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Numerical Applications of the Scaling Concept

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Abstract. We present a survey of recent developments in the applications of the scaling concept to numerical analysis. In addition, we report on some relevant topics not covered in existing surveys. Therefore, the present work updates and complements the existing surveys on the subject concerned.

Applications of the scaling concept are useful in the numerical treatment of both ordinary and partial differential problems. Applications to boundary-value problems governed by ordinary differential equations are mainly related to their transformation into initial-value problems. Within this context, special emphasis is placed on systems of governing equations, eigenvalue, and free boundary-value problems. An error analysis for a truncated boundary formulation of the Blasius problem is also reported. As far as initial-value problems governed by ordinary differential equations are concerned, we discuss the development of adaptive mesh methods. Applications to partial differential problems considered herein are related to the construction of finite-difference schemes for conservations laws, the solution structure of the Riemann problem, rescaling schemes and adaptive schemes for blow-up problems.

In writing this paper, our aim was to promote further and more important numerical applications of the scaling concept. Meanwhile, the pertinent bibliography is highlighted and is available on internet as the BIB file `sc-gita.bib` from the anonymous `ftp` area at the `URL ftp://dipmat.unime.it/pub/papers/fazio/surveys`.

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1. Introduction

Scaling analysis is concerned with the invariance of mathematical models with respect to the scaling group of transformations (s-group). Let us consider a model involving the independent variable η, the dependent variable $f(\eta; p)$, and a parameter p. Our model might be the Blasius equation with $p \neq 0$ denoting the blowing or suction parameter, or a first-order ordinary differential equation with p representing the value of the initial condition. We assume that our model is invariant with respect to the s-group

$$\eta^* = \lambda^\delta \eta, \qquad f^* = \lambda^\alpha f, \qquad p^* = \lambda^\sigma p, \tag{1.1}$$

where λ is the exponential of the group parameter, and α, δ and σ are constants determined by the required invariance or are sometimes arbitrary (see, for instance, the Blasius problem considered in Subsection 2.1.3). As a consequence of the invariance of the model with respect to (1.1), its solutions are self-similar, namely if $f(\eta; p)$ is a solution, then $f_\lambda = \lambda^{-\alpha} f(\lambda^\delta \eta; \lambda^\sigma p)$ is also a solution for any $\lambda > 0$ so that f_λ is a family of solutions depending on the parameter λ. In applications, it is useful to define invariants with respect to (1.1); for instance

$$\frac{\mathrm{d}^n f}{\mathrm{d}\eta^n}(0)\left[\frac{\mathrm{d}^k f}{\mathrm{d}\eta^k}(\infty)\right]^{(n\delta-\alpha)/(\alpha-k\delta)} \quad \text{and} \quad \frac{\mathrm{d}^n f}{\mathrm{d}\eta^n}(0)\, p^{(n\delta-\alpha)/\sigma}$$

are invariants for all $n, k \in \mathbb{N}$. Note that the derivatives of f with respect to η under (1.1) are transformed as

$$\frac{\mathrm{d}^n f^*}{\mathrm{d}\eta^{*n}} = \lambda^{\alpha-n\delta}\frac{\mathrm{d}^n f}{\mathrm{d}\eta^n} \quad \text{for} \quad n \in \mathbb{N}.$$

The book by Dresner [27] is concerned only with scaling analysis and its applications but is written in a clear and distinctive style and provides matter of interest to those readers mainly concerned with the theoretical aspects of scaling analysis.

Scaling analysis represents an extension of the classical dimensional analysis (see Bluman and Kumei [16, pp. 22–27]). On the other hand, scaling invariance may be seen as a particular branch of the group invariance theory which was initiated by the pioneering work of Lie [61] at the end of the last century. The relevance of Lie's theory was recognized by the mathematical community in the first half of this century (see Howe [49]). In the second half of this century, group invariance theory has been developed as a fundamental tool of applied mathematics (see, for instance, the books by Birkhoff [14], Sedov [84], Hansen [47], Bluman and Cole [15], Barenblatt [8], Ovsiannikov [77], Hill [48], Dresner [27], Ibragimov [50], Seshadri and Na [85], Olver [75], Sattinger and Weaver [82], Bluman and Kumei [16], Stephani [88], Gaeta [44], Olver [76], and Ibragimov [51 – 53]). As a consequence, applications of group invariance theory have been reported for almost every branch of applied sciences.

The subject of this survey belongs to the framework of the applications of group invariance theory in numerical analysis. Indeed, several items of interest that belong to this wider subject can be listed:

- the similarity methods for the solution of difference equations reported by Maeda [62, 63], Levi and Winternitz [60], Quispel and Sahadevan [78], etc. (see also the Proceedings [59]);
- the classical perturbation theory developed by Gröbner and his co-workers for the numerical solution of initial value problems (IVPs) governed by ordinary differential equations, as reported by Stetter [89, pp. 336–343];
- the construction of numerical schemes admitting symmetries for IVPs, with recent contributions by Maeda [64], Crouch and Grossman [23], Munthe-Kaas [66], Budd and Collins [19], and Iserles [54];

- the classification of finite difference schemes for partial differential equations (DEs) according to the invariance properties of their first differential approximation, see Yanenko and Shokin [94] and the book by Shokin [86];
- the construction of finite difference schemes inheriting all symmetry of the original differential model, see Dorodnitsyn [25, 26].

We decided to focus our attention on the scaling concept for several reasons:

(1) this survey is written for a reader with no previous knowledge on group invariance theory;
(2) to the best knowledge of the author no review or survey on the applications of group invariance theory to numerical analysis have been published to date;
(3) the group invariance theory (involving infinitesimal generators, vector fields, Lie' groups, Lie's algebra, etc.) is far more difficult to introduce and to apply than the scaling analysis;
(4) the number and relevance of the different applications in this survey indicate that the scaling concept is per se a valuable tool for the numerical analysis of DEs.

Because of the great versatility of the scaling analysis, several applications have been considered in connection with the numerical treatment of DEs. Some surveys on this subject can be found in the literature: Ames [2, pp. 121–127] in 1968, Klamkin [56] in 1970, and Ames [3, pp. 136–142] in 1972, discussed the noniterative numerical solution of nonlinear boundary-value problems (BVPs); Na [70] in 1979 devoted three chapters of his book (Chps 7–9) to computational methods for BVPs on the same topic; Seshadri and Na [85, Chp 9] compared two different approaches, namely the classical inspectional analysis and the infinitesimal group method; Ames [4] in 1989 concentrated on surveying his own contribution to the subject concerning essentially singular parabolic problems and eigenparameters problems; and recently the present author [39] proposed a survey on the similarity approach to the numerical solution of free BVPs. The present work updates and complements the aforementioned surveys and is addressed to readers used to solving applicative problems with a combination of theoretical and computational tools.

The goal of this paper is to survey the applications of the scaling concept to numerical analysis. In this context, we explain the leading ideas which have been used in the literature. In this introduction we have outlined the framework of the subject under concern, whereas the literature related to the topics of the survey is quoted in the following sections. In this way, the pertinent bibliography on the subject is highlighted and is available on internet as the BIB file `sc-gita.bib` from the anonymous `ftp` area at the `URL` `ftp://dipmat.unime.it/pub/papers/fazio/surveys`. Not all the references listed in the `sc-gita.bib` file are cited in this survey because the file in point results from the author's interest on the proposed subject during the last ten years.

2. Ordinary Differential Problems

In this section we discuss the applications of the scaling concept to the numerical treatment of ordinary differential problems. Several topics can be quoted:

(1) transformation of BVPs to IVPs:
  (1.1) noniterative solution of eigenvalue problems;
  (1.2) transformation of free BVPs to IVPs;
  (1.3) error analysis for a truncated boundary approach to the Blasius problem;
(2) adaptive mesh methods for IVPs.

2.1. BOUNDARY-VALUE PROBLEMS

Hereafter, a transformation method (TM) is any transformation of BVPs to IVPs resulting from the application of similar properties of a given model. The first application of a noniterative TM was given by Töpfer in [90], where by considering a series solution for the Blasius problem, he found a transformation that reduces the BVP to a pair of IVPs. This result has been quoted in several books on fluid dynamics (see, for instance, Goldstein [45, pp. 135–136]). Acrivos, Shah and Petersen [1] first and Klamkin [55] later extended Töpfer's method, respectively, to a more general problem and to a class of problems. Na [67, 68] showed the relation between the invariance properties (with respect to an s-group of transformations) of the considered problem and the applicability of a noniterative TM. Moreover, Na considered BVPs at finite intervals and invariance with respect to a different group of transformations (the spiral group: scaling in the independent variable and translation in the dependent). The invariance of one and two or more physical parameters, when they are involved in the mathematical model, were respectively proposed by Na [69] and by Scott, Rinschler and Na [83]. It is also possible to extend the applicability of noniterative TMs by a transformation of variables linking two different groups of transformations (see [28, 30]).

