

6 D Lie algebra studied by Ovsiannikov.

L . V . Ovsiannikov . The group analysis algorithms . In “Modern Group Analysis : advanced analytical and computational methods in mathematical physics”, Proceedings of the International Workshop, Acireale, October 27-- 31, 1992, Edited by N . H . Ibragimov, M . Torrisi and A . Valenti, Kluwer Academic Publishers, 277-- 289, 1993.

```
In[1]:= SetDirectory[NotebookDirectory[]];
```

```
In[2]:= << "SymboLie.wl"
```

SymboLie (v. 1.6)– A Package for determining Optimal Systems of Lie Subalgebras.

```
In[3]:= gens = {{1, 0, 0}, {0, 1, 0}, {t, 0, 0}, {0, t, 0}, {y, -x, 0}, {0, 0, 1}};
```

```
vars = {x, y, t};
```

```
pars = {{}, {}};
```

```
In[4]:= cs = StructureConstants[gens, vars];
```

```
In[5]:= ct = CommutatorTable[cs]; ct // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\mathfrak{E}_2 & 0 \\ 0 & 0 & 0 & 0 & \mathfrak{E}_1 & 0 \\ 0 & 0 & 0 & 0 & -\mathfrak{E}_4 & -\mathfrak{E}_1 \\ 0 & 0 & 0 & 0 & \mathfrak{E}_3 & -\mathfrak{E}_2 \\ \mathfrak{E}_2 & -\mathfrak{E}_1 & \mathfrak{E}_4 & -\mathfrak{E}_3 & 0 & 0 \\ 0 & 0 & \mathfrak{E}_1 & \mathfrak{E}_2 & 0 & 0 \end{pmatrix}$$

```
In[6]:= FastRun = 1;
```

```
In[7]:= Timing[alg1 = SubAlgebra[cs, pars, 1];]
```

There are 63 1-D families of subalgebras to be analyzed.

Done.

```
Out[7]= {942.654, Null}
```

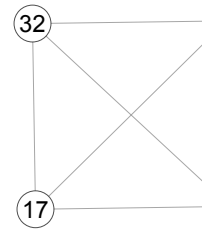
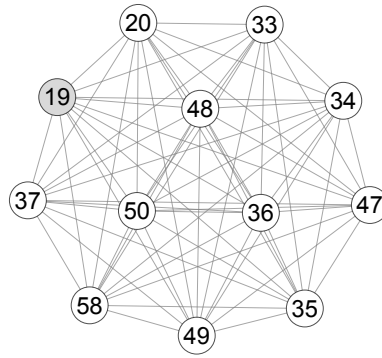
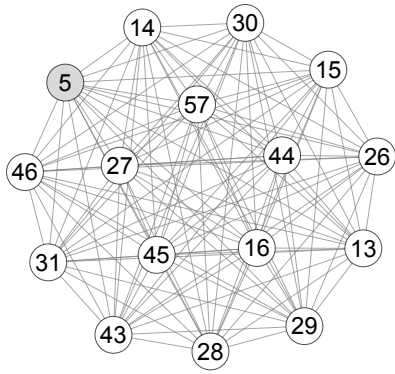
```
In[8]:= PrintOptimal[alg1]
```

There are 7 optimal families of 1-dimensional Lie subalgebras.

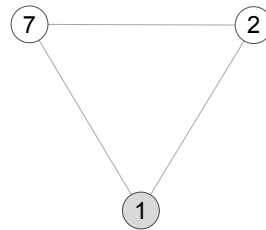
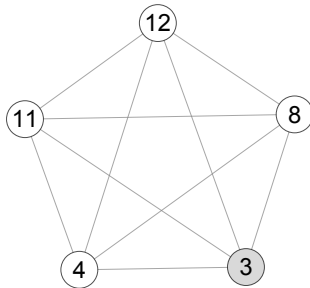
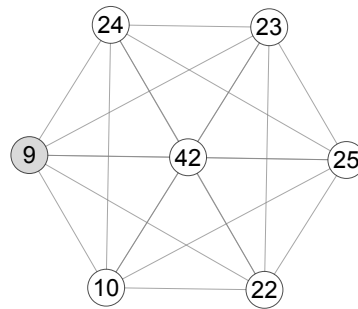
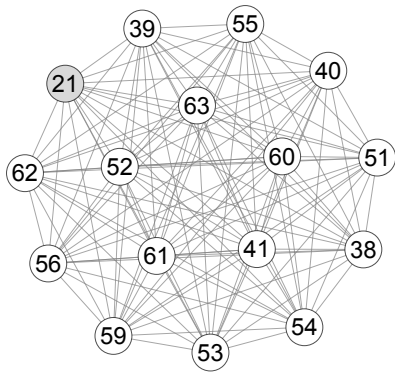
```
Out[8]= {{\mathfrak{E}_1}, {\mathfrak{E}_3}, {\mathfrak{E}_5}, {\mathfrak{E}_6}, {\mathfrak{E}_2 + a_1 \mathfrak{E}_3}, {\mathfrak{E}_3 + a_1 \mathfrak{E}_6}, {\mathfrak{E}_5 + a_1 \mathfrak{E}_6}}
```

