Xi_3\}\}$



There are 5 2-D families of subalgebras to be analyzed.

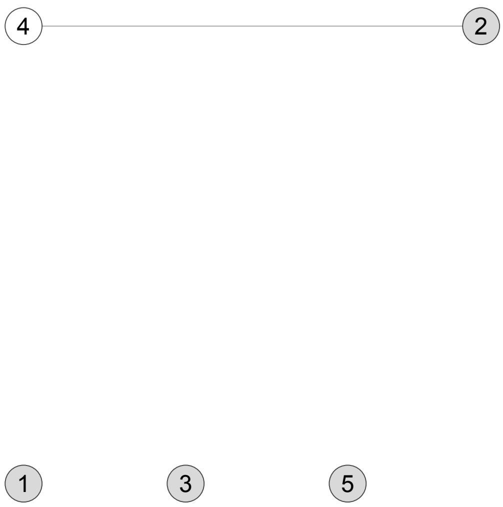
Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}\}$

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_1 + \underset{\cdot}{\alpha}_1 \Xi_2, \Xi_3\}, 5 \to \{\Xi_1 + \underset{\cdot}{a}_1 \Xi_3, \Xi_2\}\}$



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Algebra n. 3

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Xi_1 \\ 0 & -\Xi_1 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

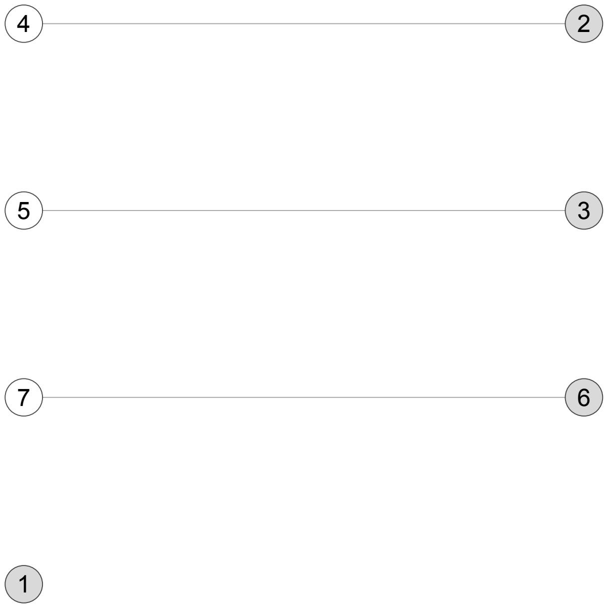
Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_2 + a_1 \Xi_3\}\}$

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + a_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$



There are 3 2-D families of subalgebras to be analyzed.

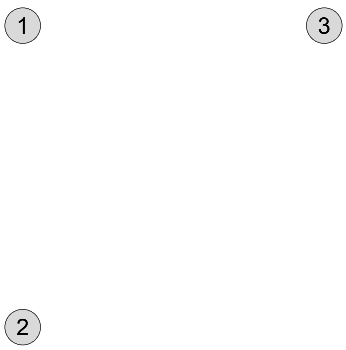
Done.

There are 3 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1 \Xi_3\}\}$

There are 3 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_2 + a_1 \Xi_3\}\}$



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Algebra n. 4

$$\begin{pmatrix} 0 & 0 & \Xi_1 \\ 0 & 0 & \Xi_1 + \Xi_2 \\ -\Xi_1 & -\Xi_1 - \Xi_2 & 0 \end{pmatrix}$$

There are 7 1–D families of subalgebras to be analyzed.