Let us remark here that, within the above context, the application of scaling properties to the numerical solution of BVPs is based upon the following conjecture:

CONJECTURE. Given an ordinary differential problem where the governing DEs are invariant under an s-group, then every consistent initial-value method (one-step, multi-step, predictor-corrector, etc.) is also invariant.

The above conjecture is easily verified in the case of the Taylor, Runge–Kutta and linear multistep methods.

As remarked in the introduction, several surveys were written on the transformation of BVPs to IVPs. For this reason we refrain from giving a complete treatment of this topic; we focus instead on two sub-topics where recent progress in our understanding of the application of the scaling concept has been made; namely, the case of simultaneous governing equations and the extension via the spiral group.

Simultaneous equations. This topic was proposed by Klamkin [55] by considering the following ordinary differential problem

$$
\begin{aligned}
&\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + pg\frac{\mathrm{d}^2 g}{\mathrm{d}\eta^2} = 0,\\
&\frac{\mathrm{d}^2 g}{\mathrm{d}\eta^2} + q\left(f\frac{\mathrm{d}g}{\mathrm{d}\eta} - \frac{\mathrm{d}f}{\mathrm{d}\eta}g\right) = 0,\\
&f(0) = \frac{\mathrm{d}f}{\mathrm{d}\eta}(0) = 0, \qquad \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) \to 2 \quad \text{as } \eta \to \infty,\\
&g(0) = 0, \qquad \frac{\mathrm{d}g}{\mathrm{d}\eta}(\eta) \to 2 \quad \text{as } \eta \to \infty,
\end{aligned}
\tag{2.1}
$$

where η, f and g are appropriate similarity variables and p and q represent two physical parameters. This problem was obtained by Greenspan and Carrier [46] via a similarity analysis of a partial differential problem in magnetohydrodynamics. Note that, as shown by Rueter and Stewartson [80], there cannot be any solution if $p > 1$. The governing DEs and the boundary conditions at the origin in (2.1) are invariant with respect to the one-parameter s-group

$$\eta^* = \lambda^{-\alpha}\eta, \qquad f^* = \lambda^{\alpha} f, \qquad g^* = \lambda^{\alpha} g.$$

Nevertheless, it is not possible to transform the BVP into an IVP in the classical way (because two noninvariant conditions are given at infinity). A two-parameter group would be sufficient, but the equations do not admit any two-parameter groups. Later on Klamkin [56] pointed out that the extension to BVPs governed by simultaneous equations is possible if the equations are invariant under a multi-parameter group of transformations. The number of parameters should be at least equal to the number of conditions at infinity plus the number of noninvariant conditions at the origin.

Na [69] solved the above problem by a TM that makes use of the invariance of p. In fact, the governing DEs and the boundary conditions at the origin are invariant with respect to the two-parameter s-group

$$\eta^* = \lambda^{-\alpha}\eta, \quad f^* = \lambda^{\alpha} f, \quad g^* = \mu^{\beta} g, \quad p^* = \lambda^{2\alpha}\mu^{-2\beta} p.$$

As a consequence, a TM can be defined as follows: fix the values of α, β and p^*, the initial conditions

$$
\begin{aligned}
&f^*(0) = \frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(0) = 0, \qquad \frac{\mathrm{d}^2 f^*}{\mathrm{d}\eta^{*2}}(0) = 1,\\
&g^*(0) = 0, \qquad \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(0) = 1
\end{aligned}
$$

are used to find, via a numerical integration, the asymptotic values

$$\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty), \qquad \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^* \to \infty);$$

the application of the scaling properties provides the relations

$$\lambda = \left[\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty)/2\right]^{1/(2\alpha)}, \qquad \mu = \left[\lambda^{-\alpha}\frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^* \to \infty)/2\right]^{1/\beta},$$
$$f(\eta) = \lambda^{-\alpha} f^*(\eta^*), \qquad \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) = \lambda^{-2\alpha}\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^*),$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}(\eta) = \lambda^{-3\alpha}\frac{\mathrm{d}^2 f^*}{\mathrm{d}\eta^{*2}}(\eta^*),$$
$$g(\eta) = \mu^{-\beta} g^*(\eta^*), \qquad \frac{\mathrm{d}g}{\mathrm{d}\eta}(\eta) = \lambda^{-\alpha}\mu^{-\beta}\frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^*),$$
$$p = \lambda^{-2\alpha}\mu^{2\beta} p^*.$$

A noniterative or an iterative TM are defined if we are interested in solving the ordinary differential problem for a range of values of p or for a particular value of p. In the latter case, fix the value of p so that we can iterate different values of p^* until the zero of the implicit defined function

$$E(p^*) = \lambda^{-2\alpha}\mu^{2\beta} p^* - p$$

is found. To this end, a root-finding method can be applied.

Yakhot *et al.* [93] have introduced an iterative extension of Töpfer's method allowing for the treatment of a more general class of problems governed by simultaneous equations. Again, the invariance of the governing DEs and of the boundary conditions at the origin under a one-parameter s-group is required. Let us show how this extension can be applied to the problem (2.1). Fix the value of α and the initial conditions

$$f^*(0) = \frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(0) = 0, \qquad \frac{\mathrm{d}^2 f^*}{\mathrm{d}\eta^{*2}}(0) = 1,$$
$$g^*(0) = 0, \qquad \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(0) = \omega^*$$

are used to find, via a numerical integration, approximate values of

$$\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty), \qquad \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^* \to \infty),$$

which are constants depending only on ω^*. The application of scaling properties shows that the solution of our problem is characterized by the zero of the following implicit function

$$F(\omega^*) = \frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty) - \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^* \to \infty).$$

In fact,

$$\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty) - \frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^* \to \infty) = \lambda^{2\alpha}\left[\frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta \to \infty) - \frac{\mathrm{d}g}{\mathrm{d}\eta}(\eta \to \infty)\right];$$

the quantity enclosed in square brackets being zero due to the boundary conditions. The scaling properties allow us to obtain

$$\lambda = \left[\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^* \to \infty)/2\right]^{1/(2\alpha)},$$
$$f(\eta) = \lambda^{-\alpha} f^*(\eta^*), \qquad \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) = \lambda^{-2\alpha}\frac{\mathrm{d}f^*}{\mathrm{d}\eta^*}(\eta^*),$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}(\eta) = \lambda^{-3\alpha}\frac{\mathrm{d}^2 f^*}{\mathrm{d}\eta^{*2}}(\eta^*),$$
$$g(\eta) = \lambda^{-\alpha} g^*(\eta^*), \qquad \frac{\mathrm{d}g}{\mathrm{d}\eta}(\eta) = \lambda^{-2\alpha}\frac{\mathrm{d}g^*}{\mathrm{d}\eta^*}(\eta^*).$$

The extension of the above method to a system of three DEs was given by Ben-Dor, Rabik and Igra [10]. In that case, the zero of a system of two implicit functions was found by the Newton–Raphson method.

Let us note that if a first integral of one of the two governing DEs can be found, then the original method of Töpfer can be applied to simultaneous DEs (e.g., the numerical study of similarity solutions for combined forced and free convection flow developed by de Hoog, Laminger and Weiss [24]).

Extension by the spiral group. The extension of noniterative TM has motivated several contributions. In this context, the paper of Na [67] has had a real impact because of the utilization of the spiral group. As a result, it has been thought that TMs ought to be generalized by introducing groups different from the stretching one, see Na and Hansen [71], Belford [9], Ames and Ibragimov [7], Ames and Adams [5, 6], Na [70, pp. 155–160] or Seshadri and Na [85, pp. 157–168]. However, it is a simple matter to exhibit DEs not admitting any group of transformations (see, for instance, Hill [48, pp. 81–82] who reported a classical example due to Bianchi [13, pp. 470–475]). Consequently, we realize that noniterative TMs cannot be extended to every BVP. Moreover, it is possible to prove that the two classes of BVPs defined by the stretching and the spiral group are equivalent [35]. In this instance, the example considered by Na [67] – and quoted by Na and Tang [72], Klamkin [56], Ames [3, p. 140], Na [70, pp. 155–158], Seshadri and Na [85, pp. 157–168] or Ames [4] – belongs to the class of problems characterized by the spiral group.

