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A SIMILARITY APPROACH TO THE NUMERICAL SOLUTION OF FREE BOUNDARY PROBLEMS*

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Abstract. The aim of this work is to point out that within a similarity approach some classes of free boundary value problems governed by ordinary differential equations can be transformed to initial value problems. The interest in the numerical solution of free boundary problems arises because these are always nonlinear problems. Furthermore we show that free boundary problems arise also via a similarity analysis of moving boundary hyperbolic problems and they can be obtained as approximations of boundary value problems defined on infinite intervals. Most of the theoretical content of this survey is original: it generalizes and unifies results already available in literature. As far as applications of the proposed approach are concerned, three problems of interest are considered and numerical results for each of them are reported.

Key words. numerical solution, similarity approach, free boundary problems

AMS subject classifications. 65L10, 34B15, 22E05, 35R35, 65P05, 76D10

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1. Introduction.

1.1. Aim of the survey. In this paper we survey the latest developments of a similarity approach to the numerical solution of free boundary value problems (BVPs) governed by ordinary differential equations (ODEs). The intended scientific readership is assumed to be formed not only by applied mathematicians with a broad knowledge of the group invariance theory, but also by physicists or by engineers who have to solve free or moving BVPs and are looking for simple resolution approaches.

In the classical numerical treatment of free BVPs a preliminary reduction to BVPs is introduced by considering a new independent variable (see, Stoer and Bulirsch [71, p. 468], Ascher and Russell [6], or Ascher, Mattheij, and Russell [5, p. 471]). Therefore, in that way free BVPs are BVPs. The first goal of this paper is to point out that within a similarity approach some classes of free BVPs are indeed initial VPs. As a consequence, when we are able to formulate BVPs on infinite intervals as free BVPs, BVPs on infinite intervals are in the same way initial VPs (this is, for instance, the case of several problems of interest in fluid dynamics as shown by Fazio [26, 32, 34]).

1.2. Free or moving BVPs. Free or moving BVPs arise in several branches of applied mathematics. Perhaps the oldest problem of this type was treated by Isaac Newton, in book II of his great "Principia Mathematica" of 1687, by considering the optimal nose-cone shape for the motion of a projectile subject to air resistance. The difficulties inherent in these problems represent an analytical as well as a numerical challenge because free or moving BVPs are always nonlinear (this was pointed out by Landau [52]). Free BVPs governed by ODEs describe real-world phenomena in biology, chemistry, mechanics, etc. Among others we can list the absorption and diffusion of oxygen in a cellular tissue (Epperson [21]), the optimal length estimation

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for tubular chemical reactors (Fazio [29, 25]), a contact problem for a string (Collatz [14]), the duration of motion for a mass subject to a nonlinear force (Meyer [55, pp. 97–98] and Na [58, pp. 88–90]), and the infinite journal bearings in lubrication theory (Pinkus and Sternlicht [63, p. 41 and p. 46]). Moreover, free BVPs governed by ODEs are derived by a similarity analysis of hyperbolic or parabolic moving BVPs. Hence, we can also list the following phenomena described by moving BVPs that can be reduced to free BVPs: phase change in heat transfer (as reported by Crank [16, pp. 2–9]), blast waves in gas dynamics (Sedov [66, Chap. IV] and Taylor [73]), thermal waves from an instantaneous plane source (Zel'dovich and Raizer [77, pp. 664–665]), transverse waves in shock-loaded elastic membrane (Dresner [19]), spreading of a viscous fluid under the influence of gravity (Meek and Norbury [54]), longitudinal waves due to a stress (Singh and Frydrychowicz [69]) or a velocity impact (Singh and Frydrychowicz [70]), and the dynamics of liquid films on a two-dimensional stretching surface (the steady and the unsteady case were studied, respectively, by Crane [15] and by Wang [75]).

1.3. Background. Similarity analysis pertains to Lie's group invariance theory [53], and it is a generalization of the classical dimensional analysis (as shown by Bluman and Kumei [11, pp. 22–27]). Similarity analysis can be considered as a fundamental tool of applied mathematics (see the following books: Birkhoff [9], Sedov [66], Hansen [44], Bluman and Cole [10], Ovsiannikov [62], Barenblatt [7], Hill [45], Dresner [20], Seshadri and Na [68], Olver [60], Bluman and Kumei [11], Olver [61], and Ibragimov [47, 48, 49]). For a discussion on the relevance of Lie's theory see Howe [46].

The application of group invariance theory to the transformation of BVPs to initial VPs is not a new topic. By considering a series solution for the Blasius problem Töpfer [74] found a transformation in 1912 that reduces that BVP to a pair of initial VPs. This result was quoted by the fluid dynamics literature (see, for instance, Goldstein [42, pp. 135–136]). Acrivos, Shah, and Petersen [1] in 1960 and Klamkin [50] in 1962 extended Töpfer's noniterative method to study more general classes of problems. The relationship between the invariance of the Blasius problem with respect to a linear group of transformations (the stretching group) and the applicability of a noniterative method was pointed out in 1967 by Na [56]. Moreover, in the same paper Na considered BVPs on finite intervals and the invariance with respect to a nonlinear group of transformations (the spiral group). A survey paper on this subject was written in 1970 by Klamkin [51].

Belford [8] in 1969 and Ames and Adams [3] in 1976 defined noniterative methods for the numerical solution of eigenvalue problems (see also the survey paper by Ames [2]). Na [58, Chap. 7–9] in 1979 devoted three chapters of his book to the noniterative numerical solution of BVPs. The contrast between the classical inspectional analysis and an infinitesimal group method was discussed in 1985 by Seshadri and Na [68, pp. 157–168]. Later, Fazio and Evans [37] used a similarity approach to define a noniterative method for the numerical solution of free BVPs.

As pointed out already in 1969 by Fox, Erickson, and Fan [39], in 1979 by Na [58, p. 137], and recently by Sachdev [64, p. 218]), noniterative methods can be applied only to specific classes of problems. To overcome this drawback several extensions have been proposed in literature. The first extension is related to the application of a different group of transformation (the spiral one) and was introduced by Na [56] in 1967. A practical application was given by Na and Tang [59] two years later. The transformation within a stretching group of a physical parameter was proposed as a second extension in 1970 by Na [57], and two years later the case of two or more physical parameters was considered by Scott, Rinschler, and Na [65]. A different way to extend noniterative methods, namely, by defining a transformation of variables linking two different groups of transformations, was proposed in 1990 by Fazio [23, 24]. Moreover, in the same way it is possible to prove that the utilization of stretching or of spiral group is equivalent (see [31] for details). Finally, an iterative extension applicable to general classes of problems has been recently developed by Fazio [27, 30, 33, 34].

The transformation of BVPs to initial VPs also has a theoretical relevance. In fact, existence and uniqueness results can be obtained as a consequence of the invariance properties. For instance, a simple existence and uniqueness theorem for the Blasius problem was given by J. Serrin as reported by Meyer [55, pp. 104–105]. On this topic a first application of a numerical test, defined within the proposed approach, to verify the existence and uniqueness of the solution of a free BVPs was considered by Fazio in [25]. A formal definition of the mentioned numerical test can be found in [35].

1.4. Organization of the survey. The remainder of the paper is organized as follows. In the next section we define the most general class of free BVPs governed by ODEs and we work out the reduction to that class of moving BVPs of hyperbolic type as well as BVPs defined on infinite intervals (provided a further independent boundary condition is available). Motivated by similarity considerations we define, in section 3, the transformation of two relevant classes of free BVPs to sequences of initial VPs. Moreover, we characterize classes of free BVPs that can be solved noniteratively via a transformation to initial value problems. The related noniterative method can be used instead of iterative methods for free BVPs (see the application of a finite difference method by Cryer [17] or by Fazio [36]).

