Scaling Invariance Theory and Numerical Transformation Method: A Unifying Framework

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Abstract

In a transformation method, the numerical solution of a given boundary value problem is obtained by solving one or more related initial value problems. Therefore, a transformation method, like a shooting method, is an initial value method. The peculiar difference between a transformation and a shooting method is that the former is conceived and formulated within scaling invariance theory. The main aim of this paper is to propose a unifying framework for numerical transformation methods. The non-iterative method is an extension of the Töpfer's non-iterative algorithm developed as a simple way to solve the celebrated Blasius problem. As many boundary value problems cannot be solved non-iteratively because they lack the required scaling invariance an iterative extension of the method has been developed.

This iterative method provides a simple numerical test for the existence and uniqueness of solutions, as shown by this author in the case of free boundary problems [Appl. Anal., **66** (1997) pp. 89-100] and proved herewith for a wide class of boundary value problems defined on a semi-infinite interval.

Key Words: Scaling invariance theory; numerical transformation method; BVPs on infinite intervals; unifying framework.
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1 Introduction.

A numerical transformation method is based on the scaling properties of the problem to be solved. In a transformation method, the numerical solution of a given boundary value problem (BVP) is obtained by solving one or more related initial value problems (IVPs). Here we show how the original algorithm used to solve the Blasius problem, due to Töpfer [39], has been extended to more complex problems of boundary layer theory.

The non-iterative transformation method (ITM) has been applied to several problems of practical interest. First of all, a non-ITM was applied to free boundary value problems in [23, 7, 6, 8]. Next, a non-ITM was used to solve a free boundary formulation of the Blasius problem [9]. The application of a non-ITM to the Blasius equation with slip boundary condition, arising within the study of gas and liquid flows at the micro-scale regime [3, 32], was considered already in [17]. A non-ITM was applied to the Blasius equation with moving wall considered by Ishak et al. [26] or surface gasification studied by Emmons [4] and recently by Lu and Law [31] or slip boundary conditions investigated by Gad-el-Hak [3] or Martin and Boyd [32], see Fazio [19] for details. In particular, we found a way to solve non-iteratively the Sakiadis problem [35, 36]. Recently, the non-ITM has been applied to the numerical solution of an extended Blasius problem as described by Schowalter [38], Lee and Ames [29], Lin and Chern [30], Kim et al. [27], or Akcay and Yükselen [1], see [22]. As far as the non-ITM is concerned, a recent review dealing with all the cited problems can be be found, by the interested reader, in [21].

Moreover, Töpfer's method has been extended to classes of problems in boundary layer theory involving a physical parameter. This kind of extension was considered first by Na [33], see also NA [34, Chapters 8-9] for an all-over survey on this topic.

Finally, an iterative extension of the transformation method has been introduced, for the numerical solution of free BVPs, by Fazio [23]. This iterative extension has been applied to several problems of interest: free boundary problems [23, 14, 15], a moving boundary hyperbolic problem [10], Homann and Hiemenz's problems governed by the Falkner-Skan equation in [11], one-dimensional parabolic moving boundary problems [16], two variants of the Blasius problem [17], namely: a boundary layer problem over moving surfaces, studied first by Klemp and Acrivos [28], and a boundary layer problem with slip boundary condition, that has found application in the study of gas and liquid flows at the micro-scale regime [3, 32], parabolic problems on unbounded domains [24] and, recently, see [18], a further variant of the Blasius problem in boundary layer theory: the so-called Sakiadis problem [35, 36], see Fazio [18]. In particular, in [16] the ITM is used to solve the sequence of free boundary problems obtained by a semi-discretization of 1D parabolic moving boundary problems, and in [24] a free boundary formulation for the reduced similarity models is used in order to propose a moving boundary formulation for the parabolic problems on unbounded domains. As far as the ITM is concerned, a recent review dealing with all the cited problems can be be found, by the interested reader, in [20].

2 Scaling invariance

Let us consider the class of BVPs defined by

$$\begin{aligned} & \frac{d^3 f}{d\eta^3} = \phi\left(\eta, f, \frac{df}{d\eta}, \frac{d^2 f}{d\eta^2}\right) \\ & f(0) = a \qquad \frac{df}{d\eta}(0) = b + c\frac{d^2 f}{d\eta^2}(0) \, , \qquad \frac{df}{d\eta}(\eta) \rightarrow d \quad \text{as} \quad \eta \rightarrow \infty \, , \end{aligned} \tag{2.1}$$

where a, b, c and d are given constants. Introducing the scaling group

$$f^* = \lambda f \ , \qquad \eta^* = \lambda^{\delta} \eta \ , \tag{2.2}$$

we require the invariance of (2.1), but the asymptotic boundary condition, as η goes to infinity, so that $\delta \neq 1$, with respect to (2.2). The requested invariance is granted on condition that $a = b = c = 0$ and

$$\phi\left(\eta, f, \frac{df}{d\eta}, \frac{d^2 f}{d\eta^2}\right) = \eta^{1-3\delta} \Phi\left(\eta^{1/\delta} f, \eta^{(1-\delta)/\delta} \frac{df}{d\eta}, \eta^{(1-2\delta)/\delta} \frac{d^2 f}{d\eta^2}\right) \ . \tag{2.3}$$