Let us report first the classical result:

THEOREM 1. *The class of two-point BVPs*

$$\begin{aligned}
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = u^{1-2\delta}\Theta\left(xu^{-\delta}, \frac{\mathrm{d}u}{\mathrm{d}x}u^{\delta-1}\right),\\
&\frac{\mathrm{d}u}{\mathrm{d}x}(0) = A[u(0)]^{1-\delta}, \qquad [u(L)]^{\zeta}\,\Phi\left(\frac{\mathrm{d}u}{\mathrm{d}x}(L)[u(L)]^{\delta-1}\right) = B,
\end{aligned} \tag{2.2}$$

where A and δ are arbitrary constants, $\Theta(\cdot,\cdot)$ and $\Phi(\cdot)$ are arbitrary functions of their arguments, $L > 0$, $\zeta \neq 0$ and $B \neq 0$, can be solved noniteratively by solving a related IVP.

Outline of the proof. The governing DE and the boundary condition at $x = 0$ in (2.2) are invariant with respect to the s-group

$$x^* = \lambda^\delta x, \qquad u^* = \lambda u. \tag{2.3}$$

Hence, it is possible to define a noniterative TM. Let δ be fixed by the particular problem under consideration, otherwise we can choose its value arbitrarily. We fix a value of $u^*(0)$, whereupon according to the boundary condition, a value of $\mathrm{d}u^*/\mathrm{d}x^*(0)$ is defined. Next, we integrate numerically the IVP

$$\frac{\mathrm{d}^2u^*}{\mathrm{d}x^{*2}} = u^{*1-2\delta}\Theta\left(x^*u^{*-\delta}, \frac{\mathrm{d}u^*}{\mathrm{d}x^*}u^{*\delta-1}\right), \qquad x^* \in (0, L^*),$$
$$u^*(0) = X, \qquad \frac{\mathrm{d}u^*}{\mathrm{d}x^*}(0) = A[u^*(0)]^{1-\delta},$$

where X can be fixed at our convenience and L^* is defined by

$$L^* = L\left\{\left[\frac{[u^*(L^*)]^\zeta}{B}\,\Phi\left(\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(L^*)[u^*(L^*)]^{\delta-1}\right)\right]^{1/\zeta}\right\}^\delta.$$

Finally, the scaling properties provides

$$\lambda = \left[\frac{[u^*(L^*)]^\zeta}{B}\Phi\left(\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(L^*)[u^*(L^*)]^{\delta-1}\right)\right]^{1/\zeta},$$
$$u(x) = \lambda^{-1}u^*(x^*), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(x) = \lambda^{\delta-1}\,\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(x^*),$$

where $x \in [0, L]$ and $x^* \in [0, L^*]$. □

We note that L^* is not given explicitly, so that it can be considered as a zero of an implicit function.

In the following we state a Theorem concerning the equivalence of two apparently different classes of BVPs defined, respectively, by the spiral group and the s-group.

THEOREM 2 (in [35]). *The class of two-point BVPs*

$$\begin{aligned}&\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = \mathrm{e}^{-2\delta v}\Omega\left(x\mathrm{e}^{-\delta v}, \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{e}^{\delta v}\right)\\ &\frac{\mathrm{d}v}{\mathrm{d}x}(0) = A\mathrm{e}^{-\delta v(0)}, \qquad \mathrm{e}^{\zeta v(L)}\Phi\left(\frac{\mathrm{d}v}{\mathrm{d}x}(L)\mathrm{e}^{\delta v(L)}\right) = B\end{aligned} \tag{2.4}$$

can be solved noniteratively by the method defined in the proof of Theorem 1.

Outline of the proof. Via the variable transformation

$$x = x, \qquad u = \mathrm{e}^{v}, \tag{2.5}$$

we can reduce the following spiral group

$$x^* = \lambda^{\delta} x, \qquad v^* = v + \mu,$$

where $\mu = \ln(\lambda)$ is the group parameter, to the stretching group (2.3). Owing to change of variable (2.5), the class of BVPs (2.4) in Theorem 2 transforms to the class of BVPs (2.2) in Theorem 1 if we identify

$$\Theta(\cdot, \cdot) = \Omega\left(x u^{-\delta}, \frac{\mathrm{d}u}{\mathrm{d}x} u^{\delta-1} \right) + \left(\frac{\mathrm{d}u}{\mathrm{d}x} u^{\delta-1} \right)^2. \qquad \square$$

COROLLARY 3 (in [35]). $u(x) > 0$ *for every* $x \in (0, L)$.

Remark. The example due to Na [67] (there $p = -1$, see Na and Tang [72] for $p = 0$ and $p = 1$)

$$\begin{aligned} &\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{p+1}{x} \frac{\mathrm{d}v}{\mathrm{d}x} + q \mathrm{e}^{v} = 0, \\ &\frac{\mathrm{d}v}{\mathrm{d}x}(0) = 0, \qquad v(1) = 0, \end{aligned} \tag{2.6}$$

belongs to the class of BVPs (2.4) characterized in Theorem 2, for

$$\Phi(\cdot) = 1, \qquad A = 0, \qquad L = 1, \qquad \zeta = 1, \qquad B = 1$$

and

$$\delta = -1/2, \qquad \Omega(\cdot, \cdot) = -\frac{p+1}{x} \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{e}^{-v} - q.$$

Problem (2.6), under the transformation of variables (2.5), becomes

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{p+1}{x} \frac{\mathrm{d}u}{\mathrm{d}x} + q u^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 u^{-1} = 0, \\ &\frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0, \qquad u(1) = 1. \end{aligned} \tag{2.7}$$

Therefore, a noniterative numerical solution of (2.6) can be obtained by solving (2.7). The key point here is that invariance properties are preserved under the transformation of variables (2.5).

2.1.1. *Eigenvalue Problems*

The application of scaling properties to the numerical solution of eigenvalue problems was initiated by Belford [9]. Here we review the general idea.

THEOREM 4 (due to Belford [9]). *The class of eigenvalue problems*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \omega^2 \Xi(u),$$
$$u(0) = A, \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0, \qquad u(L) = 0,$$

where ω *represents the positive eigenvalue,* $\Xi(\cdot)$ *is a continuously differentiable function of* u *such that* $\Xi(\cdot) > 0$ *for* $u \neq 0$ *and* $\Xi(0) = 0$*, and* $A > 0$*, can be solved by a noniterative TM.*

Outline of the proof. As a first step we eliminate ω from the governing DE by means of the following transformation of variables:

$$x^* = \omega x, \qquad u^* = u.$$

Consequently, we can numerically solve the IVP

$$\frac{\mathrm{d}^2 u^*}{\mathrm{d}x^{*2}} = \Xi(u^*),$$
$$u^*(0) = A, \qquad \frac{\mathrm{d}u^*}{\mathrm{d}x^*}(0) = 0$$

and, in this way, we approximate by x_0^* the unique zero of $u^*(x^*)$. The missing boundary condition $u^*(\omega L) = 0$ is used in order to obtain the eigenvalue

$$\omega = x_0^*/L \quad \Leftarrow \begin{cases} u^*(\omega L) = 0, \\ u^*(x_0^*) = 0. \end{cases}$$

Finally, the eigenfunction can be easily obtained by a further numerical integration. □

Even if the accuracy involved in the obtained eigenvalue depends on the error related to the computation of x_0^*, we remark that the above procedure is self-validating since, in the second numerical integration, we can verify the accuracy of the obtained numerical approximation to the missed boundary condition $u(L) = 0$. Belford [9] proposed also an iterative version of this method for the case where one of the boundary conditions at the origin is not invariant under the introduced transformation of variables.

The proposed approach was extended by Ames and Adams [5] to a class of problems. This topic has been included in a survey written by Ames [4] and therefore we refer the interested reader to it for more details and applications. An interesting application was given by Bromberg and Smullin [17], who solved an eigenvalue problem governed by a nonlinear integro-DE via a noniterative TM.

2.1.2. *Free Boundary-Value Problems*

The first application of a noniterative TM to a free BVP was given by Fazio and Evans [40]. In the past, the main drawback of noniterative TMs was that they were not widely applicable (see the critical considerations by Fox, Erickson and Fan [43], Na [70, p. 137] or Sachdev [81, p. 218]). In fact, the simplest way to verify if a TM is applicable to a particular problem is to use an inspectional analysis, as shown by Seshadri and Na [85, pp. 157–168] (cf. also the discussion on inspectional analysis by Birkhoff [14, pp. 99–103]). In relation to the transformation of free BVPs to IVPs, the present author has defined an iterative extension of the TM which is widely applicable [33, 34, 36, 37].

