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Free Boundary Formulation for Boundary Value Problems on Semi-Infinite Intervals: An Up to Date Review

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ABSTRACT

In this paper, we propose a review of the free boundary formulation for BVPs defined on semi-infinite intervals. The main idea and theorem are illustrated, for the reader convenience, by using a class of second-order BVPs. Moreover, we are able to show the effectiveness of the proposed approach using two examples where the exact solution both for the BVPs and their free boundary formulations are available. Then, we describe the free boundary formulation for a general class of BVPs governed by an n-order differential equation. In this context, we report three problems solved using the free boundary formulation. The reported numerical results, obtained by the iterative transformation method or Keller's second-order finite difference method, are found to be in very good agreement with those available in the literature. The last result of this research is that, in order to orient the interested reader, we provide an extensive bibliography. Of course, we may aspect further and more interesting applications of the free boundary formulation in the future.

MSC2020 Classification: 65L10, 34B15, 65L08

1 | Introduction

Usually, when dealing for the first time with a boundary value problem (BVP) defined on a semi-infinite interval, the applied scientist does not know the exact or even an approximate solution. As a consequence, he often is tempted to try for a numerical solution to the problem. Therefore, along the years' several approaches have been proposed in order to solve such a problem numerically.

To deal with the problems we are facing, the oldest and simplest approach is to replace the original problem by one defined on a finite interval, where a finite value, the so-called truncated boundary, is used instead of infinity. This approach was used, for instance, to get the tabulated numerical solution [1] and [2, p. 136] of the Blasius problem [3]. However, to get an accurate solution a comparison of numerical results obtained for several values of the truncated boundary is necessary as suggested by Fox [4, p. 92] and by Collatz [5, pp. 150--151]. Moreover, in some cases, accurate solutions can be found only by using very large values of the truncated boundary. This is, for instance, the case for the branches of the von Karman swirling flows where values of truncated boundaries up to several hundred were used by Lentini and Keller [6].

The limitation of the above classical approach has lead some researchers like de Hoog and Weiss [7], Lentini and Keller [8] and Markowich [9, 10] to develop a theory for defining the asymptotic boundary conditions to be imposed at a given value of the truncated boundary. Those asymptotic boundary conditions are derived by a preliminary asymptotic analysis involving the Jacobian matrix of the right-hand side of the governing equations evaluated at infinity. The main idea of this asymptotic boundary conditions approach is to project the solution into the manifold of bounded solutions. By using the same value of the truncated boundary, a more accurate numerical solution can be found by this approach than that obtained by the classical approach, because in the first case the imposed boundary conditions are obtained from the asymptotic behavior of the solution. However, we should note that this approach is not straightforward, see the remarks by Ockendon [11], and that for nonlinear problems highly nonlinear asymptotic boundary conditions usually result. Asymptotic boundary conditions have been applied successfully to the numerical approximation of the so-called "connecting orbits" problems of dynamical systems, see Beyn [12–14]. Those problems are of interest, not only in connection with dynamical systems, but also in the study of traveling wave solutions of partial differential equations of parabolic and hyperbolic type as shown by Beyn [13], Friedman [15], Bai et al. [16], and Liu et al. [17].

A different approach, for the numerical solution of BVPs defined on a semi-infinite, is to consider a free boundary formulation of the given problem, where the unknown free boundary can be identified with a truncated boundary, see [18]. In this approach, the free boundary is unknown and has to be found as part of the solution. This free boundary approach overcomes the need for an *a priori* definition of the truncated boundary. Free BVPs represent a numerical challenge because they are always nonlinear as pointed out first by Landau [19]. However, a free boundary formulation has been successfully applied to several problems in the applied sciences: namely, the Blasius problem by Fazio [18], a two-dimensional stagnation point flow by Ariel [20], the Falkner-Skan model by Fazio [21], by Zhang and Cheng [22] and by Zhu et al. [23], and the model describing a fluid flowing around a slender parabola of revolution by Fazio [24] in boundary layer theory, the computation of a two-dimensional homoclinic connecting orbit by Fazio [25], and a problem related to the deflection of a semi-infinite pile embedded in soft soil by Fazio [26]. The last problem is of interest in foundation engineering, for instance, in the design of drilling rigs above the ocean floor, see Lentini and Keller [8] and the references quoted therein.