```
In[9]:= PrintGraph[alg1, 1]
```

$\{1 \rightarrow \{\Xi_1\}, 2 \rightarrow \{\Xi_2\}, 3 \rightarrow \{\Xi_3\}, 4 \rightarrow \{\Xi_4\}, 5 \rightarrow \{\Xi_5\}, 6 \rightarrow \{\Xi_6\}, 7 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2\}, 8 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3\},$
 $9 \rightarrow \{\Xi_2 + \mathfrak{a}_1 \Xi_3\}, 10 \rightarrow \{\Xi_1 + \mathfrak{a}_1 \Xi_4\}, 11 \rightarrow \{\Xi_2 + \alpha_1 \Xi_4\}, 12 \rightarrow \{\Xi_3 + \alpha_1 \Xi_4\}, 13 \rightarrow \{\Xi_1 + \alpha_1 \Xi_5\},$
 $14 \rightarrow \{\Xi_2 + \alpha_1 \Xi_5\}, 15 \rightarrow \{\Xi_3 + \alpha_1 \Xi_5\}, 16 \rightarrow \{\Xi_4 + \alpha_1 \Xi_5\}, 17 \rightarrow \{\Xi_1 + \alpha_1 \Xi_6\}, 18 \rightarrow \{\Xi_2 + \alpha_1 \Xi_6\},$
 $19 \rightarrow \{\Xi_3 + \mathfrak{a}_1 \Xi_6\}, 20 \rightarrow \{\Xi_4 + \mathfrak{a}_1 \Xi_6\}, 21 \rightarrow \{\Xi_5 + \mathfrak{a}_1 \Xi_6\}, 22 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \mathfrak{a}_1 \Xi_3\},$
 $23 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \mathfrak{a}_1 \Xi_4\}, 24 \rightarrow \{\Xi_1 + \mathfrak{a}_1 \Xi_3 + \mathfrak{a}_2 \Xi_4\}, 25 \rightarrow \{\Xi_2 + \mathfrak{a}_1 \Xi_3 + \mathfrak{a}_2 \Xi_4\}, 26 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_5\},$
 $27 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \alpha_2 \Xi_5\}, 28 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \mathfrak{a}_1 \Xi_5\}, 29 \rightarrow \{\Xi_1 + \alpha_1 \Xi_4 + \mathfrak{a}_1 \Xi_5\}, 30 \rightarrow \{\Xi_2 + \alpha_1 \Xi_4 + \alpha_2 \Xi_5\},$
 $31 \rightarrow \{\Xi_3 + \alpha_1 \Xi_4 + \alpha_2 \Xi_5\}, 32 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_6\}, 33 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \alpha_2 \Xi_6\}, 34 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \mathfrak{a}_1 \Xi_6\},$
 $35 \rightarrow \{\Xi_1 + \alpha_1 \Xi_4 + \mathfrak{a}_1 \Xi_6\}, 36 \rightarrow \{\Xi_2 + \alpha_1 \Xi_4 + \alpha_2 \Xi_6\}, 37 \rightarrow \{\Xi_3 + \alpha_1 \Xi_4 + \mathfrak{a}_1 \Xi_6\},$
 $38 \rightarrow \{\Xi_1 + \alpha_1 \Xi_5 + \mathfrak{a}_1 \Xi_6\}, 39 \rightarrow \{\Xi_2 + \alpha_1 \Xi_5 + \mathfrak{a}_1 \Xi_6\}, 40 \rightarrow \{\Xi_3 + \mathfrak{a}_1 \Xi_5 + \mathfrak{a}_2 \Xi_6\},$