The content of this subsection is motivated by two topics of general interest: the free boundary formulation of BVPs on infinite intervals and the similarity reduction of moving boundary problems (see Fazio [39]).

Let us define first a noniterative TM.

THEOREM 5 (in [36]). *The following class of free BVPs*

$$\begin{aligned}
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = u^{1-2\delta}\Theta\left(xu^{-\delta}, \frac{\mathrm{d}u}{\mathrm{d}x}u^{\delta-1}\right),\\
&[u(0)]^{\zeta}\,\Psi\left(\frac{\mathrm{d}u}{\mathrm{d}x}(0)[u(0)]^{\delta-1}\right) = A,\\
&u(s) = Bs^{1/\delta}, \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(s) = Cs^{(1-\delta)/\delta},
\end{aligned} \tag{2.8}$$

where $\delta \neq 0, \zeta \neq 0, A \neq 0$, *$B$ and C are arbitrary constants, s represents the unknown free boundary, and $\Theta(\cdot,\cdot)$ and $\Psi(\cdot)$ are arbitrary functions of their arguments, can be solved by a noniterative TM.*

Outline of the proof. The governing DE and of the two boundary conditions at the free boundary in (2.8) are invariant with respect to the s-group

$$x^* = \lambda^\delta x, \qquad s^* = \lambda^\delta s, \qquad u^* = \lambda u.$$

The noniterative TM is defined as follows. Let the values of δ, ζ, A, B and C be fixed by the particular problem under consideration. We can fix a value for s^* and, according to the boundary conditions at the free boundary, this defines the values of $u^*(s)$ and $\mathrm{d}u^*/\mathrm{d}x^*(s)$. The governing DE in the starred variables along with the obtained end-point conditions can be integrated inwards on $[0, s^*]$ in order to obtain the values of $u^*(0)$ and $\mathrm{d}u^*/\mathrm{d}x^*(0)$. As a consequence of the scaling properties, we have

$$\lambda = \left[\frac{[u^*(0)]^{\zeta}}{A}\Psi\left(\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(0)[u^*(0)]^{\delta-1}\right)\right]^{1/\zeta},$$

$$u(x) = \lambda^{-1} u^*(x^*), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(x) = \lambda^{\delta-1} \frac{\mathrm{d}u^*}{\mathrm{d}x^*}(x^*), \qquad s = \lambda^{-\delta} s^*,$$

where $x \in [0, s]$ and $x^* \in [0, s^*]$. □

An application of this noniterative TM to a singular free BVP is given in [29].

The above noniterative TM method is not widely applicable. However, it is possible to introduce an iterative TM.

THEOREM 6 (in [36]). *Suppose that the end-point problem*

$$\begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = h^{(1-2\delta)/\sigma} f\left(h^{-\delta/\sigma} x, h^{-1/\sigma} u, h^{(\delta-1)/\sigma} \frac{\mathrm{d}u}{\mathrm{d}x}\right), \\ &(s) = h^{1/\sigma} j\big(h^{-\delta/\sigma} s\big), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(s) = h^{(1-\delta)/\sigma} \ell\big(h^{-\delta/\sigma} s\big) \end{aligned} \tag{2.9}$$

is well-posed for every value of the parameter h (*defined below*). *Then the class of free BVPs*

$$\begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}\right), \\ &g\left(u(0), \frac{\mathrm{d}u}{\mathrm{d}x}(0)\right) = A, \\ &u(s) = j(s), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(s) = \ell(s) \end{aligned} \tag{2.10}$$

can be solved numerically by an iterative TM.

Outline of the proof. The class of problems (2.10) is recovered from the following

$$\begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = h^{(1-2\delta)/\sigma} f\left(h^{-\delta/\sigma} x, h^{-1/\sigma} u, h^{(\delta-1)/\sigma} \frac{\mathrm{d}u}{\mathrm{d}x}\right), \\ &h^{\zeta/\sigma} g\left(h^{-1/\sigma} u(0), h^{(\delta-1)/\sigma} \frac{\mathrm{d}u}{\mathrm{d}x}(0)\right) = A, \\ &u(s) = h^{1/\sigma} j\big(h^{-\delta/\sigma} s\big), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(s) = h^{(1-\delta)/\sigma} \ell\big(h^{-\delta/\sigma} s\big), \end{aligned} \tag{2.11}$$

by setting $h = 1$. The governing DE and the two boundary conditions at the free boundary in (2.11) are invariant with respect to the s-group

$$x^* = \lambda^\delta x, \qquad s^* = \lambda^\delta s, \qquad u^* = \lambda u, \qquad h^* = \lambda^\sigma h.$$

The iterative method can be defined as follows. We iterate different values of h^* until we find $|h - 1|$ within a prefixed tolerance. To this end, we fix δ, σ, ζ and s^*, and integrate inwards on $[0, s^*]$ the end-point problem (2.9) in the starred variables. In this way $u^*(0)$ and $\mathrm{d}u^*/\mathrm{d}x^*(0)$ are uniquely defined. Hence, we have

$$\lambda = \left[\frac{h^{*\zeta/\sigma}}{A} g\left(h^{*-1/\sigma} u^*(0), h^{*(\delta-1)/\sigma} \frac{\mathrm{d}u^*}{\mathrm{d}x^*}(0) \right) \right]^{1/\zeta},$$
$$u(x) = \lambda^{-1} u^*(x^*), \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(x) = \lambda^{\delta-1} \frac{\mathrm{d}u^*}{\mathrm{d}x^*}(x^*),$$
$$s = \lambda^{-\delta} s^*, \qquad h = \lambda^{-\sigma} h^*,$$

where $x \in [0, s]$ and $x^* \in [0, s^*]$. □

Note that the condition $h = 1$ is equivalent to finding a root of the implicit function (the 'transformation function')

$$\Gamma(h^*) = [\lambda(h^*)]^{-\sigma} h^* - 1$$

and, consequently, to this end a root-finding method can be applied. The iterative TM was applied in [34] to the similarity reduction of the Stefan problem and to a model describing the spreading of a viscous fluid above a smooth horizontal surface.

The following theorem provides the link between the solutions of the free BVP and the roots of the transformation function $\Gamma(\cdot)$.

THEOREM 7 (in [38]). *Let s^*, δ, σ and ζ be fixed and the end-point problem* (2.9) *be well-posed for every value of h^*. The free BVP has a unique solution if and only if the transformation function has a unique real root; nonexistence* (*nonuniqueness*) *of the solution is equivalent to the nonexistence of a real root* (*existence of more than one real root*) *of $\Gamma(\cdot)$.*

Outline of the proof (*by invariance considerations*). There exists a one-to-one and onto function between the set of solutions of the free BVP and the set of real roots of the transformation function. The thesis of the theorem follows as a simple consequence.

The key idea is that the mentioned function can be defined in the following way. Given a solution of the free BVP $(s, u(x))$, we can associate to it the real root of $\Gamma(\cdot)$ defined by $h^* = (s^*/s)^{\sigma/\delta}$ and vice versa. □

A first application of this theorem was given, on an intuitive basis, for a model describing the length estimation of tubular chemical reactors in [31], further applications can be found in [38].

2.1.3. *Error Analysis*

In this subsection we consider the celebrated Blasius problem of boundary-layer theory:

$$\begin{aligned}&\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = 0,\\ &f(0) = \frac{\mathrm{d}f}{\mathrm{d}\eta}(0) = 0, \qquad \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) \to 1 \quad \text{as } \eta \to \infty.\end{aligned} \tag{2.12}$$

The governing DE and the two boundary conditions at the origin in (2.12) are invariant with respect to the s-group

$$\eta^* = \lambda^{-\alpha}, \qquad f^* = \lambda^{\alpha} f, \tag{2.13}$$

where α can be fixed at our convenience, classically $\alpha = 1/3$; but here we set $\alpha = 1$ in order to simplify the analysis. Let us remark that the invariance mentioned above has both analytical and numerical interest. From a numerical viewpoint, a noniterative TM was defined by Töpfer [90] by transforming the boundary conditions to initial conditions. Owing to this transformation, a simple existence and uniqueness theorem for the solution of the problem was given by J. Serrin, as reported by Meyer [65, pp. 104–105].

