Journal of Computational and Applied Mathematics 140 (2002) 331–344

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

A survey on free boundary identification of the truncated boundary in numerical BVPs on infinite intervals

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Received 11 September 2000; received in revised form 30 July 2001

Abstract

A free boundary formulation for the numerical solution of boundary value problems on infinite intervals was proposed recently in Fazio (SIAM J. Numer. Anal. 33 (1996) 1473). We consider here a survey on recent developments related to the free boundary identification of the truncated boundary. The goals of this survey are: to recall the reasoning for a free boundary identification of the truncated boundary, to report on a comparison of numerical results obtained for a classical test problem by three approaches available in the literature, and to propose some possible ways to extend the free boundary approach to the numerical solution of problems defined on the whole real line. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 65L10; 34B15; 73C50

1. Introduction

In this survey we consider the numerical solution of boundary value problems (BVPs) for ordinary differential equations (ODEs) on infinite intervals of the type

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{f}(x, \mathbf{u}), \quad x \in (a, \infty),$$

$$\mathbf{g}(\mathbf{u}(a), \mathbf{u}(\infty)) = \mathbf{0}, \tag{1.1}$$

where $\mathbf{u}(x)$ is an n-dimensional vector with $u_\ell(x)$ for $\ell = 1, \ldots, n$ as components, $\mathbf{f} : [a, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $a \in \{\mathbb{R} \cup \{-\infty\}\}$ and $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. These problems are solved numerically mainly by introducing truncated boundaries (say x_∞ and $x_{-\infty}$ if $a = -\infty$) which restrict the original problem to a large but finite interval and imposing "suitable" boundary conditions there.

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PII: S0377-0427(01)00526-X

The oldest and simplest approach is to replace the boundary conditions at infinity by the same conditions at chosen values of the truncated boundaries. This approach was used, for instance, by Goldstein [22, p. 136] and by Howarth [23] in 1938 for the tabulated numerical solution of the Blasius problem. However, to get an accurate solution a comparison of numerical results obtained for several values of the truncated boundaries is necessary as suggested by Fox [19, p. 92] or by Collatz [8, pp. 150–151]. Moreover, in some cases accurate solutions can be found only by using very large values of the truncated boundary, as, for instance, for the fourth branch of the von Karman swirling flows where values of x_∞ up to 200 were used by Lentini and Keller [26].

To overcome the mentioned difficulties of the classical approach described above, Lentini and Keller [25] and de Hoog and Weiss [9] have suggested to apply asymptotic boundary conditions (ABCs) at the truncated boundaries (see also the theoretical work of Markowich [28,29], Markowich and Ringhofer [30], Schmeiser [38] and Mattheij [31]). Those ABCs have to be derived by a preliminary asymptotic analysis involving the Jacobian matrix of $\mathbf{f}(x, \mathbf{u})$ evaluated at infinity. More accurate numerical solutions can be found by this approach than those obtained by the classical approach with the same values of the truncated boundaries, because the imposed conditions are obtained from the asymptotic behaviour of the solution. However, we should note that for nonlinear problems highly nonlinear ABCs may result. Moreover, it has been noticed by J.R. Ockendon that "Unfortunately the analysis is heavy and relies on much previous work, ..." see Math. Rev. 84c:34201. On the other hand, starting with the work by Beyn [3–5], the ABCs approach has been applied successfully to "connecting orbits" problems (see also: [7,12,11,20,21,34–36]). Connecting orbits are of interest in the study of dynamical systems as well as of traveling wave solutions of partial differential equations of parabolic type (see, for the latter topic [4,20,2,27,10]).