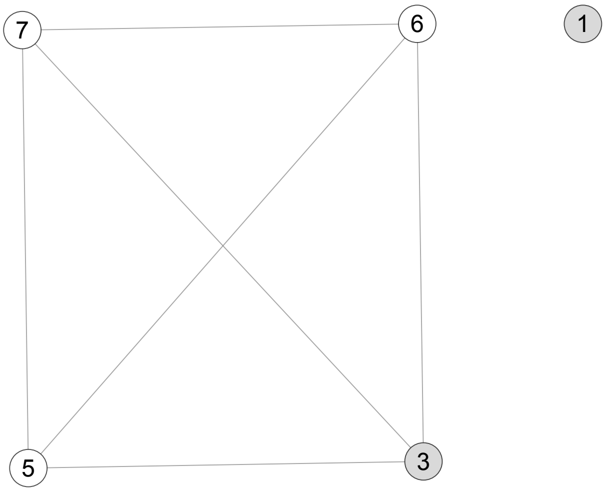
Done.

There are 3 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}\}$

There are 3 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + a_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$



There are 3 2–D families of subalgebras to be analyzed.

Done.

There are 2 optimal families of 2–dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}\}$

There are 2 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}\}$

==

Algebra n. 5

$$\begin{pmatrix} 0 & 0 & \Xi_1 \\ 0 & 0 & \Xi_2 \\ -\Xi_1 & -\Xi_2 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

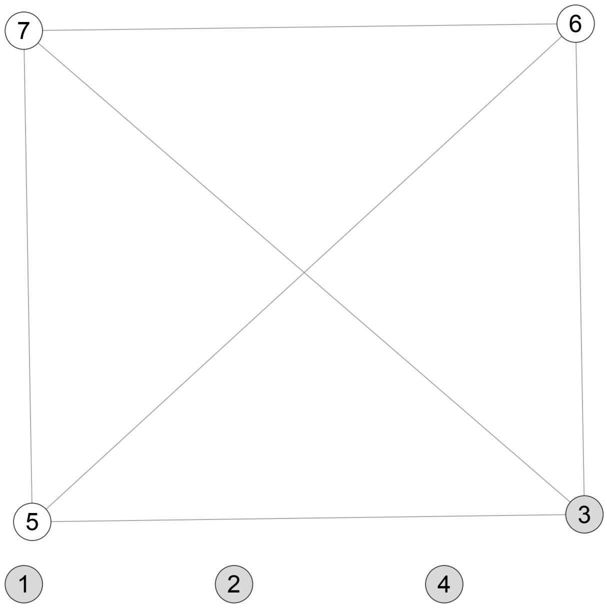
Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + a_1 \Xi_2\}\}$

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + a_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$



There are 7 2-D families of subalgebras to be analyzed.

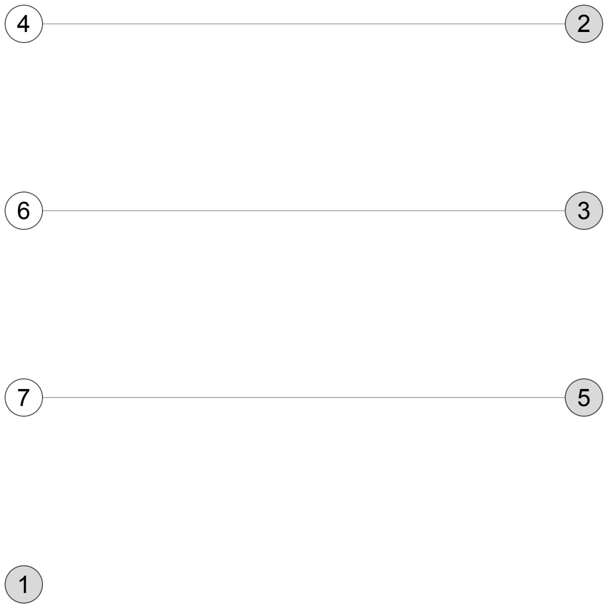
Done.