As a consequence of the above scaling invariance, we can define a non-ITM.

2.1 The non-iterative algorithm

In order to define the numerical method for the characterized class of problems, we have to consider the IVP

$$\begin{gathered}
\frac{d^3 f^*}{d\eta^{*3}} = \eta^{*(1-3\delta)} \Phi\left(\eta^{*(1/\delta)} f^*, \eta^{*(1-\delta)/\delta} \frac{df^*}{d\eta^*}, \eta^{*(1-2\delta)/\delta} \frac{d^2 f^*}{d\eta^{*2}}\right) \\
f^*(0) = \frac{df^*}{d\eta^*}(0) = 0 \ , \qquad \frac{d^2 f^*}{d\eta^{*2}}(0) = p \ ,
\end{gathered} \tag{2.4}$$

where p is defined by the user, usually I set $p = \pm 1$. We have to solve (2.4) in $[0, \eta^*_\infty]$, where η^*_∞ is a suitable truncated boundary chosen under the condition

$$\frac{df^*}{d\eta^*}(\infty) \approx \frac{df^*}{d\eta^*}(\eta^*_\infty) \ . \tag{2.5}$$

If $d \neq 0$, then we have

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta^*_\infty)/d\right]^{1/(1-\delta)} \ . \tag{2.6}$$

Computed the value of λ we can apply the inverse transformation of (2.2) to get

$$\begin{gathered}
\eta = \lambda^{-\delta} \eta^* \ , \quad f(\eta) = \lambda^{-1} f^*(\eta^*) \ , \\
\frac{df}{d\eta}(\eta) = \lambda^{\delta-1} \frac{df^*}{d\eta^*}(\eta^*) \ , \quad \frac{d^2 f}{d\eta^2}(\eta) = \lambda^{2\delta-1} \frac{d^2 f^*}{d\eta^{*2}}(\eta^*) \ .
\end{gathered} \tag{2.7}$$

In particular, we are interested to compute the missing initial condition $\frac{d^2 f}{d\eta^2}(0)$.

We are now ready to present the method of solution in the form of an algorithm.

The non-iterative algorithm.

1. Input $\frac{d^2 f^*}{d\eta^{*2}}(0)$, δ, η^*_∞, d;
2. solve the IVP (2.4) on $[0, \eta^*_\infty]$;
3. compute λ by (2.6);
4. rescale according to (2.7).

The above algorithm defines a non-ITM for the numerical solution of the class of problems characterized by (2.3) and $a = b = c = 0$. It is a common experience by numerical analysts and applied mathematicians in general that when dealing with non-linear problems, usually, we end up applying some iterative method of solution. Iteration means a large computational cost of the resolution algorithm. Therefore, non-iterative numerical methods are usually welcomed by the applied scientific community, because they mean a reduction in the computational complexity for the solution algorithm. In other words, non-iterative methods mean a reduction in the resolution complexity and execution time.

There are two simplest problems in this contest in boundary-layer theory. The first one describes the flow along with a horizontal flat motionless plate due to a constant free stream, see Blasius [2]. The second is flow induced by a horizontal flat plate moving with constant velocity inside a quiet fluid, see Sakiadis [35, 36]. In the first problem the fluid velocity increases from zero at the plate, no-slip boundary condition, to the mainstream velocity far away from the plate. In the second problem, the fluid velocity is equal to the plate velocity at the plate, no-slip condition, and decreases to zero far away from the plate. In both cases, the engineering interest is to calculate the shear at the plate (skin friction), which leads to the determination of the viscous drag on the plate, see for instance Schlichting [37]. For both problems, the governing equation is given by