A free boundary formulation for the numerical solution of BVPs on infinite intervals was recently proposed in [16]. In this approach the truncated boundary can be identified as an unknown free boundary that has to be determined as part of the solution. This eliminates the uncertainty related to the choice of the truncated boundary. To be more explicit, assume that at least one additional boundary condition is available

$$h(\mathbf{u}(a), \mathbf{u}(\infty)) = 0, \tag{1.2}$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, then a free boundary formulation for (1.1) is given by

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{f}(x, \mathbf{u}), \quad x \in [a, x_\varepsilon],$$

$$\mathbf{g}(\mathbf{u}(a;\varepsilon), \mathbf{u}(x_\varepsilon;\varepsilon)) = \mathbf{0},$$

$$h(\mathbf{u}(a;\varepsilon), \mathbf{u}(x_\varepsilon;\varepsilon)) = \varepsilon, \tag{1.3}$$

where $0 < |\varepsilon| \ll 1$, the solution of (1.3) depends on ε, that is $\mathbf{u}(x;\varepsilon)$, and x_ε is the unknown free boundary. A theorem concerning the uniform converge of the solution of (1.3) to the solution of (1.1)–(1.2), as ε goes to zero, can be found in [17]. This new formulation has already been applied to several problems: the Blasius problem [14], the Falkner–Skan equation with relevant boundary conditions [15], a model describing the flow of an incompressible fluid over a slender parabola of revolution [16], and a problem in nonlinear elasticity [13].

The goals of the present survey are: to recall the reasons leading to a free boundary identification of the truncated boundary, to report on a comparison of numerical results obtained for a classical test problem by the three approaches mentioned above, and to suggest some possible ways to extend the free boundary approach to the numerical solution of problems defined on the whole real line. To this end, the paper is organized as follows. In the next section, we consider the Blasius problem, and report on the error analysis for a truncated boundary approach to its numerical solution. The mentioned error analysis provides the justification for considering a free boundary formulation as an effective way to deal with the accuracy question. In Section 3 a classical test problem is used to show differences and similarities among the classical, the ABCs and the free boundary approaches. A graphical comparison of the related numerical results suffices for our purposes. In Section 4, we discuss some preliminary results on the extension of the free boundary approach to the numerical solution of BVPs on the whole real line. In that section we consider also an interesting test problem already used by Beyn [4]. Finally, the last section is devoted to concluding remarks.

2. On the free boundary approach

In this section, we use the simple Blasius problem to explain why a free boundary approach should be effective.

2.1. Background

The problem of determining the steady two-dimensional motion of a fluid past a flat plate placed edge-ways to the stream was formulated in general terms within the boundary layer theory by Prandtl [32], and was investigated in detail by Blasius [6]. From the original partial differential model it is possible, by a similarity analysis, to obtain the Blasius problem (for this problem the independent variable will be denoted by η and the dependent one by $f(\eta)$)

$$\begin{aligned} &\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = 0, \quad 0 < \eta < \infty, \\ &f(0) = \frac{\mathrm{d}f}{\mathrm{d}\eta}(0) = 0, \quad \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) \to 1 \text{ as } \eta \to \infty. \end{aligned} \tag{2.1}$$

The value of $\mathrm{d}^2 f/\mathrm{d}\eta^2(0)$ is of interest because it is involved in the definition of the shear at the plate (skin friction), as defined by the original partial differential model, which leads to the determination of the viscous drag on the plate (see for instance Schlichting [37, p. 138]). It has been proved by Weyl [39] that the unique solution of (2.1) has a positive second order derivative, which is monotone decreasing on $[0, \infty)$ and approaches zero as the independent variable goes to infinity.

2.2. Truncated boundary and error analysis

In order to apply to (2.1) the truncated boundary approach we replace the boundary condition at infinity by the same condition applied at a given finite value M (to simplify the notation, in this section, the truncated boundary is denoted by M according to Rubel [33]). So that, we define $f_M(\eta)$

Table 1
The "it" column indicates the number of iterations. Free boundary and missing initial condition (third and fourth columns, respectively)

ε	it	$\eta_\varepsilon \approx$	$d^2 f/d\eta^2(0)$
1D−3	7	4.62	0.4699
1D−4	7	5.24	0.4697
1D−5	8	5.77	0.4697

as the solution of

$$\frac{d^3 f_M}{d\eta^3} + f_M \frac{d^2 f_M}{d\eta^2} = 0,$$

$$f_M(0) = \frac{d f_M}{d\eta}(0) = 0, \quad \frac{d f_M}{d\eta}(M) = 1. \tag{2.2}$$

The error $e(\eta)$ related to $f_M(\eta)$ is given by

$$e(\eta) = |f(\eta) - f_M(\eta)|, \quad \eta \in [0, M].$$

The following theorem defines an upper bound for this error.