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1 \Xi_2, \Xi_3\}\}$

There are 4 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\},$
$5 \to \{\Xi_1 + a_1 \Xi_2, \Xi_3\}, 6 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + \alpha_2 \Xi_3\}\}$



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Algebra n. 6

$$\begin{pmatrix} 0 & 0 & \Xi_1 \\ 0 & 0 & -\Xi_2 \\ -\Xi_1 & \Xi_2 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

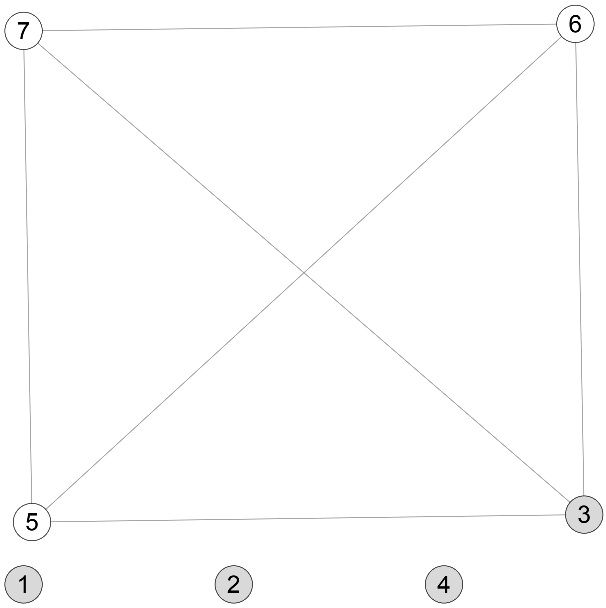
Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \underset{\circ}{\alpha}_1 \Xi_2\}\}$

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \underset{\circ}{\alpha}_1 \Xi_2\},$
$5 \to \{\Xi_1 + \underset{\circ}{\alpha}_1 \Xi_3\}, 6 \to \{\Xi_2 + \underset{\circ}{\alpha}_1 \Xi_3\}, 7 \to \{\Xi_1 + \underset{\circ}{\alpha}_1 \Xi_2 + \underset{\circ}{\alpha}_2 \Xi_3\}\}$



There are 5 2-D families of subalgebras to be analyzed.

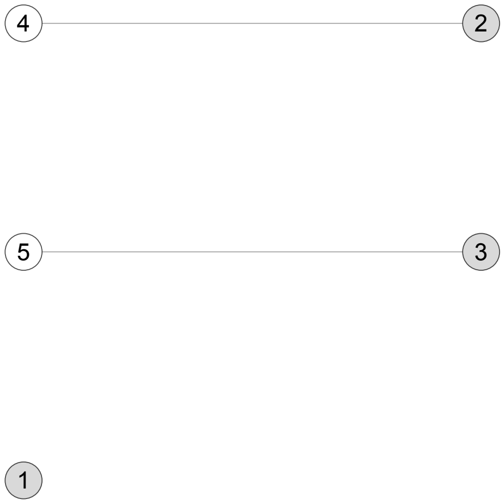
Done.

There are 3 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}\}$

There are 3 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$



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Algebra n. 7

$$\begin{pmatrix} 0 & 0 & \Xi_1 \\ 0 & 0 & a\,\Xi_2 \\ -\Xi_1 & -a\,\Xi_2 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

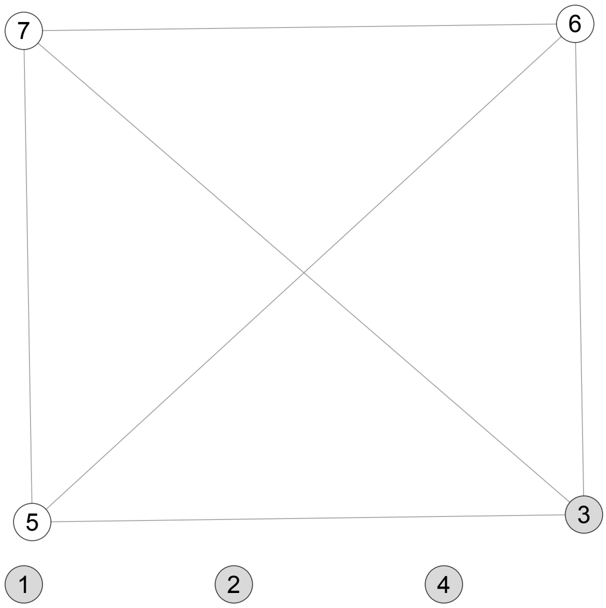
Done.