 $41 \rightarrow \{\Xi_4 + \mathfrak{a}_1 \Xi_5 + \mathfrak{a}_2 \Xi_6\}, 42 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \mathfrak{a}_1 \Xi_4\}, 43 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \alpha_3 \Xi_5\},$
 $44 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_4 + \mathfrak{a}_1 \Xi_5\}, 45 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \alpha_2 \Xi_4 + \mathfrak{a}_1 \Xi_5\}, 46 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \alpha_2 \Xi_4 + \mathfrak{a}_1 \Xi_5\},$
 $47 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \mathfrak{a}_1 \Xi_6\}, 48 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \mathfrak{a}_1 \Xi_4 + \mathfrak{a}_2 \Xi_6\}, 49 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \mathfrak{a}_1 \Xi_4 + \mathfrak{a}_2 \Xi_6\},$
 $50 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \mathfrak{a}_1 \Xi_4 + \mathfrak{a}_2 \Xi_6\}, 51 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_5 + \alpha_3 \Xi_6\}, 52 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \alpha_2 \Xi_5 + \mathfrak{a}_1 \Xi_6\},$
 $53 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \alpha_2 \Xi_5 + \mathfrak{a}_1 \Xi_6\}, 54 \rightarrow \{\Xi_1 + \alpha_1 \Xi_4 + \alpha_2 \Xi_5 + \mathfrak{a}_1 \Xi_6\}, 55 \rightarrow \{\Xi_2 + \alpha_1 \Xi_4 + \alpha_2 \Xi_5 + \mathfrak{a}_1 \Xi_6\},$
 $56 \rightarrow \{\Xi_3 + \alpha_1 \Xi_4 + \mathfrak{a}_1 \Xi_5 + \mathfrak{a}_2 \Xi_6\}, 57 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \alpha_3 \Xi_4 + \alpha_4 \Xi_5\},$
 $58 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \alpha_3 \Xi_4 + \mathfrak{a}_1 \Xi_6\}, 59 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \alpha_3 \Xi_5 + \mathfrak{a}_1 \Xi_6\},$
 $60 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_4 + \alpha_3 \Xi_5 + \mathfrak{a}_1 \Xi_6\}, 61 \rightarrow \{\Xi_1 + \alpha_1 \Xi_3 + \alpha_2 \Xi_4 + \mathfrak{a}_1 \Xi_5 + \mathfrak{a}_2 \Xi_6\},$
 $62 \rightarrow \{\Xi_2 + \alpha_1 \Xi_3 + \alpha_2 \Xi_4 + \mathfrak{a}_1 \Xi_5 + \mathfrak{a}_2 \Xi_6\}, 63 \rightarrow \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3 + \alpha_3 \Xi_4 + \alpha_4 \Xi_5 + \alpha_5 \Xi_6\}$



Out[9]=