The content of sections 2 and 3 is original and generalizes results available in literature. For the sake of brevity all the proofs are only outlined. At a first reading some readers may wish to skim section 3 and come back to it after going through section 4.

In section 4 we report the numerical results obtained via the methods defined in the previous section for three problems belonging to the characterized classes. These problems are related with the shock front propagation in a bar of rate-type material due to a stress impact (Fazio [28]), the Blasius problem (Fazio [26]), and the fluid flow past a slender parabola in boundary layer theory (Fazio [34]). The relevance of these problems will be discussed in the last section. Moreover, in the same section we indicate some topics related to the similarity approach that were omitted for brevity.

2. Free boundary problems.

A free BVP is a nonlinear BVP with an unknown boundary (the free boundary) that has to be found as part of the solution. Let us consider the most general free BVP governed by ODEs

$$\begin{aligned} &\frac{d\mathbf{y}}{dz} = \mathbf{f}(z, \mathbf{y}) \ , \quad z \in [0, s] \ , \\ &\mathbf{g}(\mathbf{y}(0), s, \mathbf{y}(s)) = \mathbf{0} \ , \end{aligned} \tag{2.1}$$

where $\mathbf{y} : [0, s] \to \mathbb{R}^n$, $\mathbf{f} : [0, s] \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n+1}$, $\mathbf{0}$ is the zero vector in $\mathbb{R}^{n+1}$, and s is the free boundary. The simplest free BVP is obtained from (2.1) by setting $n = 1$, and in that case we have to specify two boundary conditions; a simple example is reported in the next subsection.

Let us list here some topics of interest which led to a free BVP formulation: similarity reduction of moving BVPs and BVPs on infinite intervals with an auxiliary boundary condition at infinity.

2.1. Similarity reduction of moving BVPs. A moving BVP is a nonlinear initial-BVP with a moving boundary whose position has to be determined as part of the solution. In this subsection we consider the class of moving BVPs governed by

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\left[\mathbf{f}(x,t,\mathbf{u})\right] = \mathbf{g}(x,t,\mathbf{u}), \\
&x_f(0) = a, \qquad \mathbf{u}(x,0) = \begin{cases} \mathbf{u_0}(x) & \text{for} \quad 0 \le x \le a, \\ \mathbf{0} & \text{for} \quad a < x, \end{cases} \\
&\mathbf{h}\left(t, \mathbf{u}(0,t), x_f(t), \frac{dx_f}{dt}(t), \mathbf{u}(x_f(t),t)\right) = \mathbf{0},
\end{aligned} \tag{2.2}
$$

where $\mathbf{f}, \mathbf{g} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{u_0} : \mathbb{R}^+ \to \mathbb{R}^n$, $\mathbf{h} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+1}$, and $x_f(t)$ is the unknown moving boundary.

It is possible to characterize all the problems belonging to (2.2) that are invariant with respect to the stretching group of transformations

$$
\begin{aligned}
&u_i^* = \mu^{\alpha_i} u_i \quad \text{for} \quad i = 1, \ldots, n \;, \\
&{x_f}^* = \mu^{\gamma} x_f \;, \quad x^* = \mu^{\gamma} x \;, \quad t^* = \mu t \;.
\end{aligned} \tag{2.3}
$$

A simple way to proceed is to apply (2.3) to (2.2) written in the starred variables so that we get the relations

$$
\begin{aligned}
&\mu^{\alpha_i - 1}\frac{\partial u_i}{\partial t} + \mu^{-\gamma}\frac{\partial}{\partial x}\left[f_i(\mu^{\gamma}x, \mu t, \mu^{\alpha_1}u_1, \ldots, \mu^{\alpha_n}u_n)\right] = g_i(\mu^{\gamma}x, \mu t, \mu^{\alpha_1}u_1, \ldots, \mu^{\alpha_n}u_n) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 1, \ldots, n \;, \\
&\mu^{\gamma} x_f(0) = a, \quad \mu^{\alpha_i} u_i(\mu^{\gamma}x, 0) = \begin{cases} u_{0i}(\mu^{\gamma}x) & \text{for} \quad 0 \le x \le a, \\ 0 & \text{for} \quad a < x, \end{cases} \quad \text{for} \quad i = 1, \ldots, n \;, \\
&\mathbf{h}\left(\mu t, \mu^{\alpha_1}u_1(0,t), \ldots, \mu^{\gamma}x_f(t), \mu^{\gamma-1}\frac{dx_f}{dt}(t), \mu^{\alpha_1}u_1(x_f(t),t), \ldots\right) = \mathbf{0}
\end{aligned}
$$

that have to be independent from μ (note that the first ellipsis in the arguments of $\mathbf{h}(\cdot, \ldots, \cdot)$ should be replaced by the terms $\mu^{\alpha_2}u_2(0,t), \ldots, \mu^{\alpha_n}u_n(0,t)$ as well as the last one is used instead of the terms $\mu^{\alpha_2}u_2(x_f(t),t), \ldots, \mu^{\alpha_n}u_n(x_f(t),t)$). To this end we require that the arbitrary functions are transformed also. For instance, as far as the governing differential system is concerned the following conditions must hold:

$$
f_i^* = \mu^{\alpha_i + \gamma - 1} f_i \;, \quad g_i^* = \mu^{\alpha_i - 1} g_i \;, \quad \text{for} \quad i = 1, \ldots, n \;,
$$

or, in a different form,

$$
\begin{aligned}
f_i(\mu^{\gamma}x, \mu t, \mu^{\alpha_1}u_1, \ldots, \mu^{\alpha_n}u_n) &= \mu^{\alpha_i + \gamma - 1} f_i \;, \\
&\qquad\qquad \text{for} \quad i = 1, \ldots, n \;, \\
g_i(\mu^{\gamma}x, \mu t, \mu^{\alpha_1}u_1, \ldots, \mu^{\alpha_n}u_n) &= \mu^{\alpha_i - 1} g_i \;.
\end{aligned}
$$

Having made the above assumptions we can proceed in the classical way (see Cohen [13]). By differentiating the obtained relations for f_i and g_i for $i = 1, \ldots, n$ with respect to μ and setting $\mu = 1$ we get a system of first order partial differential equations (DEs). By integrating the related characteristic equations we find the following functional forms:

$$
\begin{aligned}
f_i(x, t, u_1, \ldots, u_n) &= t^{\alpha_i + \gamma - 1} F_i(t^{-\gamma}x, t^{-\alpha_1}u_1, \ldots, t^{-\alpha_n}u_n) \;, \\
&\qquad\qquad \text{for} \quad i = 1, \ldots, n \;, \\
g_i(x, t, u_1, \ldots, u_n) &= t^{\alpha_i - 1} G_i(t^{-\gamma}x, t^{-\alpha_1}u_1, \ldots, t^{-\alpha_n}u_n) \;.
\end{aligned}
$$

Also, a and $\mathbf{h}(t, u_1(0,t), \ldots, u_n(0,t), x_f(t), \frac{dx_f}{dt}(t), u_1(x_f(t),t), \ldots, u_n(x_f(t),t))$ can be characterized in a similar way in order for the invariance to hold

$$a = 0,$$
$$\mathbf{h}(\cdot, \ldots, \cdot) = \mathbf{H}\left(t^{-\alpha_1} u_1(0,t), \ldots, t^{-\gamma} x_f(t), t^{1-\gamma} \frac{dx_f}{dt}(t), t^{-\alpha_1} u_1(x_f(t),t), \ldots\right) .$$