Theorem 1 (Rubel [33]). *The error related to the truncated boundary formulation of the Blasius problem verifies the following inequality*:

$$e(\eta) \leqslant M \frac{d^2 f_M}{d\eta^2}(M)\,[f_M(M)]^{-1}.$$

An outline of the proof can be found in [18].

It follows from Theorem 1 that to control $e(\eta)$ on $[0, M]$ we can modify either the value of M or the value of $d^2 f_M/d\eta^2(M)$. Commonly, the value of M is modified to check the introduced error. For instance, several increasing values of the truncated boundary could be used to compare the numerical approximation of interest (for instance the shear value). However, as Theorem 1 shows, for the Blasius problem the error is directly proportional to M. Motivated by this consideration the following free boundary formulation of the Blasius problem:

$$\frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} = 0,$$

$$f(0) = \frac{d f}{d\eta}(0) = 0, \quad \frac{d f}{d\eta}(\eta_\varepsilon) = 1, \quad \frac{d^2 f}{d\eta^2}(\eta_\varepsilon) = \varepsilon \tag{2.3}$$

was introduced in [14]; here $0 \leqslant \varepsilon \ll 1$ and η_ε is the unknown free boundary. In (2.3) meaningful numerical results obtained by an initial value method were reported in [14].

In Table 1 some new numerical results computed by the Keller box scheme are listed [24]. To apply the box scheme we introduce the new independent variable $\xi = \eta/\eta_\varepsilon$, rewrite (2.3) in standard form and add the equation $d\eta_\varepsilon/d\xi = 0$ to the governing system. The initial iterate for the relaxation method is given by $f(\xi) = \xi$, $df/d\xi(\xi) = \xi$, $d^2f/d\xi^2(\xi) = 1 - \xi$ and $\eta_\varepsilon = 1$. As a simple termination criterion for the iteration we require that the discrete infinite norm of the difference between two successive iterations be less than a fixed tolerance, say TOL. For the numerical results of Table 1 we used $\text{TOL} = 0.5 \times 10^{-6}$ with 101 mesh-points.

3. A test problem for comparison

To describe the plane deflection of a pile in soil Lentini and Keller [25] introduced the following model:

$$\frac{d^4u}{dx^4} = -P_1(1 - e^{-P_2 u}), \quad 0 < x < \infty,$$

$$\frac{d^2u}{dx^2}(0) = 0, \quad \frac{d^3u}{dx^3}(0) = P_3,$$

$$\lim_{x\to\infty} u(x) \to 0, \quad \lim_{x\to\infty} \frac{du}{dx}(x) \to 0, \tag{3.1}$$

where x is the distance from the top of the pile, $u(x)$ represents the deflection of the pile, P_1 and P_2 are positive material constants, and the conditions $d^2u/dx^2(0) = 0$ and $d^3u/dx^3(0) = P_3 > 0$ are related to a zero moment and a positive shear at the origin, respectively. Moreover, for physical reasons we may assume that

$$\lim_{x\to\infty} \frac{d^ku}{dx^k}(x) \to 0, \quad k = 2, 3, \ldots .$$

In the following, for comparison, we set the following values:

$$P_1 = 1, \quad P_2 = \tfrac{1}{2} \quad \text{and} \quad P_3 = \tfrac{1}{2}.$$

Moreover, we define the new variables

$$u_i(x) = d^{i-1}u/dx^{i-1}(x) \quad \text{for } i = 1, 2, 3, 4,$$

and rewrite the original problem in standard form [1]

$$\frac{d\mathbf{u}}{dx} = \mathbf{f}(x, \mathbf{u}) = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ -P_1(1 - e^{-P_2 u_1}) \end{bmatrix}.$$