A different way to avoid the definition of a truncated boundary is to apply coordinate transforms. Coordinate transforms have been applied successfully to the numerical solution of ordinary and partial differential equations on unbounded domains, see Grosch and Orszag [27], Koleva[28] or Fazio and Jannelli [29].

The main idea and theorem related to the free boundary formulation are illustrated using a class of second-order BVPs. In this context, we show in full details the application of the free boundary formulation to two example of BVPs defined on semi-infinite intervals. In both cases, we are able to provide the exact solution of both the BVP and its free boundary formulation. Therefore, these problems can be used as benchmarks for the numerical methods applied to BVPs on a semi-infinite intervals and to free BVPs. In this context, sometimes, it is possible to solve a given free BVP noniteratively, see the survey by Fazio [30], whereas BVPs are usually solved iteratively. Here, for two classes of free BVPs, we define noniterative initial value methods which, in the literature, are referred to as noniterative transformation methods (ITMs). Indeed, non-ITMs can be defined within Lie's group invariance theory. For the group invariance theory, the interested reader is referred to Bluman and Cole [31], Bluman and Kumei [32], Barenblatt [33], or Dresner [34].

A review paper [30] by this authors proposed a brief description related to the main topic of this survey, the numerical example there was concerning with a fluid flow around a slender parabola of revolution mimic an airplane engine.

The main goal of this paper is to provide evidence of the effectiveness of the free boundary formulation for the class of BVPs defined on semi-infinite intervals defined by

$$\begin{aligned} &\frac{d^n u}{dx^n} + f\left(x, u, \cdots, \frac{d^{n-1}u}{dx^{n-1}}\right) = 0 \\ &\frac{d^k u}{dx^k}(0) = u_k \ , \quad \text{for} \quad k = 0, 1, \cdots, n-2 \\ &\frac{d^r u}{dx^r}(\infty) = u_\infty \ , \end{aligned} \tag{1}$$

where n is a positive integer bigger than one, $f(\cdot, \cdots, \cdot)$ is a given function of its arguments, u_k, for $k = 0, 1, \cdots, n-2$, $r \in \{0, 1, \cdots, n-1\}$, and u_∞ are given constants. However, to let the interested reader gain confidence with the free boundary formulation we are going to start, in the next section, with the simplest problem of this kind and its free boundary formulation.

2 | Free Boundary Formulation Main Idea

In order to explain the main idea behind our free boundary formulation, we consider the simplest subclass of BVPs that belongs to (1), namely,

$$\begin{aligned} &\frac{d^2u}{dx^2} + g\left(x, u, \frac{du}{dx}\right) = 0 \ , \qquad x \in [0, \infty) \\ &u(0) = u_0 \ , \qquad u(\infty) = u_\infty \end{aligned} \tag{2}$$

where $g(\cdot, \cdot, \cdot)$ is a given function of its arguments, and u_0 and u_∞ are given constants. If we can assume that the first derivative of $u(x)$ goes monotonically to zero at infinity, then we replace the problem (2) with its free boundary formulation

$$\begin{aligned} &\frac{d^2u_\epsilon}{dx^2} + g\left(x, u_\epsilon, \frac{du_\epsilon}{dx}\right) = 0 \ , \qquad x \in [0, x_\epsilon] \\ &u_\epsilon(0) = u_0 \ , \qquad u_\epsilon(x_\epsilon) = u_\infty \ , \qquad \frac{du_\epsilon}{dx}(x_\epsilon) = \epsilon \end{aligned} \tag{3}$$

where x_ϵ is an unknown free boundary and $0 \leq |\epsilon| \ll 1$ is a parameter.

We have to remark here that monotonic properties of the solution, its first and second derivative have been demonstrated by Countyman and Kannan [35], for the class of problems in (2) where g depends exclusively on u.

The following theorem provides, under suitable smoothness conditions, the order of convergence (and the uniform convergence) of the solution of (3) to the solution of (2).