Here we used the first ellipsis in $\mathbf{H}(\cdot, \ldots, \cdot)$ instead of $t^{-\alpha_2} u_2(0,t), \ldots, t^{-\alpha_n} u_n(0,t)$ and the last one instead of $t^{-\alpha_2} u_2(x_f(t),t), \ldots, t^{-\alpha_n} u_n(x_f(t),t)$. Let us note that all the introduced functions $(\mathbf{F}, \mathbf{G}, \mathbf{H})$ depend only on the following invariants:

$$\eta = t^{-\gamma} x \ , \quad \eta_f = t^{-\gamma} x_f \ , \quad \gamma\eta_f = t^{1-\gamma} \frac{dx_f}{dt} \ ,$$
$$U_i(\eta) = t^{-\alpha_i} u_i(x,t) \quad \text{for} \quad i = 1, \ldots, n \ .$$

As a consequence, taking into account that

$$\frac{\partial}{\partial t} = -\gamma t^{-1} \eta \frac{d}{d\eta} \ , \quad \frac{\partial}{\partial x} = t^{-\gamma} \frac{d}{d\eta} \ ,$$

we are allowed to rewrite the characterized class of moving BVPs

$$\begin{gathered} \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x}\left[t^{\alpha_i+\gamma-1} F_i\left(t^{-\gamma} x, t^{-\alpha_1} u_1, \ldots\right)\right] = t^{\alpha_i - 1} G_i\left(t^{-\gamma} x, t^{-\alpha_1} u_1, \ldots\right) \\ \text{for} \quad i = 1, \ldots, n, \\ x_f(0) = 0, \\ \mathbf{H}\left(t^{-\alpha_1} u_1(0,t), \ldots, t^{-\gamma} x_f(t), t^{1-\gamma} \frac{dx_f}{dt}(t), t^{-\alpha_1} u_1(x_f(t),t), \ldots,\right) = \mathbf{0} \end{gathered} \tag{2.4}$$

as free BVPs

$$\begin{gathered} \left[\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\eta, \mathbf{U}) - \gamma\eta \mathrm{I}\right] \frac{d\mathbf{U}}{d\eta} + \frac{\partial \mathbf{F}}{\partial \eta}(\eta, \mathbf{U}) = \mathbf{G}(\eta, \mathbf{U}), \\ \mathbf{H}(\mathbf{U}(0), \eta_f, \gamma\eta_f, \mathbf{U}(\eta_f)) = \mathbf{0} \ , \end{gathered} \tag{2.5}$$

where $\partial \mathbf{F}/\partial \mathbf{U}$ and I are the $n \times n$ Jacobian and identity matrix, respectively, $\mathbf{U}(\eta) = (U_1(\eta), \ldots, U_n(\eta))$ is defined on $[0, \eta_f]$ and η_f is unknown. If the matrix $\frac{\partial \mathbf{F}}{\partial \mathbf{U}} - \gamma\eta \mathrm{I}$ is nonsingular, (2.5) can be rewritten as

$$\begin{gathered} \frac{d\mathbf{U}}{d\eta} = \left[\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\eta, \mathbf{U}) - \gamma\eta \mathrm{I}\right]^{-1} \left[\mathbf{G}(\eta, \mathbf{U}) - \frac{\partial \mathbf{F}}{\partial \eta}(\eta, \mathbf{U})\right], \\ \mathbf{H}(\mathbf{U}(0), \eta_f, \gamma\eta_f, \mathbf{U}(\eta_f)) = \mathbf{0} \ . \end{gathered} \tag{2.6}$$

The points where the above mentioned matrix is singular define lines of discontinuity for the solution of (2.4).

As far as the governing differential system is concerned the proposed similarity analysis is due to Donato [18] and Ames and Donato [4]; here we have also taken into account the related moving boundary conditions. We note that (2.6) belongs to the free boundary model (2.1).

Let us summarize what we have found hitherto:

1. if a one-dimensional moving BVP admits a similarity solution, then it can be transformed into an ordinary differential problem;

2. the geometric effect of the similarity transformation can be explained in the following way: each similarity line given by $t^{-\gamma}x = constant$ in the region between $t^{-\gamma}x = 0$ and $t^{-\gamma}x_f(t) = \eta_f$ is mapped into a point laying between $\eta = 0$ and $\eta = \eta_f$;

3. finally, the functional form of the arbitrary functions $\mathbf{F}(\cdot, \ldots, \cdot)$, $\mathbf{G}(\cdot, \ldots, \cdot)$, and $\mathbf{H}(\cdot, \ldots, \cdot)$ as well as the values of α_i for $i = 1, \ldots, n$ and γ would be defined by the particular problem under consideration.

To be more specific about this last point let us consider the following example.

Example: Rainfall runoff on an impermeable surface. The simple model considered hereafter was used by Seshadri and Jagannathan in [67] to model the buildup of laminar or turbulent flow over a sloping area

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[qu^m\right] = v_0, \\ &x_f(0) = 0, \\ &u(0,t) = 0 \ , \quad u(x_f(t),t) = v_0 t, \end{aligned} \tag{2.7}$$

here x is the coordinate along the sloping plane, u is the height of the water surface above that plane, the flux is assumed to be proportional to a power of the elevation of the water (q and m are parameters related to the type of rainfall), and v_0 is the constant velocity of rainfall. As it is easily verified the moving BVP (2.7) belongs to the class of problems (2.4) (for $n = 1$). This problem can be rewritten in dimensionless form by using the new variables

$$x' = qT^m v_0^{m-1} x \ , \quad {x_f}' = qT^m v_0^{m-1} x_f \ , \quad t' = T^{-1} t \ , \quad u' = T^{-1} v_0^{-1} u,$$

where T is a reference time. A different way to work with dimensionless units is to consider the similarity variables

$$\eta = q^{-1} m^{-1} v_0^{1-m} t^{-m} x \ , \quad \eta_f = q^{-1} m^{-1} v_0^{1-m} t^{-m} x_f(t) \ , \quad U(\eta) = v_0^{-1} t^{-1} u(x,t)$$

to end up with the free BVP

$$\begin{aligned} &\left(U^{m-1} - m\eta\right) \frac{dU}{d\eta} - U = 1, \\ &U(0) = 0 \ , \quad U(\eta_f) = 1 \ . \end{aligned}$$

2.2. Free boundary formulation of BVPs on infinite intervals. Let us consider the BVP defined on an infinite interval

$$\begin{aligned} &\frac{d\mathbf{y}}{dz} = \mathbf{p}\left(z, \mathbf{y}\right) \ , \quad z \in [0, \infty) \ , \\ &\mathbf{q}\left(\mathbf{y}(0), \mathbf{y}(\infty)\right) = \mathbf{0} \ , \end{aligned} \tag{2.8}$$

where $\mathbf{y}(z)$ is an n-dimensional vector with $y_\ell(z)$ for $\ell = 1, \ldots, n$ as components, $\mathbf{p} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, and $\mathbf{q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Often in applications we are allowed to require some regular behavior of the solution of (2.8) at infinity and consequently one or more independent boundary conditions are available (see Ascher, Mattheij, and Russell [5, p. 486]). Independent boundary conditions may also be defined by applying physical considerations. Here we assume that at least one independent boundary condition is available

$$r(\mathbf{y}(0), \mathbf{y}(\infty)) = 0 \ , \tag{2.9}$$

where $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. In that case a free boundary formulation of (2.8) is as follows:

$$\begin{aligned} &\frac{d\mathbf{y}_\epsilon}{dz} = \mathbf{p}\left(z, \mathbf{y}_\epsilon\right) \ , \quad z \in [0, z_\epsilon] \ , \\ &\mathbf{q}\left(\mathbf{y}_\epsilon(0), \mathbf{y}_\epsilon(z_\epsilon)\right) = \mathbf{0} \ , \\ &r\left(\mathbf{y}_\epsilon(0), \mathbf{y}_\epsilon(z_\epsilon)\right) = \epsilon \ , \end{aligned} \tag{2.10}$$

where the subscript indicates that the solution of (2.10) depends on ϵ, $0 < |\epsilon| \ll 1$, $\mathbf{y}_\epsilon(z)$ is defined on $[0, z_\epsilon]$, and z_ϵ is unknown.