```
In[10]:= Timing[alg2 = SubAlgebra[cs, pars, 2];]
```

There are 60 2-D families of subalgebras to be analyzed.

Done.

Out[10]=

```
{8405.31, Null}
```

```
In[11]:= PrintOptimal[alg2]
```

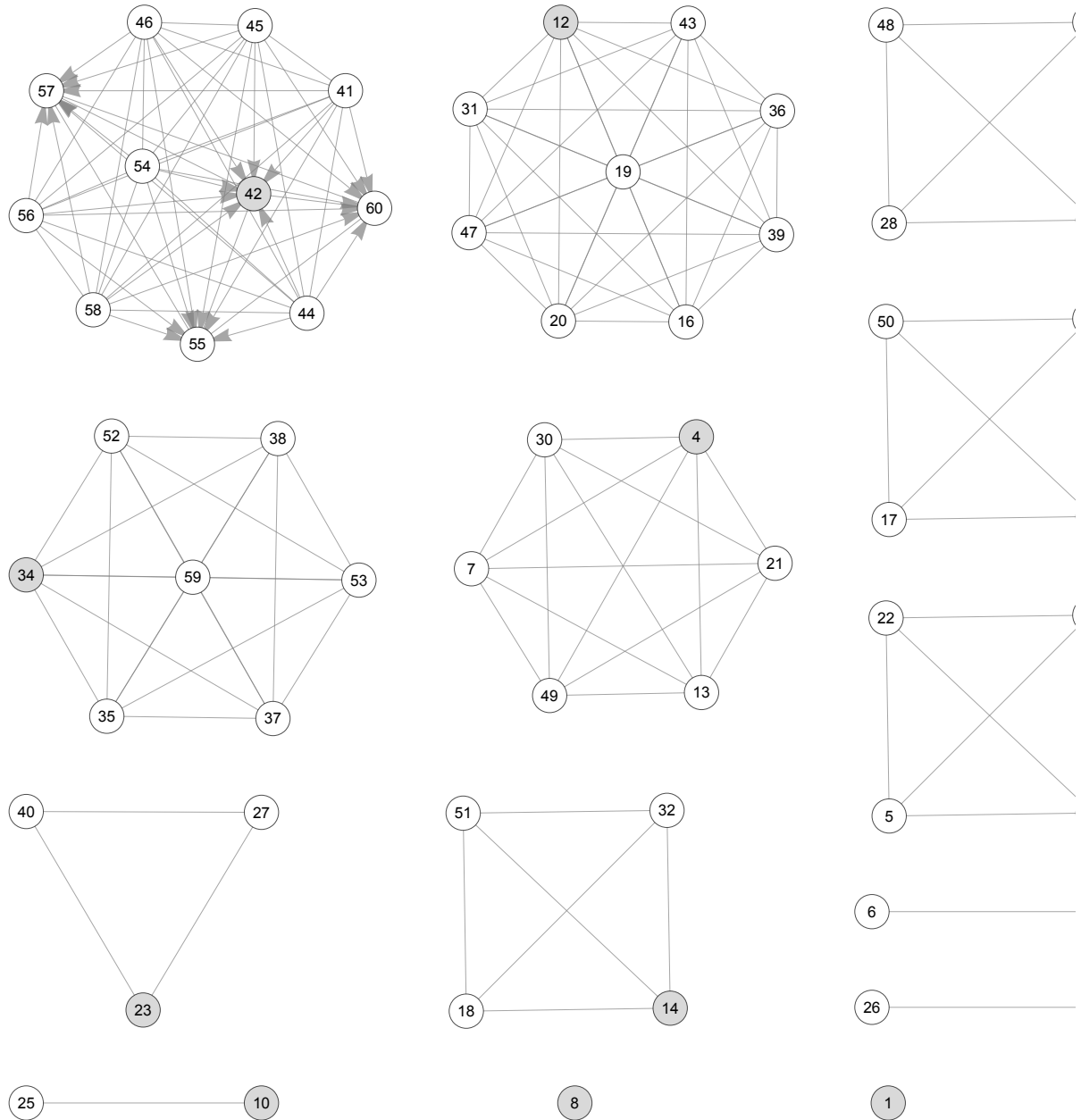
There are 14 optimal families of 2-dimensional Lie subalgebras.

Out[11]=

```
{ {e1, e2}, {e1, e3}, {e1, e4}, {e1, e6}, {e3, e4}, {e5, e6},
  {e1, e2 + a1 e3}, {e1, e3 + a1 e4}, {e1, e3 + a1 e6}, {e1, e4 + a1 e6},
  {e1 + a1 e3, e4}, {e2 + a1 e3, e4}, {e1, e3 + a1 e4 + a2 e6}, {e1 + a1 e4, e2 + a2 e3} }
```

```
In[12]:= PrintGraph[alg2, 1]
```


Out[12]=



```
In[13]:= Timing[alg3 = SubAlgebra[cs, pars, 3];]
```

There are 46 3-D families of subalgebras to be analyzed.

Done.

Out[13]=

{17579.7, Null}