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_2\}\}$

There are 4 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$



There are 5 2–D families of subalgebras to be analyzed.

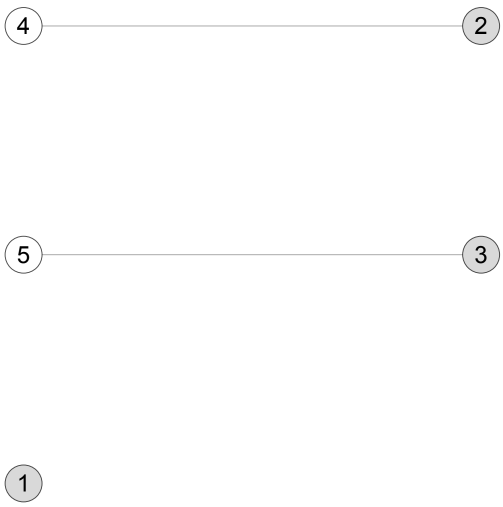
Done.

There are 3 optimal families of 2–dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}\}$

There are 3 optimal families of 2–dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_3\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\}, 5 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\}$



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Algebra n. 8

$$\begin{pmatrix} 0 & 0 & -\Xi_2 \\ 0 & 0 & \Xi_1 \\ \Xi_2 & -\Xi_1 & 0 \end{pmatrix}$$

There are 7 1–D families of subalgebras to be analyzed.

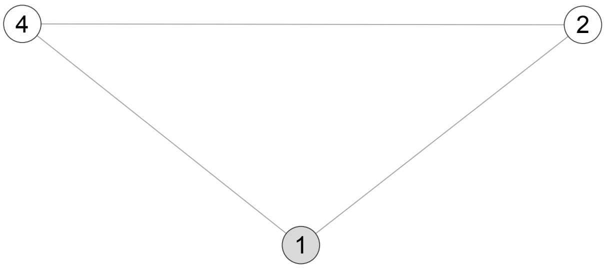
Done.

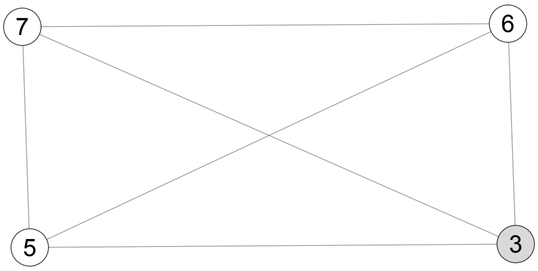
There are 2 optimal families of 1–dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_3\}\}$

There are 2 optimal families of 1–dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$





There are 1 2-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}\}$

There are 1 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}\}$

==

Algebra n. 9

$$\begin{pmatrix} 0 & 0 & a\,\Xi_1 - \Xi_2 \\ 0 & 0 & \Xi_1 + a\,\Xi_2 \\ -a\,\Xi_1 + \Xi_2 & -\Xi_1 - a\,\Xi_2 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

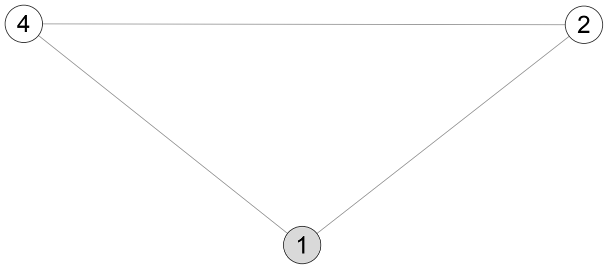
Done.