Within the defined framework, the following convergence result holds.

THEOREM 1. *Let z_ϵ be a differentiable function of ϵ on a neighborhood of $\epsilon = 0$, and let the limit of $\frac{dz_\epsilon}{d\epsilon}$ as $\epsilon \to 0$ exist. Further, let all the components of $\mathbf{y}_\epsilon(z)$ and of $\frac{d\mathbf{y}_\epsilon}{d\epsilon}(z)$ be continuous functions on the domain defined by the Cartesian product of $[0, z_\epsilon]$ and a neighborhood of $\epsilon = 0$. Then the solution of* (2.10) *converges to the solution of* (2.8) *as ϵ goes to zero.*

This theorem generalizes Theorem 1 in [34] and its proof can be developed along the lines of the proof of that theorem. To apply the result of the above theorem we solve numerically (2.10) for several decreasing values of $|\epsilon|$. Moreover, we can verify numerically whether $z_\epsilon \to \infty$ as $\epsilon \to 0$.

Let us remark here that (2.10) belongs to the free boundary model (2.1).

3. Initial value methods.

In this section we generalize and unify some theoretical results obtained elsewhere.

3.1. Free BVPs of the first class.

Here we consider the following class of free BVPs:

$$\begin{aligned} &\frac{d\mathbf{y}}{dz} = \mathbf{f}(z, \mathbf{y}), \\ &y_k(0) = A_0 \qquad (k \in \{1, \ldots, n\}), \\ &y_i(s) = \ell_i(s) \qquad \text{for} \quad i = 1, \ldots, n \ . \end{aligned} \tag{3.1}$$

(3.1) is a subclass of (2.1) obtained by setting $g_i(\mathbf{y}(0), s, \mathbf{y}(s)) = y_i(s) - \ell_i(s)$ for $i = 1, \ldots, n$ and $g_{n+1}(\mathbf{y}(0), s, \mathbf{y}(s)) = y_k(0) - A_0$ where $k \in \{1, \ldots, n\}$.

In the following we define two initial value methods for the numerical solution of problems belonging to (3.1).

THEOREM 2. *The following class of free BVPs*

$$\begin{aligned} &\frac{dy_i}{dz} = z^{(\beta_i - \delta)/\delta} \Phi_i \left(z^{-\beta_1/\delta} y_1, \ldots, z^{-\beta_n/\delta} y_n \right) \qquad \textit{for} \quad i = 1, \ldots, n \ , \\ &y_k(0) = A_0 \qquad (k \in \{1, \ldots, n\}), \\ &y_i(s) = A_i s^{\beta_i/\delta} \qquad \textit{for} \quad i = 1, \ldots, n \ , \end{aligned} \tag{3.2}$$

where $\beta_k \neq 0, A_0 \neq 0$, and $\Phi_i(\cdot, \ldots, \cdot)$ for $i = 1, \ldots, n$ are arbitrary functions of their arguments, can be solved by a noniterative method.

Outline of the proof. The class of free BVPs (3.2) can be characterized from (3.1) by requiring the invariance of the governing DEs and of the boundary conditions at the free boundary with respect to the stretching group

$$z^* = \lambda^\delta z, \qquad s^* = \lambda^\delta s, \qquad y_i^* = \lambda^{\beta_i} y_i \quad \text{for} \quad i = 1, \ldots, n \ .$$

The noniterative method is defined by the following steps:

1. let δ and β_i for $i = 1, \ldots, n$ be fixed owing to the above invariance requirement, otherwise we set their values at our convenience (e.g., see the Blasius problem discussed in the next section);

2. we fix a value for s^*, this defines the values of $y_i^*(s^*)$ for $i = 1, \ldots, n$ according to the boundary conditions at s^*;

3. the governing DEs in the starred variables can be solved inwards on $[0, s^*]$ to get the value of $y_k^*(0)$;

4. as a consequence of the similarity properties we get

$$\lambda = [y_k^*(0)/A_0]^{1/\beta_k} \ ,$$
$$y_i(z) = \lambda^{-\beta_i} y_i^*(z^*) \qquad \text{for} \quad i = 1, \ldots, n \ ,$$
$$s = \lambda^{-\delta} s^* \ ,$$

where $z \in [0, s]$ and $z^* \in [0, s^*]$. $\square$

The noniterative method defined above is not widely applicable and this is an obvious limit. To overcome this limit we can introduce the following iterative method. We will prove the convergence of the iterated solution to a solution of the free BVP provided the imbedding parameter h, introduced below to define the iterative method, goes to one.

THEOREM 3. *The class of free BVPs* (3.1)

$$\frac{d\mathbf{y}}{dz} = \mathbf{f}(z, \mathbf{y}),$$
$$y_k(0) = A_0 \quad (k \in \{1, \ldots, n\}),$$
$$y_i(s) = \ell_i(s) \qquad \textit{for} \quad i = 1, \ldots, n$$

can be solved numerically by an iterative method.

Outline of the proof. First we introduce the class of free BVPs

$$\begin{aligned} \frac{dy_i}{dz} &= h^{(\beta_i - \delta)/\sigma} f_i(h^{-\delta/\sigma} z, h^{-\beta_1/\sigma} y_1, \ldots, h^{-\beta_n/\sigma} y_n) \qquad \text{for} \quad i = 1, \ldots, n, \\ y_k(0) &= A_0 \qquad (k \in \{1, \ldots, n\}), \\ y_i(s) &= h^{\beta_i/\sigma} \ell_i(h^{-\delta/\sigma} s) \qquad \text{for} \quad i = 1, \ldots, n, \end{aligned} \tag{3.3}$$

where h is a numerical parameter. The governing differential equation and the boundary conditions at the free boundary in (3.3) are invariant with respect to the stretching group

$$z^* = \lambda^\delta z, \qquad s^* = \lambda^\delta s, \qquad y_i^* = \lambda^{\beta_i} y_i \quad \text{for} \quad i = 1, \ldots, n, \qquad h^* = \lambda^\sigma h.$$

The iterative method is defined by the following steps:

1. since (3.1) is obtained from (3.3) by setting $h = 1$, we iterate different values of h^* until we get, from (3.4) below, $|h - 1| \le \text{TOL}$ where TOL is a prefixed tolerance;

2. the values of δ, β_i for $i = 1, \ldots, n$ and σ can be set at our convenience;

3. the values of $y_i^*(s^*)$ for $i = 1, \ldots, n$ are defined according to the boundary conditions at s^* by fixing s^*;

4. the governing DEs in the starred variables can be solved inwards on $[0, s^*]$ to get the value of $y_k^*(0)$;

5. then we can apply

$$\lambda = [y_k^*(0)/A_0]^{1/\beta_k} \ ,$$
$$y_i(z) = \lambda^{-\beta_i} y_i^*(z^*) \quad \text{for} \quad i = 1, \ldots, n \ ,$$
$$s = \lambda^{-\delta} s^*$$
$$h = \lambda^{-\sigma} h^* \ , \tag{3.4}$$

where $z \in [0, s]$ and $z^* \in [0, s^*]$. □

COROLLARY 4. *Theorem* 2 *is a consequence of Theorem* 3.