The boundary condition at infinity in (2.12) is certainly not suitable for a numerical treatment. This condition has usually been replaced by the same condition applied at a truncated boundary (see Collatz [22, pp. 150–151] or Fox [42, p. 92]). In the truncated boundary formulation, $f_M(\eta)$ is defined by

$$\begin{aligned}&\frac{\mathrm{d}^3 f_M}{\mathrm{d}\eta^3} + f_M\frac{\mathrm{d}^2 f_M}{\mathrm{d}\eta^2} = 0,\\ &f_M(0) = \frac{\mathrm{d}f_M}{\mathrm{d}\eta}(0) = 0, \qquad \frac{\mathrm{d}f_M}{\mathrm{d}\eta}(M) = 1,\end{aligned} \tag{2.14}$$

where M represents the truncated boundary. It is evident that also in (2.14) the governing DE and the two boundary conditions at the origin are left invariant by (2.13).

For the error related to the truncated boundary solution $f_M(\eta)$ defined by

$$e(\eta) = |f(\eta) - f_M(\eta)|, \quad \eta \in [0, M],$$

the following theorem holds true.

THEOREM 8 (due to Rubel [79]). *A truncated boundary formulation of the Blasius problem introduces an error which verifies the following inequality*

$$e(\eta) \leqslant M\frac{\mathrm{d}^2 f_M}{\mathrm{d}\eta^2}(M)[f_M(M)]^{-1}.$$

Outline of the proof. As proved by Weyl [92], it is true that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}(\eta) > 0, \quad \text{for } \eta \in (0, \infty). \tag{2.15}$$

By (2.15) and taking into account the boundary conditions in (2.12), we have that

$$\frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) \quad \text{and} \quad f(\eta) \quad \text{are increasing functions on } \eta \in (0, \infty).$$

As a consequence, the function $\lambda^2(\mathrm{d}f/\mathrm{d}\eta)(\lambda M)$ is zero for $\lambda = 0$, increases with λ and tends to infinity as $\lambda \to \infty$. For some value of $\lambda \in (0, \infty)$, we must have

$$\lambda^2 \frac{\mathrm{d}f}{\mathrm{d}\eta}(\lambda M) = 1.$$

This value verifies $\lambda > 1$ because λM is a finite value, $\mathrm{d}f/\mathrm{d}\eta(\eta)$ is an increasing function and $\mathrm{d}f/\mathrm{d}\eta(\eta) \to 1$ as $\eta \to \infty$. For this particular value of λ, due to the scaling properties, we have found that $f_M(\eta) = \lambda f(\lambda\eta)$ because $\lambda f(\lambda\eta)$ verifies the BVP (2.14) that defines $f_M(\eta)$ uniquely.

Hence, the error for $\eta \in [0, M]$ is given by

$$e(\eta) = |\lambda f(\lambda\eta) - f(\eta)| \leqslant |(\lambda - 1) f(\lambda\eta)| + |f(\lambda\eta) - f(\eta)|.$$

By applying the mean-value theorem of differential calculus and taking into account that $\mathrm{d}f/\mathrm{d}\eta(\eta) \leqslant 1$ we get the relations $f(\lambda\eta) \leqslant \lambda\eta$ and $|f(\lambda\eta) - f(\eta)| \leqslant (\lambda - 1)\eta$. As a result

$$e(\eta) \leqslant M(\lambda^2 - 1), \qquad \eta \in [0, M],$$

where $\lambda^2 - 1 > 0$ because $\lambda > 1$. Naturally, $\mathrm{d}f_M/\mathrm{d}\eta(\eta \to \infty) = \lambda^2$, so that

$$\begin{aligned} \lambda^2 - 1 &= \frac{\mathrm{d}f_M}{\mathrm{d}\eta}(\eta \to \infty) - \frac{\mathrm{d}f_M}{\mathrm{d}\eta}(M) \\ &= \int_M^\infty \frac{\mathrm{d}^2 f_M}{\mathrm{d}\eta^2}\,\mathrm{d}\eta. \end{aligned}$$

To complete the proof, Rubel used some manipulations involving a first integral of the governing DE, to find that

$$\lambda^2 - 1 \leqslant \frac{\mathrm{d}^2 f_M}{\mathrm{d}\eta^2}(M)[f_M(M)]^{-1}. \qquad \square$$

Remark. As a consequence of this theorem, in order to control the error we can modify either the value of M or the value of $\mathrm{d}^2 f_M/\mathrm{d}\eta^2(M)$. Classically, the value of M has been chosen to this end. The above theorem shows that the error is directly proportional to M. In Fazio [32], a free-boundary formulation of the

Blasius problem was introduced where the second-order derivative of the solution with respect to η at the free boundary can be chosen as small as possible.

2.2. ADAPTIVE MESH METHODS FOR INITIAL-VALUE PROBLEMS

In this subsection, we follow Budd and Collins [19] in defining an adaptive mesh method for a model blow-up equation. Let us consider the IVP

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^m, \quad m > 1, \qquad u(0) = u_0, \tag{2.16}$$

which has the exact solution

$$u(t) = \left[u_0^{1-m} - (m-1)t\right]^{1/(1-m)},$$

blowing-up in the finite time

$$u(t) \to \infty \quad \text{as } t \to \frac{1}{m-1} u_0^{1-m}.$$

The governing DE is invariant with respect to the following s-group

$$t^* = \lambda^{1-m} t, \qquad u^* = \lambda u. \tag{2.17}$$

Note that tu^{m-1} is an invariant. Let us introduce for (2.16) the explicit Euler method with an adaptive mesh

$$U_{n+1} = U_n + \Delta t_n {U_n}^m, \tag{2.18}$$

where U_n is the numerical approximation to $u(t_n)$ and $\Delta t_n = t_{n+1} - t_n$ is to be defined. Let us call Δ the following invariant:

$$\Delta = {U_n}^{m-1} \Delta t_n,$$

which defines the adaptive mesh

$$\Delta t_n = {U_n}^{1-m} \Delta, \tag{2.19}$$

where $\Delta > 0$, being invariant, is a constant. Apart from rounding errors, the smaller the value given to Δ the more accurate the numerical solution. By substituting (2.19) into (2.18), we get

$$U_{n+1} = U_n(1 + \Delta), \tag{2.20}$$

whereupon the method (2.18), with the mesh defined by (2.19), becomes invariant under the action of the s-group (2.17). A simple way to verify the latter assertion is to write (2.20) in the starred variables and then substitute (2.17); the obtained relationship is independent on λ. As a consequence of (2.20), we have that

$$U_n = U_0(1 + \Delta)^n.$$

Hence, $U_n \to \infty$ as $n \to \infty$. Moreover,

$$t_n = \sum_{j=0}^{n-1} \Delta t_j = \Delta \sum_{j=0}^{n-1} {U_j}^{1-m} = {U_0}^{1-m} \frac{1-(1+\Delta)^{n(1-m)}}{1-(1+\Delta)^{1-m}}\, \Delta,$$

and, consequently,

$$t_n \to {U_0}^{1-m} \frac{\Delta}{1-(1+\Delta)^{1-m}} = \frac{1}{m-1} {U_0}^{1-m} + \mathrm{O}(\Delta).$$

The numerical solution blows-up for a finite value of t and estimates this value to an accuracy of $\mathrm{O}(\Delta)$.

Budd and Collins [19] reported some numerical experiments showing the remarkable performance of the adaptive mesh method and provided an interpretation of the method as a moving mesh method. Moreover, by comparing the above method with the classical Euler methods and with a method inheriting all the symmetries proposed by Dorodnitsyn [26], Budd and Collins found out that the classical methods are unable to reproduce the correct solution behavior and that the other method, requiring the same amount of work as solving the problem analytically, is useless as a practical numerical method.

Let us note here that, owing to (2.19), $\Delta t_n \propto {U_n}^{-m}$. Since ${U_n}^m$ is a first-order approximation of the gradient of the solution at t_n, the performance of the adaptive mesh method can be explained as follows: if the gradient of the solution is large – that is, the solution is rapidly varying – then the step length becomes smaller so that grid-points are introduced where they are needed.

3. Partial Differential Problems

In this section, we discuss the applications of the scaling concept to the numerical treatment of partial differential problems. Three topics are considered:

(1) finite difference schemes and their numerical properties;
(2) the solution structure of the Riemann problem;
(3) rescaling and adaptive schemes for blow-up problems.