Theorem 1. *Suppose $u_\epsilon(x)$ and $\frac{\partial u_\epsilon}{\partial \epsilon}(x)$ are continuous functions with respect to ϵ (and also with respect to x in the related free boundary domain $[0, x_\epsilon]$) and that $|\epsilon_1| < |\epsilon_2| \Rightarrow [0, x_{\epsilon_2}] \subset [0, x_{\epsilon_1}]$ at least in a nonempty interval including $\epsilon = 0$, then*

$$||u_\epsilon(x) - u(x)|| \leq L|\epsilon|$$

where $|| \cdot ||$ is the maximum norm on $[0, x_\epsilon]$ and L is a positive constant independent on ϵ.

The proof of this Theorem can be obtained along the lines of the proof for the convergence Theorem stated in Fazio [24] for a free boundary formulation for a class of problems governed by a third-order differential equation.

It is evident that, being our method an initial value method, the order of convergence of it is the same of the numerical initial value method used.

The free boundary formulation allows us to embed a BVP in (2) into a class of problems involving the control parameter ϵ. When we solve the free boundary formulation (3) numerically, we can fix a very small value of $|\epsilon|$ and apply a grid refinement to verify whether the numerical results agree within a prefixed number of significant digits. Also, it is possible to fix a step size and let ϵ goes to zero and verify whether $u_\epsilon(x) \to u(x)$ together with $x_\epsilon \to \infty$. Usually, it suffices to take $|\epsilon| \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, \ldots\}$ and compare the obtained numerical results. Let us remark here that sometimes it is possible to solve the free boundary formulation noniteratively, see the survey by Fazio [30], whereas the BVP (2) is usually solved iteratively.

3 | Two Examples for the Free Boundary Formulation

Let us consider, now, a first example of a BVP defined on an semi-infinite interval. So, we consider the linear problem

$$\begin{aligned} &\frac{d^2u}{dx^2} + P\frac{du}{dx} = 0 \,, \qquad x \in [0, \infty) \\ &u(0) = 0 \,, \qquad u(\infty) = 1 \end{aligned} \tag{4}$$

where P is a positive constant. The solution of (4) is easily found to be

$$u(x) = 1 - e^{-Px} \tag{5}$$

so that the missing initial condition is equal to P, that is $\frac{du}{dx}(0) = P$. Figure 1 shows the solution (5) of the BVP (4) for different values of P. The bigger is the value of P, the harder is to solve the BVP numerically. In fact, for large values of P, the solution has a fast and steep transient for small values of x.

Let us consider now the free boundary formulation for (4)

$$\begin{aligned} &\frac{d^2u_\epsilon}{dx^2} + P\frac{du_\epsilon}{dx} = 0 \,, \qquad x \in [0, x_\epsilon] \\ &u_\epsilon(0) = 0 \,, \qquad u_\epsilon(x_\epsilon) = 1 \,, \qquad \frac{du_\epsilon}{dx}(x_\epsilon) = \epsilon \,, \end{aligned} \tag{6}$$

with $0 \leq \epsilon \ll 1$. The solution of (6) is given by

$$u_\epsilon(x) = \frac{P+\epsilon}{P}\left(1 - e^{-Px}\right) \,, \quad x_\epsilon = -\frac{1}{P}\ln\left(\frac{\epsilon}{P+\epsilon}\right) \tag{7}$$

Therefore, we can easily verify that as ϵ goes to zero the solution $u_\epsilon(x)$ of the free boundary formulation (6) converges to the solution $u(x)$ of the original problem (4) and the free boundary x_ϵ goes to infinity. Moreover, we realize that the obtained approximation becomes the more accurate the more ϵ is near zero; see Figure 2.

Remark. the same exact solutions (5) and (7) are still valid if we replace the governing differential equation, in the BVP (4) and its free boundary formulation (6) with the nonautonomous one

$$\frac{d^2u}{dx^2} + P^2e^{-Px} = 0 \tag{8}$$

where we substitute $u = u_\epsilon$ in the free boundary case.