Proof. The class of free BVPs (3.3) introduced in the proof of Theorem 3 reduces to the class of problems (3.2) characterized in Theorem 2 if we set

$$h = z^{\sigma/\delta} \qquad \Rightarrow \qquad h = s^{\sigma/\delta} \quad \text{at} \quad z = s$$

and identify

$$f_i(1, \cdot, \ldots, \cdot) = \Phi_i(\cdot, \ldots, \cdot) \qquad \text{for} \quad i = 1, \ldots, n \ ,$$
$$\ell_i(1) = A_i \qquad \text{for} \quad i = 1, \ldots, n \ .$$

That completes the proof. □

Remark. For the class of free BVPs (3.1) the existence and uniqueness question can be reduced, within a similarity analysis, to find the number of real roots of an implicit function of h (a simple proof of this statement can be obtained along the lines of the proof of the theorem in [35]). In particular, nonexistence of solutions for a given problem is equivalent to the nonexistence of real roots of the mentioned implicit defined function. Therefore, the introduced iterative method is convergent on condition that h goes to one.

3.2. Free BVPs of the second class. Let us consider the following class of free BVPs:

$$\begin{aligned} \frac{d\mathbf{y}}{dz} &= \mathbf{f}(z, \mathbf{y}), \\ y_j(0) &= \ell_j(y_n(0)) \qquad \text{for} \quad j = 1, \ldots, n-1, \\ y_m(s) &= B_m, \\ y_{m+1}(s) &= B_{m+1} \ . \end{aligned} \qquad (m \in \{1, \ldots, n-1\}) \tag{3.5}$$

(3.5) is a subclass of (2.1) obtained by setting $g_j(\mathbf{y}(0), s, \mathbf{y}(s)) = y_j(0) - \ell_j(y_n(0))$ for $j = 1, \ldots, n-1$, $g_n(\mathbf{y}(0), s, \mathbf{y}(s)) = y_m(s) - B_m$, and $g_{n+1}(\mathbf{y}(0), s, \mathbf{y}(s)) = y_{m+1}(s) - B_{m+1}$, where $m \in \{1, \ldots, n-1\}$.

In the following we define two initial value methods for the numerical solution of problems belonging to (3.5).

THEOREM 5. *The following class of free BVPs*

$$\begin{aligned} \frac{dy_i}{dz} &= y_n^{(\beta_i-\delta)/\beta_n} \Psi_i \left(y_n^{-\delta/\beta_n} z, y_n^{-\beta_1/\beta_n} y_1, \ldots, y_n^{-\beta_{n-1}/\beta_n} y_{n-1} \right) \qquad \textit{for} \quad i = 1, \ldots, n \ , \\ y_j(0) &= C_j [y_n(0)]^{\beta_j/\beta_n} \qquad \textit{for} \quad j = 1, \ldots, n-1, \\ y_m(s) &= B_m, \\ y_{m+1}(s) &= B_{m+1}, \end{aligned} \qquad (m \in \{1, \ldots, n-1\}), \tag{3.6}$$

where $\beta_m \neq 0, B_m \neq 0,$ *and* $\Psi_i(\cdot, \ldots, \cdot)$ *for* $i = 1, \ldots, n$ *are arbitrary functions of their arguments, can be solved by a noniterative method.*

Outline of the proof. The class of free BVPs (3.6) can be characterized from (3.5) by requiring the invariance of the governing DEs and of the boundary conditions at zero under the stretching group

$$z^* = \lambda^\delta z, \qquad s^* = \lambda^\delta s, \qquad y_i^* = \lambda^{\beta_i} y_i \quad \text{for} \quad i = 1, \ldots, n \ .$$

For the noniterative method we have a variant of the one defined in Theorem 2, as follows:

1. we set a value for $y_n^*(0)$, the boundary conditions at zero define the values of $y_j^*(0)$ for $j = 1, \ldots, n-1$;
2. to get the values of $y_m^*(s^*)$ and $y_{m+1}^*(s^*)$ the governing DEs in the starred variables can be solved on $[0, s^*]$, where s^* is defined by the condition

$$[y_m^*(s^*)/B_m]^{-\beta_{m+1}/\beta_m} \, y_{m+1}^*(s^*) = B_{m+1} \quad \Leftarrow \quad y_{m+1}(s) = B_{m+1} \ ;$$

3. λ is given by

$$\lambda = [y_m^*(s^*)/B_m]^{1/\beta_m} \quad \Leftarrow \quad y_m(s) = B_m \ ,$$

and consequently

$$y_i(z) = \lambda^{-\beta_i} y_i^*(z^*) \qquad \text{for} \quad i = 1, \ldots, n \ ,$$
$$s = \lambda^{-\delta} s^* \ ,$$

where $z \in [0, s]$ and $z^* \in [0, s^*]$. ∎

An iterative extension of the above noniterative method applicable to the class of problems (3.5) is of interest. Again, the iterative method defined below is convergent on condition that h goes to one.

THEOREM 6. *The class of free BVPs* (3.5)

$$\begin{aligned} \frac{d\mathbf{y}}{dz} &= \mathbf{f}(z, \mathbf{y}), \\ y_j(0) &= \ell_j(y_n(0)) \qquad \textit{for} \quad j = 1, \ldots, n-1, \\ y_m(s) &= B_m, \\ y_{m+1}(s) &= B_{m+1} \end{aligned} \qquad (m \in \{1, \ldots, n-1\})$$

can be solved numerically by an iterative method.

Outline of the proof. In this case we have a proof similar to the one we have proposed for Theorem 3. The class of free BVPs to be considered is given by

$$\begin{aligned} \frac{dy_i}{dz} &= h^{(\beta_i - \delta)/\sigma} f_i(h^{-\delta/\sigma} z, h^{-\beta_1/\sigma} y_1, \ldots, h^{-\beta_n/\sigma} y_n) \qquad \text{for} \quad i = 1, \ldots, n, \\ y_j(0) &= h^{\beta_j/\sigma} \ell_j(h^{-\beta_n/\sigma} y_n(0)) \qquad \text{for} \quad j = 1, \ldots, n-1, \\ y_m(s) &= B_m, \\ y_{m+1}(s) &= B_{m+1} \ , \end{aligned} \qquad (m \in \{1, \ldots, n-1\}) \tag{3.7}$$

where h is the parameter. The governing differential equation and the boundary conditions at zero in (3.7) are invariant with respect to the stretching group

$$z^* = \lambda^\delta z, \qquad s^* = \lambda^\delta s, \qquad y_i^* = \lambda^{\beta_i} y_i \quad \text{for} \quad i = 1, \ldots, n, \qquad h^* = \lambda^\sigma h.$$

The iterative method is defined by the following steps:

1. since (3.5) is recovered from (3.7) by setting $h = 1$, we iterate different values of h^* until we find, from (3.8) below, that $|h - 1| \leq$ TOL where TOL is a prefixed tolerance;

2. we set a value for $y_n^*(0)$, the boundary conditions at zero define the values of $y_j^*(0)$ for $j = 1, \ldots, n-1$;

3. to get the values of $y_m^*(s^*)$ and $y_{m+1}^*(s^*)$ the governing DEs in the starred variables can be solved on $[0, s^*]$, where s^* is defined by the condition

$$[y_m^*(s^*)/B_m]^{-\beta_{m+1}/\beta_m}\, y_{m+1}^*(s^*) = B_{m+1} \quad \Leftarrow \quad y_{m+1}(s) = B_{m+1} \ ;$$

4. λ is given by

$$\lambda = [y_m^*(s^*)/B_m]^{1/\beta_m} \quad \Leftarrow \quad y_m(s) = B_m \ ,$$

and

$$\begin{gathered} y_i(z) = \lambda^{-\beta_i} y_i^*(z^*) \qquad \text{for} \quad i = 1, \ldots, n \ , \\ s = \lambda^{-\delta} s^*, \\ h = \lambda^{-\sigma} h^* \ , \end{gathered} \tag{3.8}$$

where $z \in [0, s]$ and $z^* \in [0, s^*]$. ∎

Corollary 7. *Theorem* 5 *is a consequence of Theorem* 6.