3.1. Zero boundary conditions

The truncated boundary approach leads to the following zero boundary conditions (ZBCs):

$$u_1(x_\infty) = u_2(x_\infty) = 0.$$

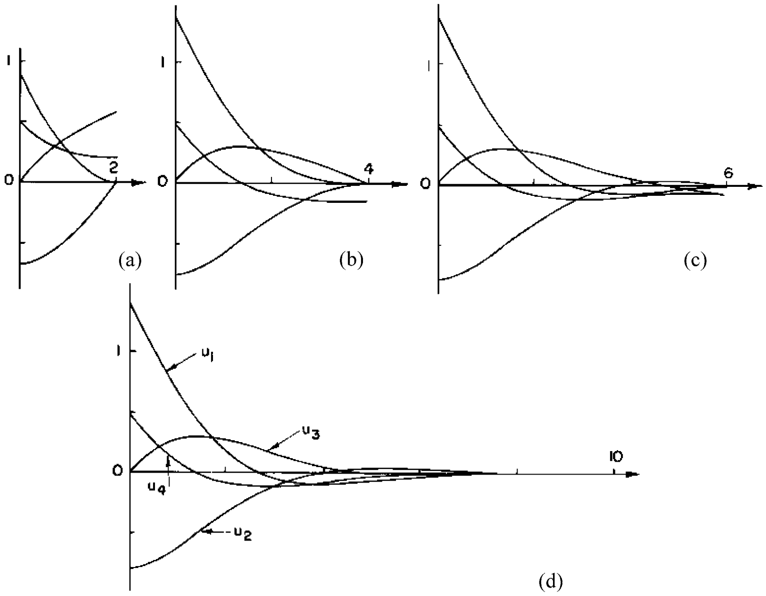

Fig. 1. Numerical solution of pile problem with ZBCs. Reprinted with permission from Lentini and Keller [25].

Fig. 1 shows four frames obtained by using the truncated boundary values: $x_\infty = 2$, 4, 6 and 10, respectively.

3.2. Asymptotic boundary conditions

Lentini and Keller [25] used (3.1) as a test problem to make evident the superiority of the ABCs approach over the classical one. They made, in fact, a comparison between the Figs. 1 and 2: the numerical value of $u_1(0)$ obtained by using the ZBCs is completely wrong for the smallest truncated boundary value used, and this can be contrasted with the results found by employing the ABCs.

For the derivation of the ABCs we follow the analysis made by Lentini and Keller [25]. The Jacobian of $\mathbf{f}$ is the following matrix:

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -P_1 P_2 \mathrm{e}^{-P_2 u_1} & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\mathbf{u}_\infty = \mathbf{0}$ is the only root of $\lim_{x\to\infty} \mathbf{f}(x, \mathbf{u}) = \mathbf{0}$. As a next step, we define the matrix

$$A_\infty = \lim_{x\to\infty} \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{u}}(x, \mathbf{u}_\infty) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -P_4^4 & 0 & 0 & 0 \end{bmatrix}$$

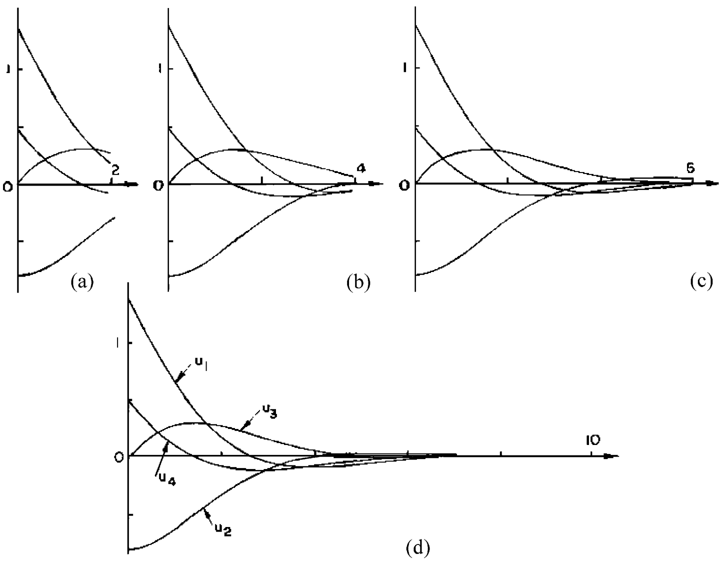

Fig. 2. Numerical solution of pile problem with ABCs. Reprinted with permission from Lentini and Keller [25].