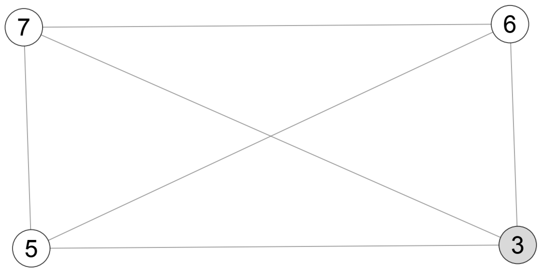
There are 2 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_3\}\}$

There are 2 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + a_1 \Xi_3\}, 6 \to \{\Xi_2 + a_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$





There are 1 2-D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 2-dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}\}$

There are 1 optimal families of 2-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}\}$

==

Algebra n. 10

$$\begin{pmatrix} 0 & \Xi_1 & -2\,\Xi_2 \\ -\Xi_1 & 0 & \Xi_3 \\ 2\,\Xi_2 & -\Xi_3 & 0 \end{pmatrix}$$

There are 7 1-D families of subalgebras to be analyzed.

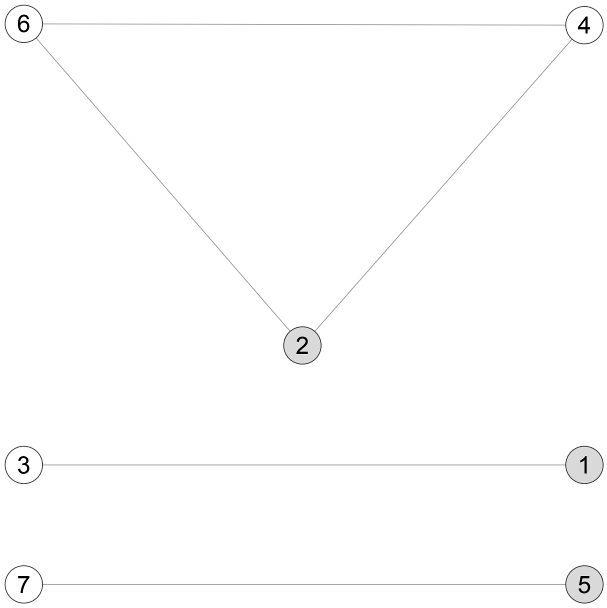
Done.

There are 3 optimal families of 1-dimensional Lie subalgebras.

$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\,\Xi_3\}\}$

There are 3 optimal families of 1-dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1\,\Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1\,\Xi_3\}, 6 \to \{\Xi_2 + \alpha_1\,\Xi_3\}, 7 \to \{\Xi_1 + \alpha_1\,\Xi_2 + \alpha_2\,\Xi_3\}\}$



There are 2 2–D families of subalgebras to be analyzed.

Done.

There are 1 optimal families of 2–dimensional Lie subalgebras.

$\{\{\Xi_1, \Xi_2\}\}$

There are 1 optimal families of 2–dimensional Lie subalgebras.

$\{1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_2, \Xi_3\}\}$

==

Algebra n. 11

$$\begin{pmatrix} 0 & \Xi_3 & -\Xi_2 \\ -\Xi_3 & 0 & \Xi_1 \\ \Xi_2 & -\Xi_1 & 0 \end{pmatrix}$$

There are 7 1–D families of subalgebras to be analyzed.