Proof. The class of free BVPs (3.7) introduced in the proof of Theorem 6 reduces to the class of problems (3.6) characterized in Theorem 5 by setting

$$h = [y_n(z)]^{\sigma/\beta_n} \qquad \Rightarrow \qquad h = [y_n(0)]^{\sigma/\beta_n} \quad \text{at} \quad z = 0$$

and identifying

$$\begin{gathered} f_i(\cdot, \ldots, \cdot, 1) = \Psi_i(\cdot, \ldots, \cdot) \qquad \text{for} \quad i = 1, \ldots, n \ , \\ \ell_j(1) = C_j \qquad \text{for} \quad j = 1, \ldots, n-1. \end{gathered}$$

That completes the proof. ∎

Remark. In the definition of the two iterative methods above we have to iterate different values of h^* until we find $|h - 1| \leq$ TOL. From a practical point of view any equivalent termination criterion can be used in the iteration as reported in the next section. In any case we have to find a root of an implicit defined function and therefore a root-finding method can be applied (for numerical details see the next section).

4. Applications (with numerical results).

4.1. Rate-type materials. It is required to find $x_f(t), v(x,t)$, and $\sigma(x,t)$ such that

$$\begin{gathered} \rho_0 \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0, \\ \frac{\partial \sigma}{\partial t} - \Phi_0 \sigma^{1/p} \frac{\partial v}{\partial x} = -\Psi_0 \sigma^{1/q}, \\ x_f(0) = 0, \qquad v(x,0) = 0 \ , \qquad \sigma(x,0) = 0 \ , \\ \sigma(0,t) = \sigma_0 t^\delta H(t) \ , \\ \rho_0 \frac{dx_f}{dt} [\![v(x_f(t),t)]\!] + [\![\sigma(x_f(t),t)]\!] = 0 \ , \\ \frac{p}{\Phi_0(p-1)} \frac{dx_f}{dt} [\![\sigma(x_f(t),t)^{(p-1)/p}]\!] + [\![v(x_f(t),t)]\!] = 0 \ , \end{gathered} \tag{4.1}$$

where — denoting by $d(x,t)$ the displacement along a thin rod of rate-type material at time t and reference position x — $v(x,t) = \frac{\partial d}{\partial t}(x,t)$ is the tensile velocity, $\sigma(x,t)$ is the tensile stress, $\rho_0, \Phi_0, \Psi_0, p, q, \sigma_0 > 0$ and $\delta > -1$ are constants, $H(t)$ is the Heaviside step function, $x_f(t)$ represents the shock front, and the notation $[\![\cdot]\!]$ indicates a jump across it.

The problem (4.1) describes a shock front propagation in a constant state. The shock front propagation into the known state ahead of the shock is due to the time dependent stress impact at the end of the thin rod (Taulbee, Cozzarelli, and Dym [72], Singh and Frydrychowicz [70], Fusco [41], and Frydrychowicz and Singh [40]). The given moving boundary conditions are the Rankine–Hugoniot conditions at the shock.

Within the framework developed in section 2, a similarity analysis leads us to introduce the similarity variables

$$\eta = \sigma_0^{-1/2p}\Phi_0^{-1/2}\rho_0^{1/2}xt^{-\gamma} , \qquad \eta_f = \sigma_0^{-1/2p}\Phi_0^{-1/2}\rho_0^{1/2}x_f(t)t^{-\gamma} ,$$
$$V(\eta) = \sigma_0^{(1-2p)/2p}\,\Phi_0^{1/2}\rho_0^{1/2}t^{-\alpha}v(x,t) , \qquad \Sigma(\eta) = \sigma_0^{-1}t^{-\delta}\sigma(x,t) ,$$

where $\alpha = \delta(1-1/2p), \gamma = 1+\delta/2p$, and δ is an arbitrary constant if $\Psi_0 = 0$ or $q = 1$ while $\delta = q/(q-1)$ otherwise. Consequently we get the reduced problem

$$\begin{aligned} \frac{dV}{d\eta} &= \frac{\alpha\gamma\eta V - \delta\Sigma - \Sigma_0\Sigma^{1/q}}{\Delta}, \\ \frac{d\Sigma}{d\eta} &= \frac{\gamma\eta(\delta\Sigma + \Sigma_0\Sigma^{1/q}) - \alpha V\Sigma^{1/p}}{\Delta}, \\ \Sigma(0) &= 1, \\ \Sigma(\eta_f) &= \left[\frac{p-1}{p\,(\gamma\eta_f)^2}\right]^{-p} , \qquad V(\eta_f) = -\frac{\Sigma(\eta_f)}{\gamma\eta_f}, \end{aligned} \tag{4.2}$$

where $\Sigma_0 = \Psi_0\sigma_0^{(1-q)/q}$ and $\Delta = (\gamma\eta)^2 - \Sigma^{1/p}$.

We notice that if $\Psi_0 = 0$ or $q = 1$, then (4.2) is a free BVP that belongs to (3.2). In fact, in these cases the governing DEs and the boundary conditions at the free boundary are invariant with respect to the stretching group

$$\eta^* = \lambda\eta , \quad {\eta_f}^* = \lambda\eta_f , \quad V^* = \lambda^{2p-1} , \quad \Sigma^* = \lambda^{2p}\Sigma . \tag{4.3}$$

Therefore, in the homogeneous (nondissipative) case we can apply the noniterative initial value method. We report in Table 1 the numerical results obtained by the noniterative method.

Next we consider the dissipative case corresponding to $\Sigma_0 \neq 0$. By extending the stretching group (4.3) by

$$h^* = \lambda h ,$$

we have to consider the problem

$$\begin{aligned} \frac{dV}{d\eta} &= \frac{\alpha\gamma\eta V - \delta\Sigma - \Sigma_0 h^{2p(q-1)/q}\Sigma^{1/q}}{\Delta}, \\ \frac{d\Sigma}{d\eta} &= \frac{\gamma\eta(\delta\Sigma + \Sigma_0 h^{2p(q-1)/q}\Sigma^{1/q}) - \alpha V\Sigma^{1/p}}{\Delta}, \end{aligned}$$

TABLE 1

Results related to a nonlinear model ($p = 3$) *in the nondissipative case* $\Sigma_0 = 0$. *Reprinted from Wave Motion, Vol.* 16, *R. Fazio,* A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material, *pp.* 299–305, 1992 *with kind permission from Elsevier Science - NL, Sara Burgerhatstraat* 25, 1055 *KV Amsterdam, The Netherlands.*

δ	$-V(0)$	η_f
-0.5	1.286499	1.077350
0.5	1.211072	0.669950
1.0	1.206158	0.571939
1.5	1.203906	0.500607
2.0	1.202708	0.445920

TABLE 2

Results obtained for $p = 3, \delta = 2, q = 2, \Sigma_0 = -1,$ *and* ${\eta_f}^* = 1$. *Reprinted from Wave Motion, Vol.* 16, *R. Fazio,* A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material, *pp.* 299–305, 1992 *with kind permission from Elsevier Science - NL, Sara Burgerhatstraat* 25, 1055 *KV Amsterdam, The Netherlands.*

k	$h^{*(k)}$	$\lambda^{-1}h^{*(k)} - 1$	$-V(0)$	${\eta_f}^{(k)}$
0	2.000000	-0.151110		
1	4.000000	0.431889		
2	2.518389	$0.29\text{E}-1$	1.441716	0.408756
3	2.410143	$-0.64\text{E}-2$	1.419068	0.412262
4	2.429466	$0.73\text{E}-4$	1.423061	0.411643
5	2.429248	$0.18\text{E}-6$	1.423015	0.411650
6	2.429248	$-0.50\text{E}-11$	1.423015	0.411650

$$\Sigma(0) = 1,$$
$$\Sigma(\eta_f) = \left[\frac{p-1}{p\,(\gamma\eta_f)^2}\right]^{-p}, \qquad V(\eta_f) = -\frac{\Sigma(\eta_f)}{\gamma\eta_f} .$$

In Table 2 we report a sample iteration obtained by the iterative method defined in subsection 3.1.