```
In[14]:= PrintOptimal[alg3]
```

There are 12 optimal families of 3-dimensional Lie subalgebras.

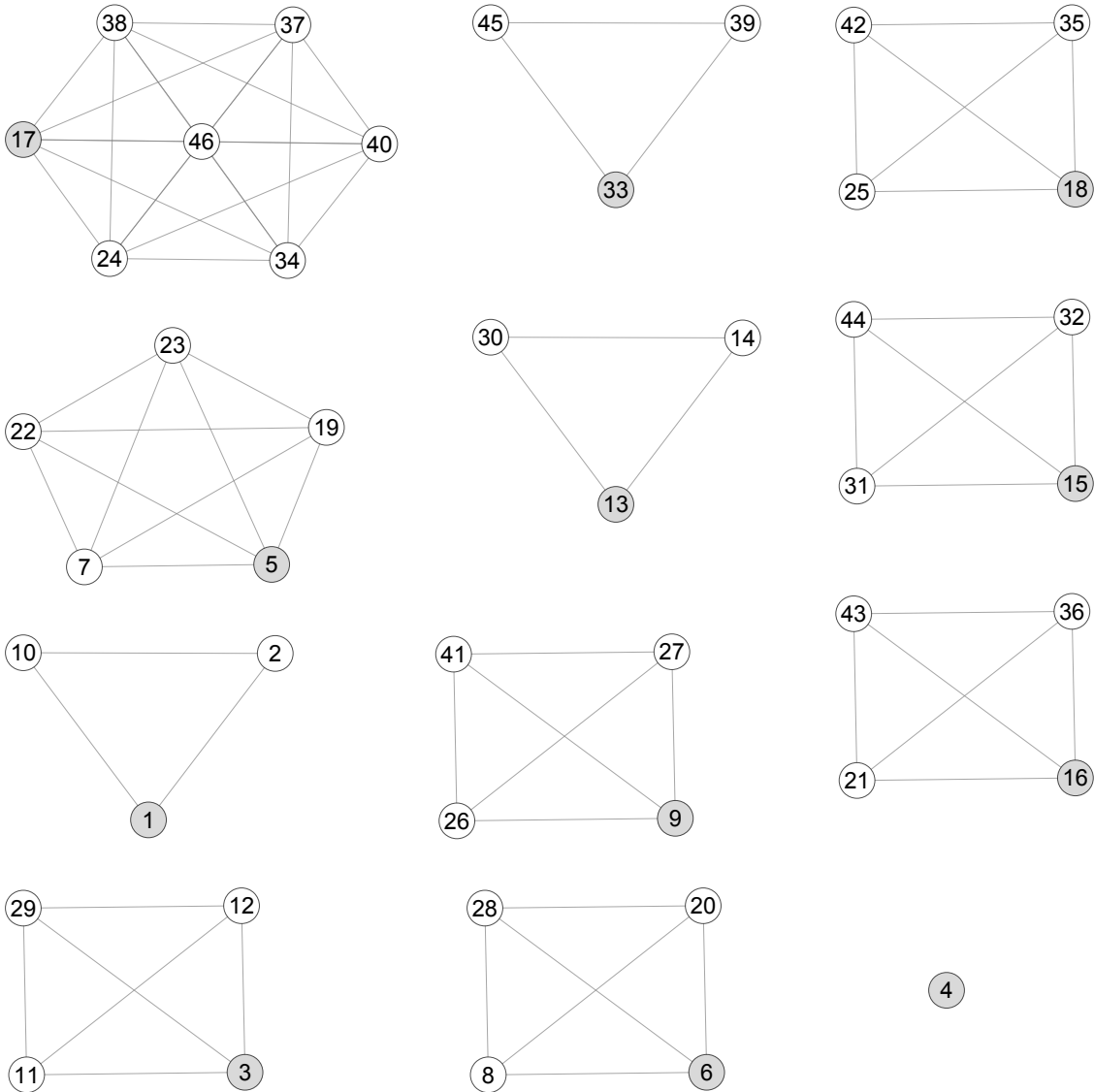
Out[14]=

{ $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_5\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_6\}$, $\{\mathfrak{e}_1, \mathfrak{e}_3, \mathfrak{e}_4\}$, $\{\mathfrak{e}_1, \mathfrak{e}_3, \mathfrak{e}_6\}$,
 $\{\mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_5\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 + a_1 \mathfrak{e}_6\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_5 + a_1 \mathfrak{e}_6\}$, $\{\mathfrak{e}_1, \mathfrak{e}_3, \mathfrak{e}_4 + a_1 \mathfrak{e}_6\}$,
 $\{\mathfrak{e}_1, \mathfrak{e}_2 + a_1 \mathfrak{e}_3, \mathfrak{e}_4\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2 + a_1 \mathfrak{e}_3, \mathfrak{e}_6\}$, $\{\mathfrak{e}_1, \mathfrak{e}_2 + a_1 \mathfrak{e}_3, \mathfrak{e}_4 + a_2 \mathfrak{e}_6\}$ }

In[15]:= **PrintGraph[alg3, 1]**

$1 \rightarrow \{E_1, E_2, E_3\}$, $2 \rightarrow \{E_1, E_2, E_4\}$, $3 \rightarrow \{E_1, E_2, E_5\}$, $4 \rightarrow \{E_1, E_2, E_6\}$,
 $5 \rightarrow \{E_1, E_3, E_4\}$, $6 \rightarrow \{E_1, E_3, E_6\}$, $7 \rightarrow \{E_2, E_3, E_4\}$, $8 \rightarrow \{E_2, E_4, E_6\}$, $9 \rightarrow \{E_3, E_4, E_5\}$,
 $10 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_4\}$, $11 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_5\}$, $12 \rightarrow \{E_1, E_2, E_4 + \alpha_1 E_5\}$,
 $13 \rightarrow \{E_1, E_2, E_3 + a_1 E_6\}$, $14 \rightarrow \{E_1, E_2, E_4 + a_1 E_6\}$, $15 \rightarrow \{E_1, E_2, E_5 + a_1 E_6\}$,
 $16 \rightarrow \{E_1, E_3, E_4 + a_1 E_6\}$, $17 \rightarrow \{E_1, E_2 + a_1 E_3, E_4\}$, $18 \rightarrow \{E_1, E_2 + a_1 E_3, E_6\}$,
 $19 \rightarrow \{E_1, E_2 + \alpha_1 E_4, E_3\}$, $20 \rightarrow \{E_1, E_2 + \alpha_1 E_6, E_3\}$, $21 \rightarrow \{E_2, E_3 + a_1 E_6, E_4\}$,
 $22 \rightarrow \{E_1 + \alpha_1 E_2, E_3, E_4\}$, $23 \rightarrow \{E_1 + \alpha_1 E_3, E_2, E_4\}$, $24 \rightarrow \{E_1 + a_1 E_4, E_2, E_3\}$,