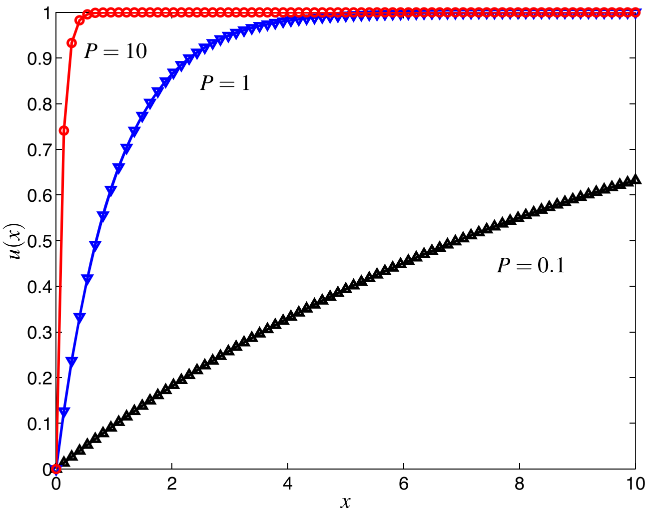

FIGURE 1 | The solution (5) for different values of P. The symbols stand for: ∘ $P = 10$, ▿ $P = 1$, and ▵ $P = 0.1$. [Colour figure can be viewed at wileyonlinelibrary.com]

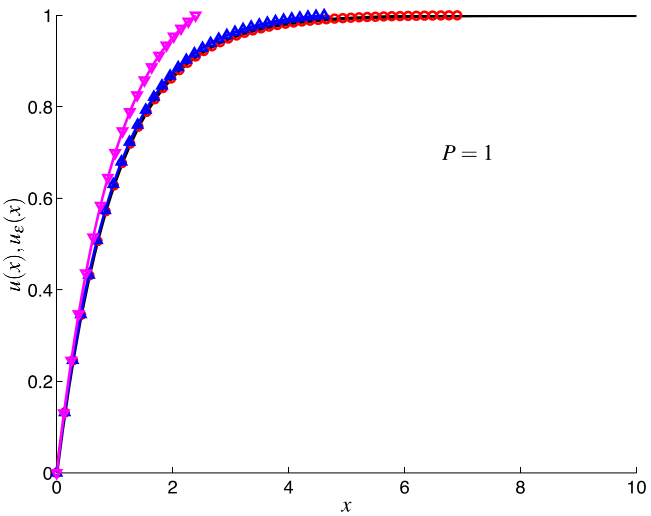

FIGURE 2 | The solution (7) for $P = 1$ and different values of ϵ. The symbols stand for: – the exact solution, ▿, ▵ and ∘ the free boundary solution u_ϵ with $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$, respectively. [Colour figure can be viewed at wileyonlinelibrary.com]

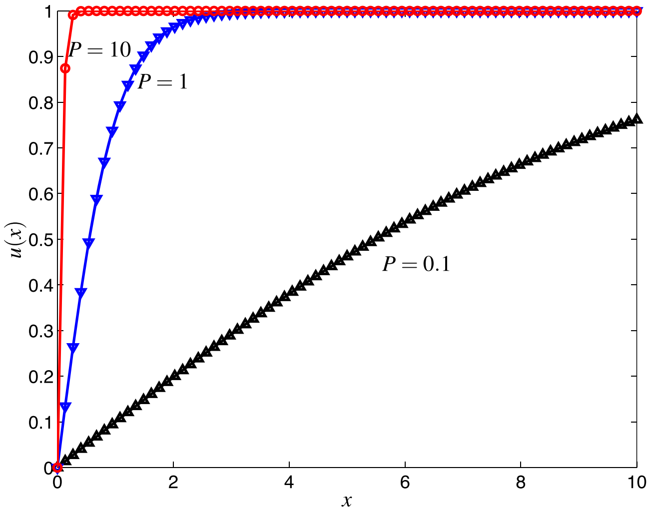

FIGURE 3 | The solution (10) for different values of P. The symbols stand for: $\circ$ $P = 10$, $\triangledown$ $P = 1$, and $\vartriangle$ $P = 0.1$. [Colour figure can be viewed at wileyonlinelibrary.com]