Here and in the following the E (D) notation indicates a simple (double) precision arithmetic. The numerical results related to different values of the dissipative coefficient Σ_0 are reported in Table 3.

For the numerical iterations we used the secant method with $h^{*(0)} = 2$ and $h^{*(1)} = 4$. We adopted the simple termination criterion

$$|{\eta_f}^{(k)} - {\eta_f}^{(k-1)}| \leq \text{TOL},$$

where k is the iteration index and TOL is a fixed tolerance. By setting $\text{TOL} = 1\text{D}-6$ the above condition was always verified within six iterations.

4.2. Fluid flow past an obstacle. Next, we consider the mathematical model related to the study of the flow of an incompressible fluid past a slender parabola of revolution (as reported by Na [58, pp. 217–221]). By introducing the Levy–Lees transformation to the relevant governing equations in boundary layer theory, the following BVP on an infinite interval is obtained

$$(4.4)\qquad \begin{gathered} (1 + P_1 z)\frac{d^3y}{dz^3} + \left(\frac{1}{2}y + P_1\right)\frac{d^2y}{dz^2} = 0 , \\ y(0) = P_2, \quad \frac{dy}{dz}(0) = 0, \quad \frac{dy}{dz}(z) \to 1 \quad \text{as} \quad z \to \infty , \end{gathered}$$

TABLE 3

Results connected with $p = 3, \delta = 2, q = 2$, and ${\eta_f}^* = 1$. Reprinted from Wave Motion, Vol. 16, *R. Fazio,* A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material, *pp.* 299–305, 1992 *with kind permission from Elsevier Science - NL, Sara Burgerhatstraat* 25, 1055 *KV Amsterdam, The Netherlands.*

Σ_0	$-V(0)$	η_f
-1.0	1.423015	0.411650
-0.5	1.316325	0.428250
0.0	1.202708	0.445920
0.5	1.081360	0.464515
1.0	0.951518	0.483819

where P_1 is the transverse curvature parameter and P_2 represents suction ($P_2 < 0$) or blowing ($P_2 > 0$). The Blasius problem is recovered by setting $P_1 = 0$ and $P_2 = 0$ in (4.4).

As far as the Blasius problem is concerned, Weyl [76] proved that the only solution of the problem has a second-order derivative that is positive, monotone decreasing and goes to zero at infinity. Taking into account the mentioned properties, the following free boundary formulation of the Blasius problem

$$\begin{gathered} \frac{d^3 y_\epsilon}{dz^3} + \frac{1}{2} y_\epsilon \frac{d^2 y_\epsilon}{dz^2} = 0 \ , \\ y_\epsilon(0) = 0 \ , \quad \frac{dy_\epsilon}{dz}(0) = 0 \ , \quad \frac{dy_\epsilon}{dz}(z_\epsilon) = 1 \ , \quad \frac{d^2 y_\epsilon}{dz^2}(z_\epsilon) = \epsilon \end{gathered} \tag{4.5}$$

was considered in [26]. The governing DE and the two boundary conditions at the origin in (4.5) are invariant with respect to the stretching group

$$z^* = \lambda^\delta z, \qquad {z_\epsilon}^* = \lambda^\delta z_\epsilon, \qquad {y_\epsilon}^* = \lambda^{-\delta} y_\epsilon \ .$$

The noniterative method defined in subsection 3.2 can be applied in this case. Here we choose $\delta = -1$ instead of the classical value $\delta = -1/3$ (cf. Töpfer [74], Klamkin [50], or Na [58, pp. 138–142]). The initial value problem

$$\begin{gathered} \frac{d^3 {y_\epsilon}^*}{dz^{*3}} + \frac{1}{2} {y_\epsilon}^* \frac{d^2 {y_\epsilon}^*}{dz^{*2}} = 0 \ , \\ {y_\epsilon}^*(0) = 0 \ , \quad \frac{d{y_\epsilon}^*}{dz^*}(0) = 0 \ , \quad \frac{d^2 {y_\epsilon}^*}{dz^{*2}}(0) = 1\mathrm{D}+3 \ , \end{gathered} \tag{4.6}$$

on $[0, {z_\epsilon}^*]$, where ${z_\epsilon}^*$ is defined by the condition

$$\left[\frac{d{y_\epsilon}^*}{dz^*}({z_\epsilon}^*)\right]^{-3/2} \frac{d^2 {y_\epsilon}^*}{dz^{*2}}({z_\epsilon}^*) = \epsilon \quad \Leftarrow \quad \frac{d^2 y_\epsilon}{dz^2}(z_\epsilon) = \epsilon$$

is solved numerically to define

$$\lambda = \left[\frac{d{y_\epsilon}^*}{dz^*}({z_\epsilon}^*)\right]^{1/2} \quad \Leftarrow \quad \frac{dy_\epsilon}{dz}(z_\epsilon) = 1,$$

$$y_\epsilon(z) = \lambda^{-1} {y_\epsilon}^*(z^*), \qquad \frac{dy_\epsilon}{dz}(z) = \lambda^{-2} \frac{d{y_\epsilon}^*}{dz^*}(z^*), \qquad \frac{d^2 y_\epsilon}{dz^2}(z) = \lambda^{-3} \frac{d^2 {y_\epsilon}^*}{dz^{*2}}(z^*),$$
$$z_\epsilon = \lambda {z_\epsilon}^* \ ,$$

TABLE 4

Numerical results for (4.5). *Reprinted from Acta Mech., Vol.* 95, *R. Fazio,* The Blasius problem formulated as a free boundary value problem, *pp.* 1–7, 1992 *with kind permission from Springer-Verlag Wien, Sachsenplatz* 4-6, *P.O. Box* 89, A-1201 *Wien, Austria.*

ϵ	$z_\epsilon{}^*$	z_ϵ	$\frac{d^2y_\epsilon}{dz^2}(0)$
1D−6	0.61	8.752700	0.332057
1D−9	0.73	10.500242	0.332057336
1D−12	0.83	11.953565	0.332057336215

TABLE 5

Numerical results for (4.4), *where* $P_1 = 2, P_2 = 2$. *Reprinted with permission from SIAM J. Numer. Anal., Vol.* 33, *R. Fazio,* A novel approach to the numerical solution of boundary value problems on infinite intervals, *pp.* 1473–1483, 1996.

ϵ	h^*	$\lambda^{-1}h^* - 1$	$z_\epsilon{}^*$	z_ϵ	$\frac{d^2y_\epsilon}{dz^2}(0)$
1D−6	1.907248998	−1.0D − 13	19.52	37.229499817	1.441377749
1D−7	1.907251352	7.5D − 14	23.92	45.621452332	1.441372413
1D−8	1.907251591	−9.0D − 15	28.39	54.146873474	1.441371870
1D−9	1.907251615	2.5D − 14	32.90	62.748577118	1.441371815

where $z \in [0, z_\epsilon]$ and $z^* \in [0, z_\epsilon{}^*]$. The value of $\frac{d^2y_\epsilon{}^*}{dz^{*2}}(0) = 1\mathrm{D}+3$ is related to the condition $z_\epsilon{}^* \ll z_\epsilon$ that was considered to reduce the error introduced by the numerical integration (we refer here to Fazio [26]).

The numerical results reported in Table 4 are in good agreement with the corresponding results available in literature and obtained by analytical or numerical methods (see [26]).