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \,, \tag{2.8}$$

to be considered along with the related transformed boundary conditions

$$f(0) = \frac{df}{d\eta}(0) = 0 \, , \qquad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty \, ,$$

for the Blasius flow, and

$$f(0) = 0 \, , \qquad \frac{df}{d\eta}(0) = 1 \, , \qquad \frac{df}{d\eta}(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \, ,$$

for the Sakiadis flow, respectively. Let us notice here, that the governing equation (2.8) belongs to the characterized class of problems (2.1) where the condition (2.3) holds true with $\delta = -1$, and $\Phi(\cdot,\cdot,\cdot) = -\frac{1}{2} f \frac{d^2 f}{d\eta^2}$. Moreover, recently our non-ITM has been used to solve numerically an extended Blasius problem, see Fazio [22], given by

$$\begin{aligned} & \frac{d^3 f}{d\eta^3} \frac{d^2 f}{d\eta^2}^{(P-1)} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \\ & f(0) = \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty \, , \end{aligned} \tag{2.9}$$

where P verifies the conditions $1 \le P < 2$. Let us notice again, how the governing equation (2.9) belongs to the characterized class of problems (2.1) where the condition (2.3) holds true with $\delta = \frac{2-P}{1-2P}$, and $\Phi(\cdot,\cdot,\cdot) = -\frac{1}{2} f \frac{d^2 f}{d\eta^2}^{2-P}$.

2.2 The iterative algorithm

We turn now to define the numerical method for the class of problems (2.1). To this end, we consider an embedding parameter h and the extended class of problems

$$\begin{aligned} & \frac{d^3 f}{d\eta^3} = h^{(1-3\delta)/\sigma} \phi \left(h^{-(\delta/\sigma)} \eta, h^{-1/\sigma} f, h^{(\delta-1)/\sigma} \frac{df}{d\eta}, h^{(2\delta-1)/\sigma} \frac{d^2 f}{d\eta^2} \right) \\ & f(0) = a h^{1/\sigma} \, , \qquad \frac{df}{d\eta}(0) = b h^{(1-\delta)/\sigma} + c h^{(-\delta)/\sigma} \frac{d^2 f}{d\eta^2}(0) \, , \\ & \frac{df}{d\eta}(\eta) \rightarrow d \quad \text{as} \quad \eta \rightarrow \infty \, . \end{aligned} \tag{2.10}$$

Let us remark here that (2.1) is recovered from (2.10) by setting $h = 1$. Moreover, the governing differential equation and the two initial conditions in (2.10) are invariant, the asymptotic boundary condition is not invariant, with respect to the extended scaling group of transformations

$$f^* = \lambda f \ , \qquad \eta^* = \lambda^{\delta} \eta \ , \qquad h^* = \lambda^{\sigma} h \ , \tag{2.11}$$

with $\delta \neq 1$ and $\sigma \neq 0$.

We have to consider now the auxiliary IVP

$$\begin{aligned} & \frac{d^3 f^*}{d\eta^{*3}} = h_j^{*(1-3\delta)/\sigma} \phi \left(h_j^{*-(\delta/\sigma)} \eta, h_j^{*-1/\sigma} f, h_j^{*(\delta-1)/\sigma} \frac{df^*}{d\eta^*}, h_j^{*(2\delta-1)/\sigma} \frac{d^2 f^*}{d\eta^{*2*}} \right) \\ & f^*(0) = h_j^{*1/\sigma} a \ , \quad \frac{df^*}{d\eta^*}(0) = h_j^{*(1-\delta)/\sigma} b + h_j^{*(-\delta)/\sigma} p \ , \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = p \ , \end{aligned} \tag{2.12}$$

where p is defined by the user, usually I set $p = \pm 1$. We have to solve (2.12) in $[0, \eta_\infty^*]$, where η_∞^* is a suitable truncated boundary chosen under the usual asymptotic condition (2.5). Once again, if $d \neq 0$, then λ is given by (2.6). Let us remark that we are able now to dismiss the above request for d. In fact, if $d = 0$, then we can substitute do d the value $1 - h_j^{*(1-\delta)/\sigma}$ in (2.10) and compute λ_j by

$$\lambda_j = \left[\frac{df^*}{d\eta^*}(\eta_\infty^*) + h_j^{*(1-\delta)/\sigma} \right]^{1/(1-\delta)} \ . \tag{2.13}$$

It is evident that setting h_j^* arbitrarily the transformed value of h_j under (2.11) can be different from one, the target value. Therefore, we can apply a root-finder method; I usually use the secant method but, of course, bisection or regula falsi or Newton or quasi-Newton methods can be considered. For instance, by starting with suitable values of h_0^* and h_1^* the secant method is used to define the sequences h_j^* and λ_j for $j = 2, 3, \ldots,$. A related sequence $\Gamma(h_j^*)$, for $j = 0, 1, 2, \ldots,$ is defined by

$$\Gamma(h_j^*) = \lambda_j^{-\sigma} h_j^* - 1 \ , \tag{2.14}$$

where $\Gamma(\cdot)$ is defined implicitly by the solution of the IVP (2.12). Here and in the following $\Gamma(h^*)$ will be called the (implicit) transformation function. We can use the notation $\Gamma_j = \Gamma(h_j^*)$. If the computed values of λ_j are convergent to a value of λ, that is, if $|\Gamma_j|$ goes to zero, then we can apply the inverse transformation of