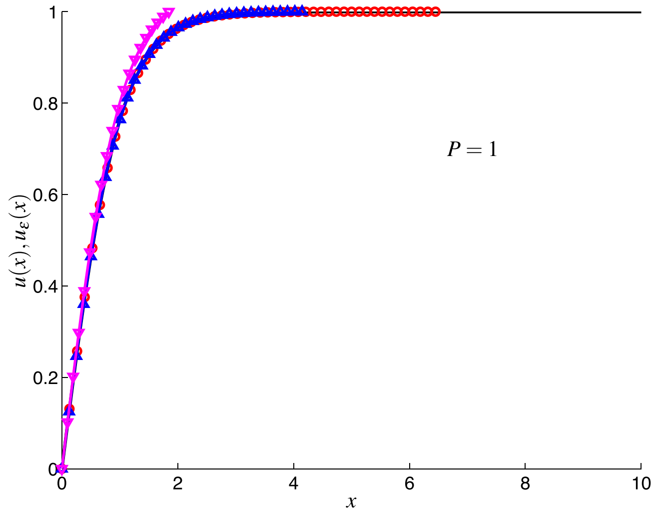

FIGURE 4 | The solution (12) for $P = 1$ and different values of ϵ. The symbols stand for: $-$ the exact solution, $\triangledown$, $\vartriangle$ and $\circ$ the free boundary solution u_ϵ with $\epsilon = 0.1$, $\epsilon = 0.001$ and $\epsilon = 0.00001$, respectively. [Colour figure can be viewed at wileyonlinelibrary.com]

Replacing a linear problem with a nonlinear, one can be justified, from a numerical viewpoint, only by considering that in this way we overcome the singularity related to the boundary condition prescribed at infinity. Of course, when the original problem is a nonlinear one, a free boundary formulation for it can be really convenient to solve numerically.

As a second example of a BVP defined on an semi-infinite interval, we consider the nonlinear problem

$$\begin{aligned} &\frac{d^2u}{dx^2} + 2Pu\frac{du}{dx} = 0 \,, \qquad x \in [0, \infty) \\ &u(0) = 0 \,, \qquad u(\infty) = 1 \,, \end{aligned} \tag{9}$$

where, again, P is a positive constant. The solution of (9) is given by

$$u(x) = \tanh(Px) \tag{10}$$

and, again, $\frac{du}{dx}(0) = P$. Figure 3 shows the solution (10) of the BVP (9) for different values of P. Again, for large values of P, the solution has a fast transient for small values of x. It can be easily verified that, for instance, by comparing Figure 1 with Figure 3, for the same value of the parameter P, the BVP (9) is more challenging than the BVP (4).

Let us consider now the free boundary formulation for (9)

$$\begin{aligned} &\frac{d^2u_\epsilon}{dx^2} + 2Pu_\epsilon\frac{du_\epsilon}{dx} = 0 \,, \qquad x \in [0, x_\epsilon] \\ &u_\epsilon(0) = 0 \,, \qquad u_\epsilon(x_\epsilon) = 1 \,, \qquad \frac{du_\epsilon}{dx}(x_\epsilon) = \epsilon \,, \end{aligned} \tag{11}$$

with $0 \le \epsilon \ll 1$. The positive solution of (11) is given by

$$u_\epsilon(x) = -\frac{1}{C}\tanh(Px) \,, \qquad x_\epsilon = \frac{1}{2P}\ln\left(\frac{1-C}{1+C}\right) \tag{12}$$

where $C = \left(\epsilon - \sqrt{\epsilon^2 + 4P^2}\right)/2P$. Also, in this case, as ϵ goes to zero the solution $u_\epsilon(x)$ of the free boundary formulation (11) converges to the solution $u(x)$ of the original problem (9) and the free boundary x_ϵ goes to infinity. Moreover, also, in this case, the obtained approximation becomes the more accurate the more ϵ is close to zero; see Figure 4.

Golden rule. The free boundary formulation would be most effective when we are able to infer that as the values of ϵ goes to zero, then the free boundary x_f goes to infinity. To this end, we can verify numerically that if $\epsilon_1 < \epsilon_2$ then $x_{f1} > x_{f2}$.