with $P_4 = (P_1 \; P_2)^{1/4}$. Since $P_4^4 > 0$ (recall that in our case $P_1 > 0$ and $P_2 > 0$), the eigenvalues of A_∞ are given by

$$\lambda_\ell = P_4 e^{i(2\ell-1)\pi/4}, \quad \ell = 1, 2, 3, 4,$$

and the matrix of related eigenvectors by the matrix product RS, where

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & P_4/\sqrt{2} & 0 & 0 \\ 0 & 0 & iP_4^2 & 0 \\ 0 & 0 & 0 & P_4^3/\sqrt{2} \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ (1+i) & -(1-i) & (1+i) & (1-i) \\ 1 & -1 & 1 & -i \\ -(1-i) & (1+i) & (1-i) & -(1+i) \end{bmatrix}.$$

Since λ_1 and λ_4 have positive real parts, after introducing the projection matrix

$$P^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the asymptotic conditions

$$\lim_{x\to\infty} P^+(RS)^{-1}\mathbf{f}(x, \mathbf{u}) = \mathbf{0}$$

yield the ABCs at the truncated boundary

$$u_2(x_\infty) + P_4^{-1}/\sqrt{2}u_3(x_\infty) + P_4^{-3}/\sqrt{2}P_1[1 - \mathrm{e}^{-P_2 u_1(x_\infty)}] = 0,$$
$$P_4^{-1}/\sqrt{2}u_3(x_\infty) - P_4^{-2}u_4(x_\infty) - P_4^{-3}/\sqrt{2}P_1[1 - \mathrm{e}^{-P_2 u_1(x_\infty}\,)] = 0.$$

3.3. Free boundary formulation

Let us consider now a free boundary formulation for problem (3.1), and rewrite the obtained free BVP in standard form

$$\begin{aligned}
&\frac{\mathrm{d}u_1}{\mathrm{d}z} = u_5 u_2,\\
&\frac{\mathrm{d}u_2}{\mathrm{d}z} = u_5 u_3,\\
&\frac{\mathrm{d}u_3}{\mathrm{d}z} = u_5 u_4,\\
&\frac{\mathrm{d}u_4}{\mathrm{d}z} = -\,u_5 P_1(1 - \mathrm{e}^{-P_2 u_1}),\\
&\frac{\mathrm{d}u_5}{\mathrm{d}z} = 0,\\
&u_3(0) = 0, \quad u_4(0) = P_3, \quad u_1(1) = 0, \quad u_2(1) = 0,\\
&|u_3(1)| + |u_4(1)| = \varepsilon,
\end{aligned} \tag{3.2}$$

where $u_5 \equiv x_\varepsilon$ and $z = x/u_5$. For the numerical solution of (3.2) we apply the Keller box scheme [24]. The related details: initial iterate, termination criteria, etc., can be found by the interested reader in [13].

A graphical comparison of the numerical solutions can be made with the help of the Figs. 1–3. By the first frame of Fig. 3 it is evident that already in the case $\varepsilon = 10^{-1}$ the free boundary approach reaches a qualitative correct value of $u_1(0)$. Moreover, the numerical solutions shown on the second and third frame of Fig. 3 are in good agreement within the common domain. On the contrary, the first and second frame solutions slightly differ on the interval $[3, 6.46]$.

4. BVPs on the whole real line

We would like to discuss here two different ways to extend the free boundary approach to BVPs defined on the whole real line. For these problems we have to consider all boundary conditions imposed at plus or minus infinity, so that $a = -\infty$ in (1.1)–(1.2). A free boundary formulation can be defined by replacing the boundary condition (1.2) with