(2.2) to get the same formulae (2.7). A convergence criterion should be enforced, and usually, I use the condition

$$|\Gamma_j| \leq \text{Tol} \ , \tag{2.15}$$

where Tol is a user-defined tolerance. Once again, our goal is to find the missing initial condition $\frac{d^2f}{d\eta^2}(0)$.

We are now ready to present the iterative method of solution in the form of an algorithm.

The iterative algorithm.

1. Input $\frac{d^2f^*}{d\eta^{*2}}(0)$, δ, h_0^*, h_1^*, η_∞^*, d, Tol;
2. for $j = 2, 3, \ldots$; **repeat** through step 5 **until** condition (2.15) is satisfied;
3. solve (2.12) in $[0, \eta_\infty^*]$;
4. if $d \neq 0$, then compute λ_j by the value of λ given by (2.6) otherwise apply (2.13);
5. use equation (2.14) to get Γ_j;
6. rescale the solution and its domain according to (2.7).

The above algorithm defines an ITM for the numerical solution of the class of problems (2.1). We notice that if we apply as a root finder Newton's method then the value of h_1^* can be dismissed from the input parameters within our iterative algorithm. On the other hand, as far as the application of Newton's method is concerned we have to remark that in this case, as a result, we need to evaluate the derivative of the transformation function and for this purpose, we end up by doubling the computational complexity of our algorithm, see the case of the Sakiadis problem [18].

Of course, the simple Sakiadis problem (2.8)-(2.9) belongs to the class of problems that can be solved by our ITM. As a further example, we recall the Falkner-Skan model problem [5] given by

$$\begin{gathered}
\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0 \\
f(0) = \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta) \to 1 \quad \text{as} \quad \eta \to \infty
\end{gathered} \tag{2.16}$$

where $\beta = 2m/(1+m)$. In fact, the governing equation in (2.16) belongs to (2.1) when we select $\phi(\cdot,\cdot,\cdot) = -f\dfrac{d^2f}{d\eta^2} - \beta\left[1-\left(\dfrac{df}{d\eta}\right)^2\right]$.

2.3 Back to the non-iterative algorithm

Here we prove that the non-iterative algorithm is a special instance of the iterative one. This is the main goal of this paper and it may be considered as a unifying framework for the numerical transformation methods.

Teorema 1 *The non-iterative algorithm can be derived from the iterative one.*

Proof. In the proof of this theorem we consider the class of BVPs (2.10) and replace h with η and, of course, σ with δ to get the class of BVPs (2.1) when we identify

$$\begin{aligned}\phi\left(1, \eta^{-1/\delta} f, \eta^{(\delta-1)/\delta}\frac{df}{d\eta}, \eta^{(2\delta-1)/\delta}\frac{d^2f}{d\eta^2}\right) = \\ \Phi\left(\eta^{-1/\delta} f, \eta^{(\delta-1)/\delta}\frac{df}{d\eta}, \eta^{(2\delta-1)/\delta}\frac{d^2f}{d\eta^2}\right) .\end{aligned} \tag{2.17}$$

This ends the proof. □

We can notice that the boundary conditions are transformed correctly because at the first boundary we have to set $\eta = 0$.

3 Existence and uniqueness

Let us discuss now, the relation between the real zero of the transformation function $\Gamma(h^*)$ and the number of solutions of the considered BVP.

4 Existence and uniqueness

A consequence of the ITM can be stated as follows: for a given BVP the existence and uniqueness question is reduced to finding the number of real zeros of the transformation function. This result is proved below.

Teorema 2 *Let us assume that δ and σ are fixed and that for every value of h^* the auxiliary IVP (2.12) is well posed on $[0, \eta^*_\infty]$. Then, the BVP (2.1) has a unique solution if and only if the transformation function has a unique real zero; non-existence (non-uniqueness) of the solution of (2.1) is equivalent to non-existence of real zeros (existence of more than one real zero) of $\Gamma(h^*)$.*

Proof by invariance considerations. In the proof, we show that there exists a one-to-one and onto correspondence between the set of solutions of (2.1) and the set of real zeros of the transformation function. Moreover, if one of the two sets is empty the other one is empty too. Our thesis follows from this result.