4 | Free Boundary Formulation for the BVPs (1)

Let us come, now, to the free boundary formulation for the class of BVPs (1).

$$\begin{aligned} &\frac{d^nu}{dx^n} + f\left(x, u, \cdots, \frac{d^{n-1}u}{dx^{n-1}}\right) = 0 \\ &\frac{d^ku}{dx^k}(0) = u_k \,, \qquad \text{for} \qquad k = 0, 1, \cdots, n-2 \\ &\frac{d^ru}{dx^r}(x_\epsilon) = u_\infty \,, \qquad \frac{d^{r+1}u}{dx^{r+1}}(x_\epsilon) = \epsilon \,, \end{aligned} \tag{13}$$

where x_ϵ is an unknown free boundary and $0 \le |\epsilon| \ll 1$ is a parameter.

The following theorem provides, under suitable smoothness conditions, the order of convergence (and the uniform convergence) of the solution of (13) to the solution of (1).

Theorem 2. *Assume that* $\frac{\partial^k u_\epsilon}{\partial \epsilon^k}(x)$, *for* $k = 0, 1, \cdots, n-1$, *are continuous functions with respect to* ϵ *(and also with respect to* x *in the related free boundary domain* $[0, x_\epsilon]$*) and that* $|\epsilon_1| < |\epsilon_2| \Rightarrow [0, x_{\epsilon_2}] \subset [0, x_{\epsilon_1}]$ *at least in a nonempty interval including* $\epsilon = 0$, *then*

$$||u_\epsilon(x) - u(x)|| \le L|\epsilon|$$

where $||\cdot||$ *is the maximum norm on* $[0, x_\epsilon]$ *and* L *is a positive constant independent on* ϵ.

TABLE 1 | Numerical results, related to the free boundary value x_ϵ and the missing initial condition $\frac{d^2 f}{d\eta^2}(0)$, for the BVP (15) when $P_1 = P_2 = 2$ for different values of ϵ.

ϵ	x_ϵ	$\frac{d^2u}{dx^2}$
10^{-6}	37.23	1.441377749
10^{-7}	45.62	1.441372413
10^{-8}	54.15	1.441371875
10^{-9}	62.75	1.441371815

TABLE 2 | Comparison of the velocity gradient at the plate at the wall and truncated boundary x_∞ for the Sakiadis problem.

[37]		**[39] Finite difference**[1]		**[40] Simple shooting**		**ITM**[2]	
x_∞	$\frac{d^2u}{dx^2}(0)$	x_∞	$\frac{d^2u}{dx^2}(0)$	x_∞	$\frac{d^2u}{dx^2}(0)$	x^*_∞	$\frac{d^2u}{dx^2}(0)$
	−0.44375		−0.4438	20	−0.443747	10	−0.443761

[1] Keller's second order box scheme [41] and [2], ITM stands for iterative transformation method.

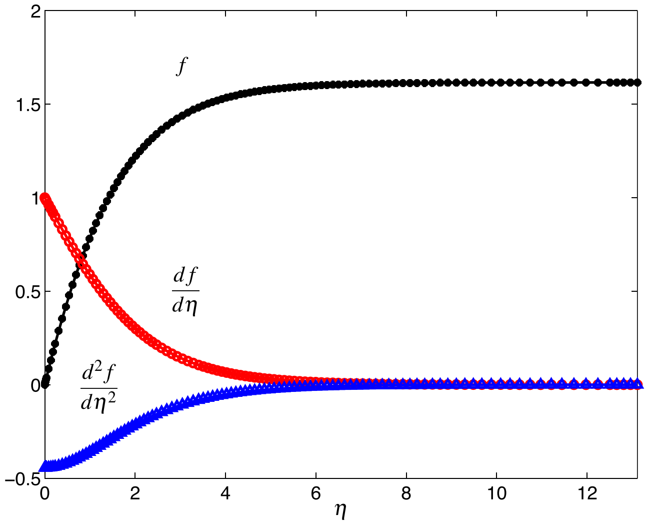

FIGURE 5 | Numerical solution for the Sakiadis problem via the iterative transformation method. [Colour figure can be viewed at wileyonlinelibrary.com]

For the proof of Theorem 2 we can follow the lines of the proof of the convergence Theorem stated in Fazio [24] for a free boundary formulation for a class of problems governed by a third-order differential equation.

Once again, the free boundary formulation allows us to embed a BVP in (2) into a class of problems involving the control parameter ϵ.