3.1. FINITE DIFFERENCE SCHEMES AND THEIR NUMERICAL PROPERTIES

The analysis given below can be also applied to models governed by parabolic or elliptic equations or to higher-order or implicit methods. For the sake of simplicity, we consider the simplest model and a simple numerical scheme.

Let us consider the one-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big[f(u)\big] = 0 \qquad (3.1)$$

along with the related initial condition

$$u(x,0) = u_0(x), \qquad (3.2)$$

where $t > 0, x \in \mathbb{R}$, $f(\cdot) \in C^2(\mathbb{R})$ and $u_0(\cdot)$ is a given function of its argument. The IVP (3.1)–(3.2) has been used as a prototype model problem in order to develop numerical methods that can be extended to systems and to multi-dimensional problems (see LeVeque [58, p. 12]).

For the numerical solution of (3.1) let us consider the upwind method given by the explicit finite-difference formula

$$U_{i,j+1} = U_{i,j} - cF(U_{i-1,j}, U_{i,j}, U_{i+1,j}),$$
$$F(\cdot,\cdot,\cdot) = \begin{cases} f(U_{i,j}) - f(U_{i-1,j}) & \text{if } \frac{\mathrm{d}f}{\mathrm{d}u}(U_{i,j}) > 0, \\ f(U_{i+1,j}) - f(U_{i,j}) & \text{if } \frac{\mathrm{d}f}{\mathrm{d}u}(U_{i,j}) < 0, \end{cases}$$

where $U_{i,j} \approx u(i\Delta x, j\Delta t)$ and $c = \Delta t/\Delta x$ is the Courant number. The order of the local truncation error [te] and the local stability condition for the upwind method are respectively

$$[\mathrm{te}] = \mathrm{O}(\Delta t) + \mathrm{O}(\Delta x) \quad \text{and} \quad \frac{\Delta t}{\Delta x} \leqslant \left| \frac{\mathrm{d}f}{\mathrm{d}u}(U_{i,j}) \right|^{-1}.$$

Given a finite difference scheme, a mesh refinement is usually applied in order to verify the numerical convergence of the obtained results. A mesh refinement can be introduced by assuming that the mesh lengths transform according to the following definition:

$$\Delta t^* = \lambda \Delta t, \qquad \Delta x^* = \lambda \Delta x,$$

where $\lambda \in (0, 1)$ (e.g., $\lambda \in \{2/3, 3/4, 4/5, \ldots, k/(k+1)\}$ in [74, p. 122] or $\lambda \in \{1/2, 1/4, 1/8, \ldots, 1/2^k\}$ in [41]). The stability condition and the order of the truncation error are left-invariant by the proposed refinement, because

$$\frac{\Delta t^*}{\Delta x^*} = \frac{\Delta t}{\Delta x} \leqslant \left| \frac{\mathrm{d}f}{\mathrm{d}u}(U_{i,j}) \right|^{-1}$$

and

$$[\mathrm{te}]^* = \lambda[\mathrm{te}]. \tag{3.3}$$

The relation (3.3) represents a different way to verify the consistency of the considered method, because it means that the truncation error goes to zero as λ tends to zero.

As far as properly posed IVPs are concerned, the Lax–Richtmyer theorem states that for a consistent scheme convergence and stability are equivalent (see Lax and Richtmyer [57]). This result justifies the numerical convergence that can be obtained for finite difference schemes by a mesh refinement.

3.2. THE SOLUTION STRUCTURE OF THE RIEMANN PROBLEM

Let us now consider the simple Riemann problem where the initial condition (3.2) takes the form

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & 0 < x. \end{cases}$$

Here u_l and u_r are constants satisfying the condition $u_l \neq u_r$. Equation (3.1) is invariant with respect to the s-group

$$t^* = \lambda t, \qquad x^* = \lambda x, \qquad u^* = u,$$

where $\lambda \in \mathbb{R}^+ - \{0\}$. As a consequence, the variable transformation

$$\xi = xt^{-1}, \qquad u = u(\xi)$$

allows us to rewrite (3.1) as follows

$$\left[\frac{\mathrm{d}f}{\mathrm{d}u}(u(\xi)) - \xi\right]\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0. \tag{3.4}$$

It is now evident that the solution of the Riemann problem has to be related to the following three types of solution of (3.4):

(1) 'Constant states': these solutions verify the condition $\mathrm{d}u/\mathrm{d}\xi = 0$ and therefore are classical solutions given by $u(\xi) = \text{const}$.

(2) 'Shock waves': (nonclassical) solutions of the form

$$u(\xi) = \begin{cases} u_0, & \xi < s, \\ u_1, & s < \xi, \end{cases}$$

where u_0 and u_1 are constants and $s = [f(u_0) - f(u_1)]/(u_0 - u_1)$. Note that the definition of s follows from the Rankine–Hugoniot jump condition.

(3) 'Rarefaction waves': these are continuous solutions of the DE

$$\frac{\mathrm{d}f}{\mathrm{d}u}(u(\xi)) = \xi. \tag{3.5}$$

In fact, by assuming $\mathrm{d}^2f/\mathrm{d}u^2(u) > 0$ (or < 0) on a given range of u, we have that $\mathrm{d}f/\mathrm{d}u(\cdot)$ is a strictly monotone function of u and, therefore, invertible there. Moreover, if we differentiate (3.5) with respect to ξ, we find that

$$\frac{\mathrm{d}^2f}{\mathrm{d}u^2}(u(\xi))\frac{\mathrm{d}u}{\mathrm{d}\xi}(\xi) = 1$$

and, consequently, $\mathrm{d}u/\mathrm{d}\xi(\xi)$ is finite and always has the same sign. In this case, a rarefaction wave might connect continuously two constant states, say u_0 and u_1, when ξ lies between $\mathrm{d}f/\mathrm{d}u(u_0)$ and $\mathrm{d}f/\mathrm{d}u(u_1)$. On the other hand, if $\mathrm{d}^2f/\mathrm{d}u^2(u) = 0$ somewhere, then $\mathrm{d}u/\mathrm{d}\xi(\xi)$ goes to infinity and, consequently, at those points we have a shock.

The similarity analysis reported above has been used for the solution of the Riemann problem (see Smoller [87, pp. 266–303]).

3.3. RESCALING AND ADAPTIVE SCHEMES FOR BLOW-UP PROBLEMS

A large class of nonlinear evolution equations with scale-invariant structure and blowing-up solutions has been studied extensively during the last decades. Some examples include reaction-diffusion equations such as $u_t - \Delta u = u^m$ or $u_t - \Delta u = e^u$, which have been used as models of combustion and the nonlinear Schrödinger equation $iu_t - \Delta u = |u|^{m-1}u$ which is a model for problems in plasma physics and nonlinear optics. Berger and Kohn [11] were the first to use a scaling concept to define numerical schemes for blowing-up solutions. For the sake of simplicity, let us consider the simple blow-up equation

$$u_t = u_{xx} + u^m, \quad m > 1. \tag{3.6}$$

This equation is invariant with respect to the following s-group

$$x^* = \lambda^{(1-m)/2}x, \qquad t^* = \lambda^{1-m}t, \qquad u^* = \lambda u. \tag{3.7}$$

As a consequence, we have the following scale-invariance property: if $u(x, t)$ is a solution of (3.6), then $u_\lambda(x, t)$ defined by

$$u_\lambda(x, t) = \lambda^{-1}u(\lambda^{(1-m)/2}x, \lambda^{1-m}t)$$

is also a solution of (3.6) for any $\lambda > 0$. The key idea of rescaling algorithms is that we can set λ to be large whenever $u(x, t)$ is large, keeping the rescaled solution $u_\lambda(x, t)$ bounded. However, in order to avoid a loss of accuracy, due to the fact that the independent variables are stretched by the rescaling, additional points can be introduced for the computation of $u_\lambda(x, t)$ (for details, see Berger and Kohn [11]).

The above approach is more general and was introduced for a different context: Chorin [21] defined a rescaling algorithm for the solution of the three-dimensional Euler and Navier–Stokes equations.

An alternative to rescaling is to define adaptive mesh schemes. We again follow Budd and Collins [19] and consider Equation (3.6) which is invariant with respect to the s-group (3.7). Denote by U_j^n the numerical approximation to $u(X_j^n, t_n)$, where t_n is the nth time step and X_j^n defines the spatial mesh. For any scheme to be scaling-invariant, we have to require that if (X_j^n, t_n, U_j^n) is a numerical solution, then $(\lambda^{(1-m)/2}X_j^n, \lambda^{1-m}t_n, \lambda U_j^n)$ is also a solution.