 $25 \rightarrow \{E_1 + a_1 E_4, E_2, E_6\}$, $26 \rightarrow \{E_1 + \alpha_1 E_5, E_3, E_4\}$, $27 \rightarrow \{E_2 + \alpha_1 E_5, E_3, E_4\}$,
 $28 \rightarrow \{E_1 + \alpha_1 E_6, E_2, E_4\}$, $29 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_4 + \alpha_2 E_5\}$, $30 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_4 + a_1 E_6\}$,
 $31 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_5 + a_1 E_6\}$, $32 \rightarrow \{E_1, E_2, E_4 + \alpha_1 E_5 + a_1 E_6\}$, $33 \rightarrow \{E_1, E_2 + a_1 E_3, E_4 + a_2 E_6\}$,
 $34 \rightarrow \{E_1, E_2 + \alpha_1 E_4, E_3 + a_1 E_4\}$, $35 \rightarrow \{E_1, E_2 + \alpha_1 E_6, E_3 + a_1 E_6\}$, $36 \rightarrow \{E_1, E_2 + \alpha_1 E_4 + \alpha_2 E_6, E_3\}$,
 $37 \rightarrow \{E_1 + a_1 E_3, E_2 + a_2 E_3, E_4\}$, $38 \rightarrow \{E_1 + a_1 E_4, E_2, E_3 + \alpha_1 E_4\}$, $39 \rightarrow \{E_1 + a_1 E_4, E_2, E_3 + a_2 E_6\}$,
 $40 \rightarrow \{E_1 + a_1 E_4, E_2 + a_2 E_4, E_3\}$, $41 \rightarrow \{E_1 + \alpha_1 E_2 + \alpha_2 E_5, E_3, E_4\}$, $42 \rightarrow \{E_1 + \alpha_1 E_6, E_2, E_4 + a_1 E_6\}$,
 $43 \rightarrow \{E_1 + \alpha_1 E_3 + \alpha_2 E_6, E_2, E_4\}$, $44 \rightarrow \{E_1, E_2, E_3 + \alpha_1 E_4 + \alpha_2 E_5 + \alpha_3 E_6\}$,
 $45 \rightarrow \{E_1 + a_1 E_4, E_2, E_3 + \alpha_1 E_4 + a_2 E_6\}$, $46 \rightarrow \{E_1 + \alpha_1 E_4, E_2 + \alpha_2 E_4, E_3 + a_1 E_4\}$

Out[15]=



In[16]:= **Timing[alg4 = SubAlgebra[cs, pars, 4];]**

There are 11 4-D families of subalgebras to be analyzed.

Done.

Out[16]=

{3667.72, Null}

In[17]:= **PrintOptimal[alg4]**

There are 4 optimal families of 4-dimensional Lie subalgebras.

Out[17]=

$\{\{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4\}, \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_6\}, \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_5, \mathfrak{E}_6\}, \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \alpha_1 \mathfrak{E}_6, \mathfrak{E}_4 + \mathfrak{a}_1 \mathfrak{E}_6\}\}$

In[18]:= **PrintGraph[alg4, 1]**

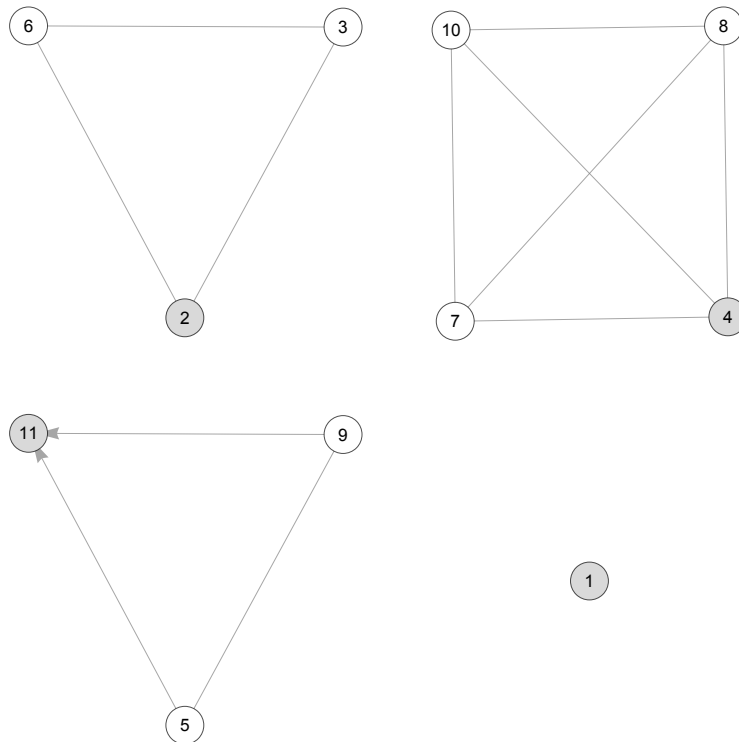