$$h(\mathbf{u}(-x_\varepsilon), \mathbf{u}(x_\varepsilon)) = \varepsilon.$$

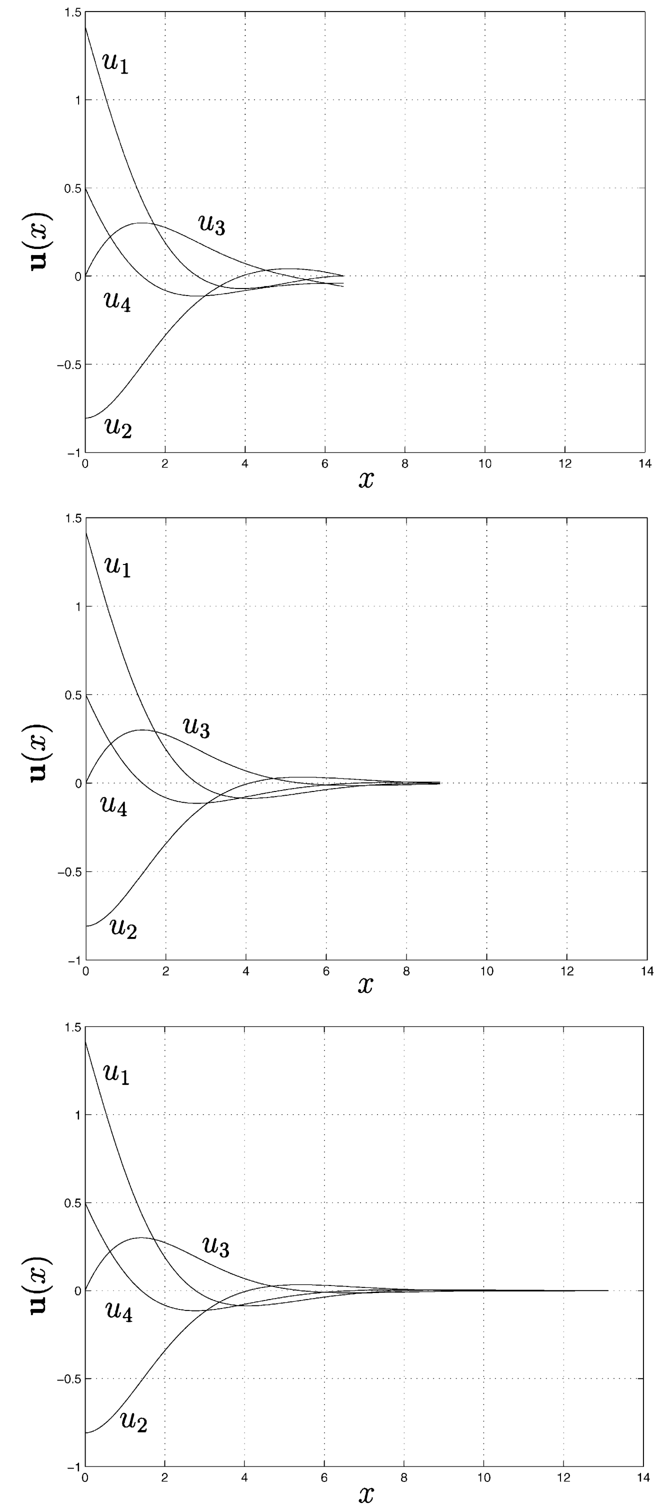

Fig. 3. Numerical solutions of the pile problem by the free boundary approach. Top: for $\varepsilon = 10^{-1}$ we found $x_\varepsilon \approx 6.46$, middle: here $\varepsilon = 10^{-2}$ and $x_\varepsilon \approx 8.84$, and bottom: by setting $\varepsilon = 10^{-3}$ we have $x_\varepsilon \approx 13.13$.

Moreover, the resulting free BVP can be rewritten in standard form by setting $u_{n+1} = x_\varepsilon$, introducing the independent variable transformation

$$z = \frac{x + u_{n+1}}{2u_{n+1}} \tag{4.1}$$

and appending

$$\frac{\mathrm{d}u_{n+1}}{\mathrm{d}z} = 0$$

to the governing system. The above transformation maps $[-x_\varepsilon, x_\varepsilon]$ to $[0, 1]$.