5 | Examples With Numerical Results

As a first example, we recall the model describing the flow of an incompressible fluid around a slender parabola of revolution, mimic an airplane engine, as described by Na [36, pp. 217--221]

$$
\begin{aligned}
&(1 + P_1 x)\frac{d^3u}{dx^3} + \left(\frac{1}{2}u + P_1\right)\frac{d^2u}{dx^2} = 0\\
&u(0) = P_2 \,, \qquad \frac{du}{dk}(0) = 0 \,,\\
&\frac{du}{dx}(\infty) = 1 \,,
\end{aligned}
\tag{14}
$$

where P_1 is the transverse curvature parameter and P_2 is a parameter related to suction (when $P_2 < 0$) or blowing (when $P_2 > 0$). By setting $P_1 = 0$ in (14) we end up with Blasius problem in the case of suction or blowing. A free boundary formulation for this problem is as follows:

$$
\begin{aligned}
&(1 + P_1 x)\frac{d^3u}{dx^3} + (\frac{1}{2}u + P_1)\frac{d^2u}{dx^2} = 0\\
&u(0) = P_2 \,, \qquad \frac{du}{dk}(0) = 0 \,,\\
&\frac{du}{dx}(x_\epsilon) = 1 \,, \qquad \frac{d^2u}{dx^2}(x_\epsilon) = \epsilon \,,
\end{aligned}
\tag{15}
$$

where x_ϵ is the introduced free boundary and ϵ is a small value. We list in Table 1 some numerical results, concerning the missing initial condition, obtained by an iterative transformation method for the case where $P_1 = P_2 = 2$. From the results in Table 1, we can conclude that the missing initial condition ha the value $\frac{d^2 f}{d\eta^2}(0) = 1.4413718$.

As a second example, we consider the Sakiadis BVP [37, 38]

$$
\begin{aligned}
&\frac{d^3u}{dx^3} + \frac{1}{2}u\frac{d^2u}{dx^2} = 0 \,,\\
&u(0) = 0 \,, \qquad \frac{du}{dx}(0) = 1 \,,\\
&\frac{du}{dx}(\infty) = 0 \,.
\end{aligned}
\tag{16}
$$

A free boundary formulation for this problem can be easily obtained and is given by

$$\frac{d^3u}{dx^3} + \frac{1}{2}u\frac{d^2u}{dx^2} = 0 \,,$$
$$u(0) = 0 \,, \qquad \frac{du}{dx}(0) = 1 \,, \qquad (17)$$
$$\frac{du}{dx}(x_\epsilon) = 0 \,, \qquad \frac{d^2u}{dx^2}(x_\epsilon) = \epsilon \,, \,.$$

where x_ϵ is the introduced free boundary and ϵ is a small positive value.

In Table 2, we propose a comparison between our results and those reported in the literature. As it can be easily seen, our numerical results compare very well with those obtained by other authors.

Figure 5 shows the numerical approximation.

As our last example, we consider a BVP that was already used by Lentini and Keller [8] to test the asymptotic boundary conditions