Let us see how we can define this correspondence. For every values of $\frac{df^*}{d\eta^*}(0)$, δ and σ different from zero, given a solution $f(\eta)$ of (2.1), which specify a particular value of $\frac{df}{d\eta}(0)$, we can associate to it the real zero of Γ given by

$$h^* = \left[\frac{\frac{df^*}{d\eta^*}(0)}{\frac{df}{d\eta}(0)} \right]^{\sigma/(1-2\delta)} .$$

The related value of $\lambda = h^{*1/\sigma}$, allows us to verify by substitution in (2.14) that we have found a real zero of the transformation function.

According to the definition of the transformation function, in general to each real zero h^* of Γ we have fixed a solution $f^*(\eta^*)$, defined on $[0, \eta^*_\infty]$, of the auxiliary IVP (2.12). Now, the condition for $f^*(\eta^*), h^*$ to be transformed by (2.11) to $f(\eta), 1$ (where $f(\eta)$ is defined on $[0, \eta_\infty]$) is given by $\lambda = h^{*1/\sigma}$. Since $\lambda = h^{*1/\sigma}$ we have $f^*(0) = h^{*1/\sigma} f(0)$ and $\frac{df^*}{d\eta^*}(0) = h^{*(1-\delta)/\sigma} \frac{df}{d\eta}(0)$, so that the relation (2.11) implies that $f(\eta)$ verifies the boundary conditions at zero in (2.1). Hence, for each real zero of Γ we get a solution of (2.1). Again $\lambda = h^{*1/\sigma}$, and consequently $f(\eta) = h^{*-1/\sigma} f^*(h^{*-\delta/\sigma} \eta^*)$.

In this way we have defined both a right and left inverse of our correspondence. Therefore, the correspondence is one-to-one and onto. □

We came now to some remarks. The theory of well-posed IVPs is developed in detail in several classical books, as an example see [25, Chapters 2, 3 and 5]. In particular, the solution continuous dependence on parameters holds true provided suitable regularity conditions on $\phi(\eta, f, \frac{df}{d\eta}, \frac{df^2}{d^2\eta})$ are fulfilled. Second, if for every value of h^* we assume $\lambda(h^*) > 0$, then for $\delta \neq 0$ and each fixed value of h^* the scaling $[\lambda(h^*)]^{-\delta}\eta^* : [0, \eta_\infty^*] \to [0, \eta_\infty]$ is one-to-one and onto whereas the function of h^* defined by $[\lambda(h^*)]^{-\sigma}h^* : \mathbb{R} \to \mathbb{R}$ may not be one-to-one for $\sigma \neq 0$. Therefore, since $\Gamma(h^*) = [\lambda(h^*)]^{-\sigma}h^* - 1$, the transformation function may not be one-to-one. Third, it is possible to test the existence and uniqueness question by studying the behaviour of the transformation function.

5 Concluding remarks

The applicability of a non-ITM to the Blasius problem is a consequence of the invariance of the governing differential equation and initial conditions with respect to a scaling group and the non-invariance of the asymptotic boundary condition, as η goes to infinity. Several problems in boundary-layer theory lack this kind of invariance plus non-invariance and cannot be solved by non-ITMs. To overcome this drawback, we can modify the problem at hand by introducing a numerical parameter h, and require the invariance of the modified problem with respect to an extended scaling transformation group involving h, see [13, 14] for the application of this idea to classes of problems.

The main interest, when solving a boundary layer problem, is to find the so-called missed initial condition, which is the correct value of $\frac{d^2f}{d\eta^2}(0)$. This was the main goal of Blasius, Töpfer and many scientists dealing with these kind of problems. In table 1 for the reader convenience just in case he would like to face this question, we list the value for the problems considered in this manuscript. Note that, as well known, Homman and Hiemenz's flows are two special cases of the Falkner-Skan model problem for a specific value of the involved parameter. At the end of this work, we can conclude by saying that our numerical transformation methods allow us to deal with BVPs, free BVPs or defined on a semi-infinite interval, of applied mathematics that are of great practical interest. Moreover, the ITM provides a simple numerical test for the existence and uniqueness of solutions, as shown by this author in the case of free boundary problems [14]

Problem	$\frac{d^2f}{d\eta^2}(0)$	reference
Blasius	0.332057	Fazio [9]
Sakiadis	-0.443761	Fazio [18]
extended Blasius	0.618922	Fazio [22]
Homman's flow	0.927680	Fazio [12]
Hiemenz's flow	1.232588	Fazio [12]

Table 1: Missing initial condition: problems, numerical results by the non-ITM or the ITM and references.

and proved herewith for a wide class of boundary value problems defined on a semi-infinite interval.

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