A free boundary formulation of the problem (4.4) was considered in [34]; however the mentioned formulation does not belong to the class (3.5). As a consequence, in the present case we have to apply the iterative method of subsection 3.2. To this end we can introduce the following stretching group

$$z^* = \lambda^\delta z, \qquad z_\epsilon{}^* = \lambda^\delta z_\epsilon, \qquad y_\epsilon{}^* = \lambda^{-\delta} y_\epsilon \ , \qquad h^* = \lambda^{-\delta} h$$

and the related initial value problem

$$\begin{gathered} (1 + h^* P_1 z^*) \frac{d^3 y_\epsilon{}^*}{dz^{*3}} + \left(\frac{1}{2} y_\epsilon{}^* + h^* P_1 \right) \frac{d^2 y_\epsilon{}^*}{dz^{*2}} = 0 \ , \\ y_\epsilon{}^*(0) = h^* P_2 \ , \quad \frac{dy_\epsilon{}^*}{dz^*}(0) = 0 \ , \quad \frac{d^2 y_\epsilon{}^*}{dz^{*2}}(0) = 1\mathrm{D}+1 \ . \end{gathered} \tag{4.7}$$

Here we again set $\delta = -1$. In Table 5 we report the numerical results for particular values of the parameters involved in (4.4). As far as the missing initial condition is concerned, the last column of Table 5 indicates the achievement of six decimal digits of accuracy. Table 5 permits a comparison of the values of $\frac{d^2y_\epsilon}{dz^2}(0)$ with the value of 1.4426 reported by Na [58, p. 220]. The obtained values of $\frac{d^2y_\epsilon}{dz^2}(0)$ and z_ϵ were used in a further integration to verify the numerical results for $\frac{dy_\epsilon}{dz}(z_\epsilon)$ and $\frac{d^2y_\epsilon}{dz^2}(z_\epsilon)$. This validation is reported in Table 6.

For the results reported in Table 5 we applied the secant method with $h^{*(0)} = 3/2$ and $h^{*(1)} = 2$. After some numerical tests we adopted the simple termination criterion

$$|h^{*(k)} - h^{*(k-1)}| \leq \mathrm{TOL},$$

TABLE 6

A convergence numerical test for the results reported in Table 5. *Reprinted with permission from SIAM J. Numer. Anal., Vol.* 33, *R. Fazio,* A novel approach to the numerical solution of boundary value problems on infinite intervals, *pp.* 1473–1483, 1996.

$\frac{d^2y_\epsilon}{dz^2}(0)$	z_ϵ	$\frac{dy_\epsilon}{dz}(z_\epsilon)$	$\frac{d^2y_\epsilon}{dz^2}(z_\epsilon)$
1.441377749	37.229499817	1.000000000010	9.97226D − 7
1.441372413	45.621452332	1.000000000016	9.9747D − 8
1.441371870	54.146873474	0.999999999825	9.961D − 9
1.441371815	62.748577118	0.999999999686	1.000D − 9

where k is the iteration index and TOL a fixed tolerance. By setting TOL = 5D − 7 the above condition was always verified within six iterations.

5. Discussion. This survey was based on recent work available in literature. In particular we have extended and included in the same framework initial value methods developed in [37, 27, 30, 26, 33, 32, 34].

As the numerical solution of free BVPs is a challenging topic, it is distressing that noniterative methods are often overlooked. In this context our contribution was related to the definition of two classes of free BVPs that can be solved noniteratively. For the present survey we focused our attention on the similarity approach. However, ad hoc extensions of noniterative methods can be obtained within group invariance theory. For instance, the following class of free BVPs

$$\begin{aligned} &\frac{d^2v}{dw^2} = \Omega\left(v, \frac{dv}{dw}\right), \\ &v(0) = A, \\ &v(s) = B, \qquad \frac{dv}{dw}(s) = C \ , \end{aligned} \tag{5.1}$$

where $\Omega(\cdot,\cdot)$ is an arbitrary function of its arguments and A, B, and C are arbitrary constants (with $A \neq B$ and $C \neq A - B$), can be transformed to a subclass of (3.2). In fact, within group invariance theory we can introduce the transformation of variables

$$w = \ln(y), \qquad v = (B-A)z/y$$

that, by assuming $y = y(z)$, allows us to rewrite (5.1) as follows

$$\begin{aligned} &\frac{d^2y}{dz^2} = \frac{z}{y^2}\left(\frac{dy}{dz}\right)^3 - \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{(B-A)y}\left(\frac{dy}{dz}\right)^3 \Omega\left((B-A)\frac{z}{y}, (B-A)\frac{y - z\frac{dy}{dz}}{y\frac{dy}{dz}}\right), \\ &y(0) = 1, \\ &y(e^s) = e^s, \qquad \frac{dy}{dz}(e^s) = \frac{B-A}{C+B-A} \ , \end{aligned} \tag{5.2}$$

which, by assuming e^s as a new free boundary and setting

$$\begin{aligned} &y_1(z) = y(z) \ , \quad y_2(z) = \frac{dy}{dz}(z) \ , \quad \Phi_1(\ldots) = y_2 \ , \\ &\Phi_2(\ldots) = \frac{z^2}{(y_1)^2}(y_2)^3 - \frac{z}{y_1}(y_2)^2 - \frac{z}{(B-A)y_1}(y_2)^3 \, \Omega(\cdot,\cdot) \, , \\ &\beta_1 = 1 \quad \beta_2 = 0 \quad \text{and} \quad \delta = 1 \ , \end{aligned}$$

can be rewritten as a subclass of (3.2). The details of the group invariance analysis can be found in [24]. Let us note here that the governing differential equation and the two boundary conditions at the free boundary in (5.1) are invariant with respect to the following translation group of transformations:

$$w^* = w + \omega, \qquad s^* = s + \omega, \qquad v^* = v,$$

where ω is the group parameter. Under the introduced transformation of variables the translation group is transformed into the stretching group

$$z^* = \lambda z, \qquad e^{s^*} = \lambda e^s, \qquad y^* = \lambda y,$$

where $\lambda = e^\omega$. The governing differential equation and the two boundary conditions at the free boundary in (5.2) are invariant with respect to the above stretching group.

To show different applications of the similarity approach, we have reported the numerical results obtained for three problems. The mathematical formulation of the first problem can be found in [28]. That problem describes an impact problem at one end of a thin rod. In recent years a great deal of attention has been devoted to apply the similarity analysis to impact problems in nonlinear elasticity (see Taulbee, Cozzarelli, and Dym [72], Singh and Frydrychowicz [69], Singh and Frydrychowicz [70], Dresner [20, pp. 77–87], Frydrychowicz and Singh [40], Donato [18], Fazio [28], and Fazio and Evans [38]). Moreover, the relevance of models describing the evolution of a shock at the initial time cannot be underestimated. In fact, impact problems in nonlinear elasticity, as well as the point explosion problem in gas dynamics (Sedov [66] and Taylor [73]), and the dam breaking problem in shallow water theory (Grundy and Rottman [43]), are due to the occurrence of a shock at the initial time. The Blasius problem is well known and its physical background and formulation can be found in many books among those quoted in the references. The model formulation for the fluid flow past a slender parabola is reported in the book by Na [58, pp. 217–221]).

More applications of the similarity approach can be found in the literature. As a relevant example the numerical solution within the proposed approach of the Falkner–Skan equation subject to the classical boundary conditions (see Falkner and Skan [22]) has been considered by Fazio in [32]. This particular problem was used by Na to discuss the limitations of the group invariance approach. Indeed neither a direct transformation (see Na [58, pp. 146–147]) nor a transformation of the physical parameters (see Na [58, Chaps. 8–9]) is applicable to the Falkner–Skan model. The sensitivity of the classical initial value method (the shooting method) for the Falkner–Skan model has been discussed by Cebeci and Keller (see Fig. 1 in [12]).

In this survey we have proposed a useful approach to the numerical solution of free BVPs in view of the frequent occurrence of these problems in many fields of applied sciences.

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