Let us consider an application of the above treatment to a two-dimensional homoclinic orbit example. The governing dynamical system is given by

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} = u_2, \quad x \in \mathbb{R},$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}x} = u_1 - u_1^2 + \lambda u_2 + \mu u_1 u_2. \tag{4.2}$$

As reported by Beyn [4], this system has two stationary points: $(0, 0)$ and $(1, 0)$. For fixed $\mu > 0$, a supercritical Hopf bifurcation from $(1, 0)$ occurs at $\lambda = -\mu$, then at some $\lambda = \lambda_c(\mu) > -\mu$ the periodic orbit becomes a homoclinic orbit with saddle point $(0, 0)$. For more details on the bifurcation diagram and related phase portraits details the interested reader is referred to the references reported in [4]. Our concern here is not the study of the dynamical system by itself but the numerical approximation of the homoclinic orbit. Therefore, according to Beyn we set the values of $\mu = 0.5$, and $\lambda = -0.429505849$.

As far as boundary condition (1.2) is concerned, in the case of this homoclinic orbit we can consider

$$h(\mathbf{u}(-\infty), \mathbf{u}(\infty)) = u_2(\infty). \tag{4.3}$$

Then, we introduce a free boundary formulation for (4.2), rewrite the obtained free BVP in standard form

$$\frac{\mathrm{d}u_1}{\mathrm{d}z} = 2u_3 u_2,$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}z} = 2u_3(u_1 - u_1^2 + \lambda u_2 + \mu u_1 u_2),$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}z} = 0 \tag{4.4}$$

with boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(1) = \varepsilon. \tag{4.5}$$

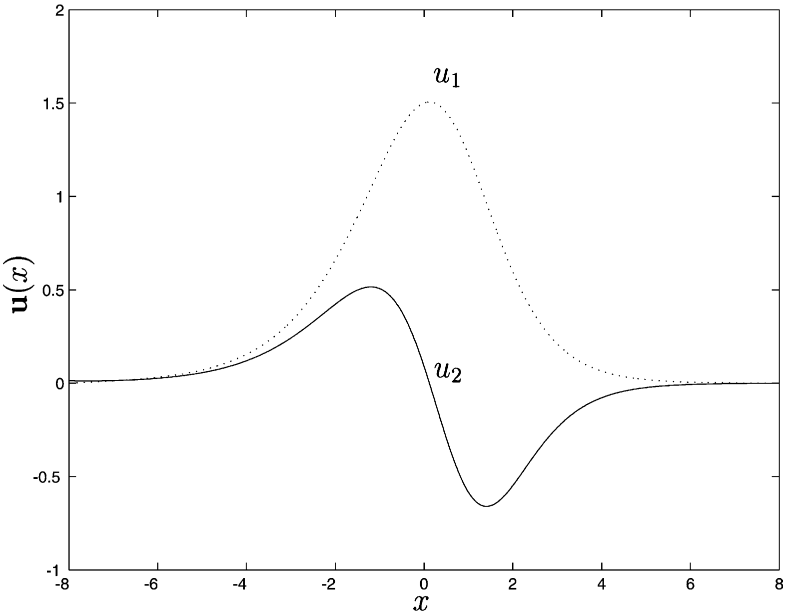

Fig. 4. Homoclinic orbit solution by the free boundary approach.

For the numerical solution of (4.4) we apply a Keller box scheme with 1001 mesh-points. The initial iterate for the computation is given by

$$u_1(z) = 10z(1-z),$$
$$u_2(z) = z(\tfrac{1}{2} - z)(1-z),$$
$$u_3(z) = 10.$$

Our preliminary numerical results are computed for the case $\varepsilon = -10^{-3}$ with $\mathrm{TOL} = 10^{-6}$ and are shown in Fig. 4.

Fig. 5 can be compared with the homoclinic orbit shown by Fig. 3 in [4]. It is easily seen that a better control of the numerical error is achieved at the free boundary x_ε than at the other end of the domain. This is also evident from Fig. 4. Moreover, connecting orbits solutions may be asymmetric (cf. Fig. 8(a) in Beyn [4]) and therefore the above treatment could result to be too expensive from a computational viewpoint.