Note that tu^{m-1} and $xu^{(m-1)/2}$ are invariant. By defining

$$\Delta t_n = t_{n+1} - t_n \quad \text{and} \quad \Delta X_j^n = X_{j+1}^n - X_j^n,$$

a scaling-invariant mesh is obtained by setting

$$\Delta t_n = \Delta \max_j \left[U_j^n\right]^{1-m} \quad \text{and} \quad \Delta X_j^n = \Delta X\left[\frac{1}{2}\left(U_{j+1}^n + U_j^n\right)\right]^{(1-m)/2},$$

where $\Delta > 0$ and $\Delta X > 0$, being invariant, are constants. Note that, if we assume that $m \geqslant 2$, then Δt_n and ΔX_j^n are small when U_j^n is large. Apart from rounding errors, the smaller the values given to Δ and ΔX the more accurate the numerical solution.

For numerical results, related to the application of self-similar adaptive mesh schemes, the interested reader is referred to Berger and Kohn [11] and Budd, Huang and Russell [20]. Moreover, Budd and Collins [18] used an invariant moving-mesh scheme for the nonlinear diffusion equation and a further application of self-similar mesh scheme has been used in the study of singularity formation in lubrication models of fluid flows (see Bertozzi [12] and the references quoted therein).

4. Conclusions

In the introduction we explained in a very simple way the meaning and the consequences of the scaling concept. Sections 2 and 3 were devoted to surveying the applications of the concept to the numerical treatment of ordinary and partial differential problems, respectively. Hence, we have tried to present a simple and comprehensive account of the applications of the scaling concept to numerical analysis. Although several of these applications are already known to researchers working in the field, some of them have received minimal attention in the literature. As an example, the error analysis for a truncated boundary formulation of the Blasius problem dating back to 1955 shows that the introduced error is directly proportional to the value of the truncated boundary, but it is a simple matter to report on recent studies where it is suggested that an increment of the truncated boundary should be used in order to assess the accuracy of the computed solution (see Nasr, Hassanien and El-Hawary [73] and the references quoted therein, in particular Wadia and Paine [91]).

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References

1. Acrivos, A., Shah, M. J., and Petersen, E. E.: Momentum and heat transfer in laminar boundary-layer flows of non-Newtonian fluids past external surfaces, *AIChE J.* **6** (1960), 312–317.
2. Ames, W. F.: *Nonlinear Ordinary Differential Equations in Transport Processes*, Academic Press, New York, 1968.
3. Ames, W. F.: *Nonlinear Partial Differential Equations in Engineering*, Vol. II, Academic Press, New York, 1972.
4. Ames, W. F.: Applications of group theory in computation, a survey, in: W. F. Ames (ed.), *Numerical and Applied Mathematics*, Vol. 1.1, J. C. Baltzer AG, Basel, 1989, pp. 47–55.
5. Ames, W. F. and Adams, E.: Exact shooting and eigenparameter problems, *Nonlinear Anal.* **1** (1976), 75–82.

6. Ames, W. F. and Adams, E.: Non-linear boundary and eigenvalue problems for the Emden–Fowler equations by group methods, *Internat. J. Nonlinear Mech.* **14** (1979), 35–42.
7. Ames, W. F. and Ibragimov, N. H.: Utilization of group properties in computation, in: N. H. Ibragimov and L. V. Ovsiannikov (eds), *Group Theoretical Methods in Mechanics*, Proc. Joint IUTAM/IMV Sympos., Novosibirsk, 1978, pp. 9–23.
8. Barenblatt, G. I.: *Similarity, Self-Similarity and Intermediate Asymptotics*, Consultant Bureau, New York, 1979.
9. Belford, G. G.: An initial value problem approach to the solution of eigenvalue problems, *SIAM J. Numer. Anal.* **6** (1969), 99–103.
10. Ben-Dor, G., Rakib, Z., and Igra, O.: A new method of solution of the boundary layer equations of a singly ionized frozen flow over a flat plate, *Israel J. Tech.* **21** (1983), 62–75.
11. Berger, M. and Kohn, R. V.: A rescaling algorithm for the numerical calculation of blowing-up solutions, *Comm. Pure Appl. Math.* **41** (1988), 841–863.
12. Bertozzi, A. L.: Symmetric singularity formation in lubrication-type equations for interface motion, *SIAM J. Appl. Math.* **56** (1996), 681–714.
13. Bianchi, L.: *Lezioni sulla Teoria dei Gruppi Continui di Trasformazioni*, Spoerri, Pisa, 1918.
14. Birkhoff, G.: *Hydrodynamics: A Study on Logic, Fact and Similitude*, 2nd edn, Princeton University Press, Princeton, 1960.
15. Bluman, G. W. and Cole, J. D.: *Similarity Methods for Differential Equations*, Springer, Berlin, 1974.
16. Bluman, G. W. and Kumei, S.: *Symmetries and Differential Equations*, Springer, Berlin, 1989.
17. Bromberg, L. and Smullin, L. D.: Gas discharges in planar low-pressure thermionic diodes. I. Space-charge regime, *J. Appl. Phys.* **50** (1979), 2630–2633.
18. Budd, C. J. and Collins, G. J.: An invariant moving mesh scheme for the nonlinear diffusion equation, *Appl. Numer. Math.* **26** (1998), 23–39.
19. Budd, C. J. and Collins, G. J.: Symmetry based numerical methods for partial differential equations, in: D. F. Griffiths, D. J. Higham and G. A. Watson (eds), *Numerical Analysis 1997*, Pitman Res. Notes in Math. 380, Longman, Harlow, 1998, pp. 16–36.
20. Budd, C. J., Huang, W. and Russell, R. D.: Moving mesh methods for problems with blow-up, *SIAM J. Sci. Comput.* **17** (1996), 305–327.
21. Chorin, A.: Estimates of intermittency, spectra, and blow-up in developed turbulence, *Comm. Pure Appl. Math.* **34** (1981), 853–866.
22. Collatz, L.: *The Numerical Treatment of Differential Equations*, 3rd edn, Springer, Berlin, 1960.
23. Crouch, P. E. and Grossman, R.: Numerical integration of ordinary differential equations on manifolds, *J. Nonlinear Sci.* **3** (1993), 1–33.
24. de Hoog, F. R., Laminger, B. and Weiss, E.: A numerical study of similarity solutions for combined forced and free convection, *Acta Mech.* **51** (1984), 139–149.
25. Dorodnitsyn, V. A.: Transformation groups in net spaces, *J. Soviet Math.* **55** (1991), 1490–1517.
26. Dorodnitsyn, V. A.: Finite difference models entirely inheriting symmetry of original differential equations, in: N. H. Ibragimov, M. Torrisi, and A. Valenti (eds), *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*, Kluwer Acad. Publ., Dordrecht, 1993, pp. 191–201.
27. Dresner, L.: *Similarity Solutions of Non-linear Partial Differential Equations*, Pitman Res. Notes in Math. 88, Pitman, London, 1983.
28. Fazio, R.: A noniterative transformation method applied to two-point boundary-value problems, *Appl. Math. Comput.* **39** (1990), 79–87.
29. Fazio, R.: A nonlinear hyperbolic free boundary value problem, *Acta Mech.* **81** (1990), 221–226.
30. Fazio, R.: Normal variables transformation method applied to free boundary value problems, *Internat. J. Comput. Math.* **37** (1990), 189–199.