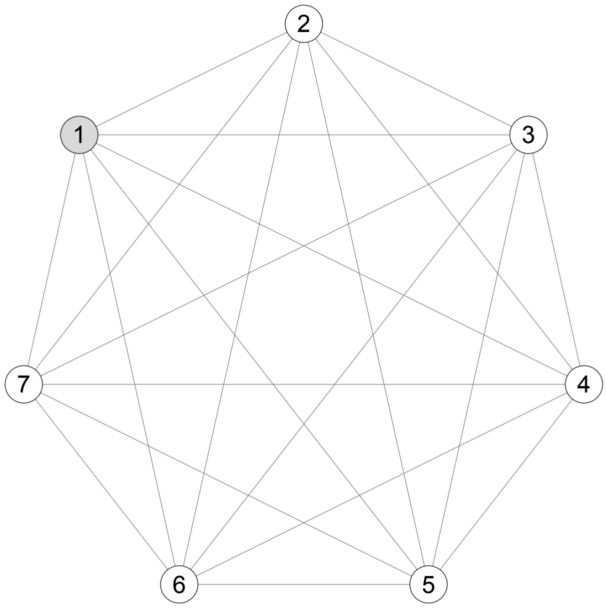
Done.

There are 1 optimal families of 1–dimensional Lie subalgebras.

$\{\{\Xi_1\}\}$

There are 1 optimal families of 1–dimensional Lie subalgebras.

$\{1 \to \{\Xi_1\}, 2 \to \{\Xi_2\}, 3 \to \{\Xi_3\}, 4 \to \{\Xi_1 + \alpha_1 \Xi_2\},$
$5 \to \{\Xi_1 + \alpha_1 \Xi_3\}, 6 \to \{\Xi_2 + \alpha_1 \Xi_3\}, 7 \to \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}\}$




```
There is no subalgebra.

Null

There is no subalgebra.

Empty graph.

{}

==========================================
```

In[36]:= **allalg1**

Out[36]=

```
{{{{1, 0, 0, 0, 0, 0, 0},
    {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1}},
   {{1}, {2}, {4}, {3}, {5}, {6}, {7}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
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  {{{1, 0, 0, 1, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 1,
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   {{1}, {2}, {4}, {3}, {5}, {6}, {7}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
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  {{{1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0}, {0, 0, 1, 0, 1, 0, 0}, {0, 1, 0, 1,
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   {{1}, {2}, {4}, {3}, {5}, {6}, {7}}, {{{0, 0, 0}, {0, 0, 1}, {0, -1, 0}},
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    {{0, 0, 0}, {0, 0, 1}, {0, -1, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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     0, 0, 0}, {0, 0, 1, 0, 1, 1, 1}, {0, 0, 1, 0, 1, 1, 1}, {0, 0, 1, 0, 1, 1, 1}},
   {{1}, {2}, {4}, {3}, {5}, {6}, {7}}, {{{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}},
    {{0, 0, 0}, {0, 0, 1}, {0, -1, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
  {{{1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 1, 1, 1}, {0, 0, 0, 1,
```

```
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```

In[37]:= **allalg2**

Out[37]=

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    {0, 0, 0, 0, 0, 0, 1}}, {{1, 2}, {1, 4}, {2, 4}, {1, 6}, {3, 4}, {5, 2}, {5, 6}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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  {{1, 2}, {1, 4}, {2, 4}, {3, 4}, {5, 2}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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  {{{0, 0, 0}, {0, 0, 1}, {0, -1, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
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    {0, 0, 0, 0, 1, 0, 1}}, {{1, 2}, {1, 4}, {2, 4}, {1, 6}, {3, 4}, {5, 2}, {5, 6}},
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    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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  {{1, 2}, {1, 4}, {2, 4}, {1, 6}, {5, 2}}, {{{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}},
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  {{1, 2}, {1, 4}, {2, 4}, {1, 6}, {5, 2}}, {{{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}},
    {{0, 0, 0}, {0, 0, -1/2}, {0, 1/2, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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    {{0, 0, -1}, {0, 0, 0}, {1, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
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 {{{}}, {}, {{{0, 0, 0}, {0, 0, 1}, {0, -1, 0}}, {{0, 0, -1}, {0, 0, 0}, {1, 0, 0}},
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```

In[38]:= **SessionTime[]**

Out[38]=

45.393866