$1 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4\}, 2 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_6\}, 3 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_4, \mathfrak{E}_6\},$

$4 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_5, \mathfrak{E}_6\}, 5 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4 + \mathfrak{a}_1 \mathfrak{E}_6\}, 6 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \alpha_1 \mathfrak{E}_4, \mathfrak{E}_6\},$

$7 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \alpha_1 \mathfrak{E}_5, \mathfrak{E}_6\}, 8 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_4 + \alpha_1 \mathfrak{E}_5, \mathfrak{E}_6\}, 9 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \mathfrak{a}_1 \mathfrak{E}_6, \mathfrak{E}_4\},$

$10 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \alpha_1 \mathfrak{E}_4 + \alpha_2 \mathfrak{E}_5, \mathfrak{E}_6\}, 11 \rightarrow \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3 + \alpha_1 \mathfrak{E}_6, \mathfrak{E}_4 + \mathfrak{a}_1 \mathfrak{E}_6\}$

Out[18]=



In[19]:= **Timing[alg5 = SubAlgebra[cs, pars, 5];]**

There are 3 5-D families of subalgebras to be analyzed.

Done.

Out[19]=

{19705., Null}

In[20]:= **PrintOptimal[alg5]**

There are 3 optimal families of 5-dimensional Lie subalgebras.

Out[20]=

$\{\{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4, \mathfrak{E}_5\}, \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4, \mathfrak{E}_6\}, \{\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4, \mathfrak{E}_5 + \mathfrak{a}_1 \mathfrak{E}_6\}\}$


```

{{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, {{1, 2, 4, 8, 16},
{1, 2, 4, 8, 32}, {1, 2, 4, 8, 48}},
{{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, -1},
{0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}},
{{0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, -1}, {1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0}},
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 1, 0}, {0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0},
{0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}}}

```

In[23]:= SessionTime[]

Out[23]=

55.233.206726