A better way to deal with the error at both sides of the domain, as well as with asymmetric solutions, is to introduce two free boundaries, say $u_{n+1} = x_{\varepsilon+}$ and $u_{n+2} = x_{\varepsilon-}$. This can be done only if we are able to define two supplementary boundary conditions

$$h_1(\mathbf{u}(-\infty), \mathbf{u}(\infty)) = 0, \qquad h_2(\mathbf{u}(-\infty), \mathbf{u}(\infty)) = 0. \tag{4.6}$$

In this case, a different transformation from the one used above, namely

$$z = \frac{x - u_{n+2}}{u_{n+1} - u_{n+2}} \tag{4.7}$$

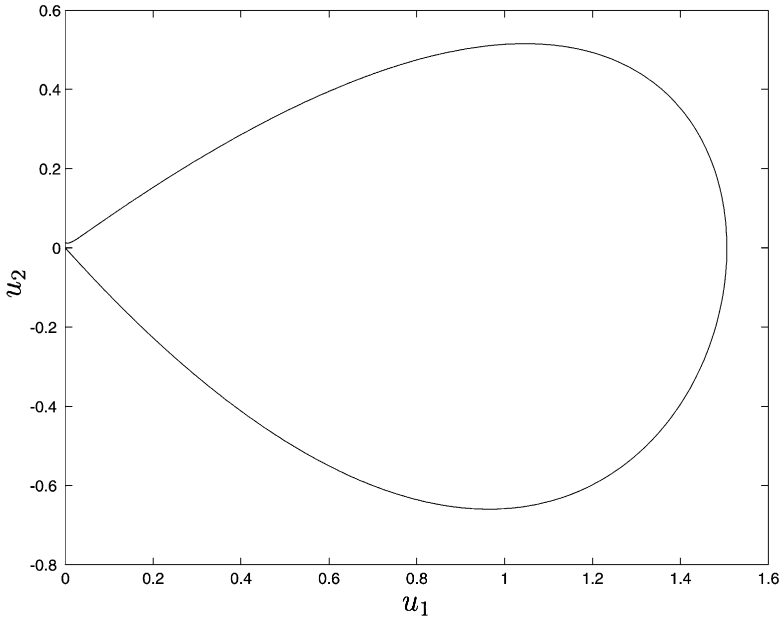

Fig. 5. The homoclinic orbit in phase plane approximated on $[-8, 8]$.

can be used to write the resulting free boundary problem in standard form. To this end, we have to append the equations

$$\frac{du_{n+1}}{dz} = 0, \quad \frac{du_{n+2}}{dz} = 0$$

to the governing system. Note that the above transformation maps $[x_{\varepsilon-}, x_{\varepsilon+}]$ to $[0, 1]$.

As an example, for the homoclinic orbit considered above we can use the supplementary boundary conditions

$$h_1(\mathbf{u}(-\infty), \mathbf{u}(\infty)) = u_2(\infty), \qquad h_2(\mathbf{u}(-\infty), \mathbf{u}(\infty)) = u_2(-\infty). \tag{4.8}$$

Then, we can introduce a new free boundary formulation and rewrite the obtained free BVP in standard form

$$\begin{aligned}
\frac{du_1}{dz} &= (u_3 - u_4)u_2, \\
\frac{du_2}{dz} &= (u_3 - u_4)(u_1 - u_1^2 + \lambda u_2 + \mu u_1 u_2), \\
\frac{du_3}{dz} &= 0, \\
\frac{du_4}{dz} &= 0
\end{aligned} \tag{4.9}$$

with boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = \varepsilon-, \quad u_2(1) = \varepsilon + . \tag{4.10}$$

5. Conclusions

The introduction was written to orient the reader within the available numerical methods for BVPs on infinite intervals which replace the infinite domains by finite ones. Then we recalled the original idea of the free boundary approach. In particular, this approach was conceived by considering the numerical solution of the simple Blasius problem. We believe that the comparison of the theoretical preliminary work, completed by graphical results, between the two classical approaches and the free boundary formulation model of a pile in soil is of relevant interest for the inexperienced reader as well as for the skillful researcher. Finally, in the previous section we considered a new subject, namely the possibility to extend the free boundary approach to the large class of problems known as connecting orbits. The reported numerical results show the validity of the free boundary approach when applied to those problems.

In conclusion, we surveyed several motivations for considering a free boundary identification of the truncated boundary as an effective way to deal with the accuracy requirement in the numerical approximation of BVPs defined on infinite intervals.

Acknowledgements

The research of this work was partially supported by the Italian “MURST”.

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