31. Fazio, R.: The iterative transformation method and length estimation for tubular flow reactors, *Appl. Math. Comput.* **42** (1991), 105–110.
32. Fazio, R.: The Blasius problem formulated as a free boundary value problem, *Acta Mech.* **95** (1992), 1–7.
33. Fazio, R.: The iterative transformation method: Free or moving boundary value problems, in: C. Brezinski and P. J. U. Kulisch (eds), *Computational and Applied Mathematics, Algorithms and Theory*, Vol. I, Elsevier, New York, 1992, pp. 147–152.
34. Fazio, R.: Similarity analysis for moving boundary parabolic problems, in: W. F. Ames and P. J. van der Houwen (eds), *Computational and Applied Mathematics, Differential Equations*, Vol. II, Elsevier, New York, 1992, pp. 153–162.
35. Fazio, R.: Non-iterative transformation methods equivalence, in: N. H. Ibragimov, M. Turrisi and A. Valenti (eds), *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*, Kluwer Acad. Publ., Dordrecht, 1993, pp. 217–221.
36. Fazio, R.: Numerical transformation methods: a constructive approach, *J. Comput. Appl. Math.* **50** (1994), 299–303.
37. Fazio, R.: A novel approach to the numerical solution of boundary value problems on infinite intervals, *SIAM J. Numer. Anal.* **33** (1996), 1473–1483.
38. Fazio, R.: A numerical test for the existence and uniqueness of solution of free boundary problems, *Appl. Anal.* **66** (1997), 89–100.
39. Fazio, R.: A similarity approach to the numerical solution of free boundary problems, *SIAM Rev.* **40** (1998), 616–635.
40. Fazio, R. and Evans, D. J.: Similarity and numerical analysis for free boundary value problems, *Internat. J. Comput. Math.* **31** (1990), 215–220; **39** (1991), 249.
41. Fazio, R. and Evans, D. J.: An implicit difference scheme for a moving boundary hyperbolic problem, *Appl. Numer. Math.* **12** (1993), 485–496.
42. Fox, L.: *Numerical Solution of Two-point Boundary Value Problems in Ordinary Differential Equations*, Clarendon Press, Oxford, 1957.
43. Fox, V. G., Erickson, L. E. and Fan, L. I.: The laminar boundary layer on a moving continuous flat sheet in a non-Newtonian fluid, *AIChE J.* **15** (1969), 327–333.
44. Gaeta, G.: *Nonlinear Symmetries and Nonlinear Equations*, Kluwer Acad. Publ., Dordrecht, 1994.
45. Goldstein, S.: *Modern Developments in Fluid Dynamics*, Clarendon Press, Oxford, 1938.
46. Greenspan, H. P. and Carrier, G. F.: The magnetohydrodynamic flow past a flat plate, *J. Fluid Mech.* **6** (1959), 77–96.
47. Hansen, A. G.: *Similarity Analyses of Boundary Value Problems in Engineering*, Prentice-Hall, Englewood Cliffs, 1964.
48. Hill, J. M.: *Solution of Differential Equations by means of One-parameter Groups*, Pitman Res. Notes in Math. 63, Pitman, London, 1982.
49. Howe, R.: Very basic Lie theory, *Amer. Math. Monthly* **90** (1983), 600–623.
50. Ibragimov, N. H.: *Groups of Transformations in Mathematical Physics*, D. Reidel, Dordrecht, 1985.
51. Ibragimov, N. H.: *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 1: Symmetries, Exact Solutions, and Conservation Laws*, CRC Press, Boca Raton, FL, 1994.
52. Ibragimov, N. H.: *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 2: Applications in Engineering and Physical Sciences*, CRC Press, Boca Raton, FL, 1994.
53. Ibragimov, N. H.: *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends in Theoretical and Computational Methods*, CRC Press, Boca Raton, FL, 1996.
54. Iserles, A.: Lie groups and the computation of invariants, DAMTP Technical Report 1998/NA07.
55. Klamkin, M. S.: On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations, *SIAM Rev.* **4** (1962), 43–47.

56. Klamkin, M. S.: Transformation of boundary value problems into initial value problems, *J. Math. Anal. Appl.* **32** (1970), 308–330.
57. Lax, P. D. and Richtmyer, R. D.: Survey of the stability of linear finite difference equations, *Comm. Pure Appl. Math.* **9** (1956), 267–293.
58. LeVeque, R. J.: *Numerical Methods for Conservation Laws*, 2nd edn, Birkhäuser, Basel, 1992.
59. Levi, D., Vinet, L. and Winternitz, P. (eds): *Symmetries and Integrability of Difference Equations, Providence, 1994*, Lecture Notes Amer. Math. Soc., Proc. Workshop held Esterel, Canada, 22–29 May, 1994.
60. Levi, D. and Winternitz, P.: Continuous symmetries of discrete equations, *Phys Lett. A* **152** (1991), 335–338.
61. Lie, S. and Engel, F.: *Theorie der Transformations-gruppen*, Vol. 1–3, Taubner, Leipzig, 1890.
62. Maeda, S.: Canonical structure and symmetries for discrete systems, *Math. Japon.* **25** (1980), 405–420.
63. Maeda, S.: The similarity method for difference equations, *IMA J. Appl. Math.* **38** (1987), 129–134.
64. Maeda, S.: Application of numerical schemes admitting symmetries, *Electron. Comm. Japan* **73** (1990), 107–113.
65. Meyer, R. E.: *Introduction to Mathematical Fluid Dynamics,* Wiley, New York, 1971.
66. Munthe-Kaas, H.: Lie–Butcher theory for Runge–Kutta methods, *BIT* **35** (1995), 572–587.
67. Na, T. Y.: Transforming boundary conditions to initial conditions for ordinary differential equations, *SIAM Rev.* **9** (1967), 204–210.
68. Na, T. Y.: Further extension on transforming from boundary value to initial value problems, *SIAM Rev.* **20** (1968), 85–87.
69. Na, T. Y.: An initial value method for the solution of a class of nonlinear equations in fluid mechanics, *J. Basic Engrg. Trans. ASME* **92** (1970), 91–99.
70. Na, T. Y.: *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York, 1979.
71. Na, T. Y. and Hansen, A. G.: General group-theoretic transformations from boundary to initial value problems, NASA Rep. CR-61218, 1968.
72. Na, T. Y. and Tang, S. C.: A method for the solution of heat conduction with nonlinear heat generation, *Z. Angew. Math. Mech.* **49**(1/2) (1969), 45–52.
73. Nasr, H., Hassanien, I. A. and El-Hawary, H. M.: Chebyshev solution of laminar boundary layer flow, *Internat. J. Comput. Math.* **33** (1990), 127–132.
74. Noye, J.: Finite difference techniques for partial differential equations, in: J. Noye (ed.), *Computational Techniques for Differential Equations*, North-Holland, Amsterdam, 1984, pp. 95–354.
75. Olver, P. J.: *Applications of Lie Groups to Differential Equations,* Springer, Berlin, 1986.
76. Olver, P. J.: *Equivalence, Invariants and Symmetry*, Cambridge University Press, Cambridge, 1994.
77. Ovsiannikov, L. V.: *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
78. Quispel, G. R. W. and Sahadevan, R.: Lie symmetries and the integration of difference equations, *Phys. Lett. A* **184** (1993), 64–70.
79. Rubel, L. A.: An estimation of the error due to the truncated boundary in the numerical solution of the Blasius equation, *Quart. Appl. Math.* **13** (1955), 203–206.
80. Rueter, G. E. H. and Stewartson, K.: A non-existence theorem in magnetofluid dynamics, *Phys. Fluids* **4** (1961), 276–277.
81. Sachdev, P. L.: *Nonlinear Ordinary Differential Equations and their Applications*, Marcel Dekker, New York, 1991.
82. Sattinger, D. H. and Weaver, O. L.: *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics*, Springer, New York, 1986.

83. Scott, T. C., Rinschler, G. L. and Na, T. Y.: Further extensions of an initial value method applied to certain nonlinear equations in fluid mechanics, *J. Basic Engrg. Trans. ASME* **94** (1972), 250–251.
84. Sedov, L. I.: *Similarity and Dimensional Methods in Mechanics*, Academic Press, New York, 1959.
85. Seshadri, R. and Na, T. Y.: *Group Invariance in Engineering Boundary Value Problems*, Springer, New York, 1985.
86. Shokin, J. Yu.: *The Method of Differential Approximation*, Springer, Berlin, 1983.
87. Smoller, J.: *Shock Waves and Reaction-Diffusion Equations*, Springer, New York, 1983.
88. Stephani, H.: *Differential Equations: Their Solutions Using Symmetries*, Cambridge University Press, Cambridge, 1989.
89. Stetter, H. J.: *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer-Verlag, Berlin, 1973.
90. Töpfer, K.: Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Phys.* **60** (1912), 397–398.
91. Wadia, A. R. and Payne, F. R.: A boundary value technique for the analysis of laminar boundary layer flows, *Internat. J. Comput. Math.* **9** (1981), 163–172.
92. Weyl, H.: On the differential equation of the simplest boundary-layer problems, *Ann. Math.* **43** (1942), 381–407.
93. Yakhot, A., Ben-Dor, G., Rakib, Z. and Ingra, O.: A new approach to the solution of the boundary layer equations of an ideal compressible flow over a flat plate, *Aeronaut. J.* **85** (1981), 34–35.
94. Yanenko, N. N. and Shokin, J. Yu.: Group classification of difference schemes for a system of one-dimensional equations of gas dynamics, *Amer. Math. Soc. Transl.* **104** (1976), 259–265.