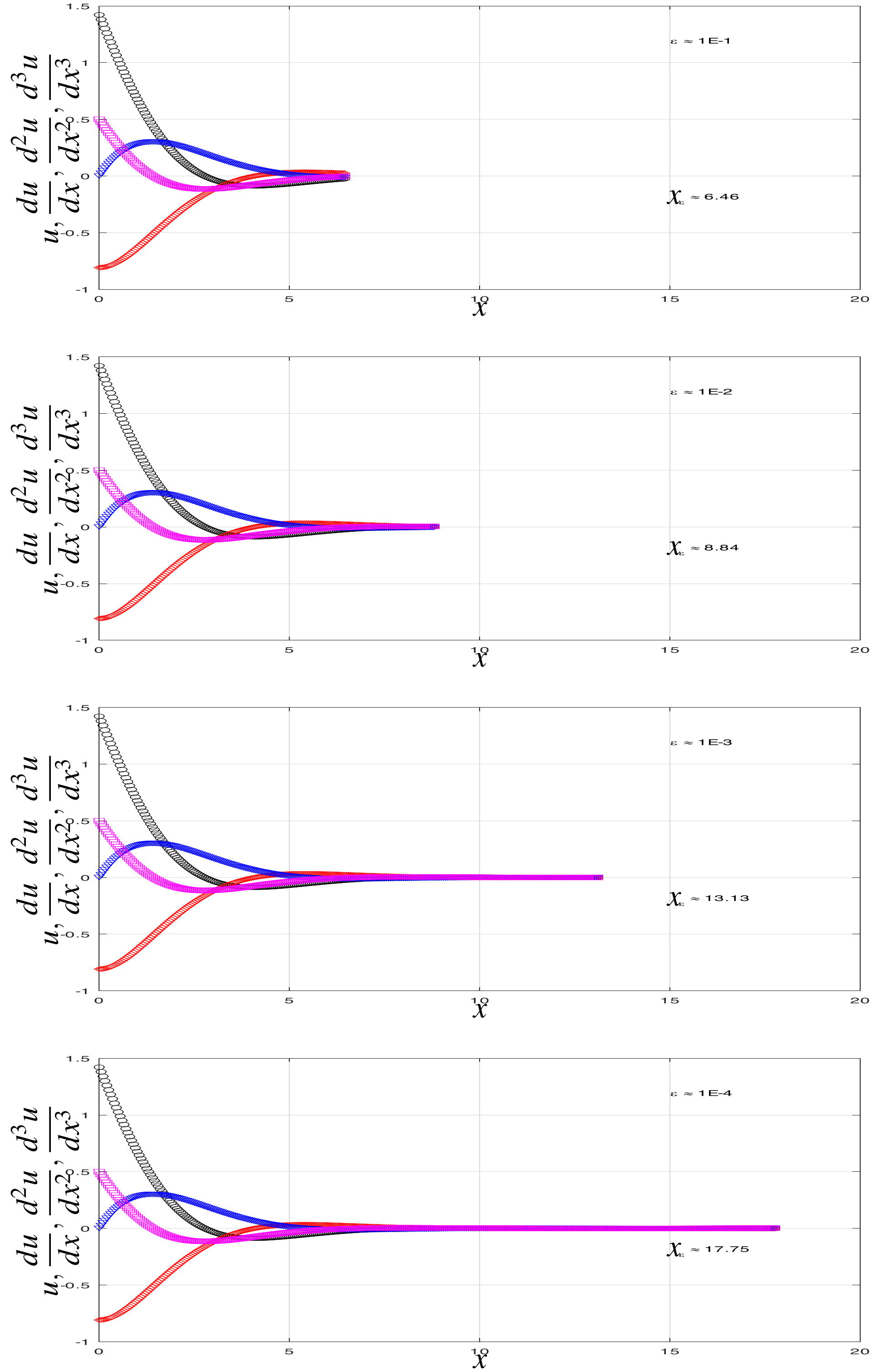

FIGURE 6 | The numerical solution of the pile problem by the free boundary approach. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 | Numerical results for the BVP (18), for these results we used 1001 grid-points.

ϵ	x_ϵ	$u(0)$	$\frac{du}{dx}(0)$	$\frac{d^2u}{dx^2}(x_\epsilon)$	$\frac{d^3u}{dx^3}(x_\epsilon)$
$1E-1$	6.46	1.41566	−0.805665	$-5.9E-2$	$-4.1E-2$
$1E-2$	8.84	1.42148	−0.808104	$-4.4E-3$	$5.6E-3$
$1E-3$	13.13	1.42154	−0.808146	$8.9E-4$	$1.1E-4$
$1E-4$	17.75	1.42154	−0.808144	$-7.0E-5$	$-3.0E-5$

TABLE 4 | Numerical results computed via a mesh refinement.

2^kM+1	$u(0)$	$\frac{du}{dx}(0)$
126	1.421166	−0.807913
251	1.421450	−0.808089
501	1.421521	−0.808133
1001	1.421539	−0.808144
2001	1.421543	−0.808147
4001	1.421544	−0.808148
8001	1.421545	−0.808148
16001	1.421545	−0.808148

approach. That problem is of special interest here because none of the solution components is a monotone function (see the bottom frame of Figure 6).

Let $u(x)$ be the deflection of a semi-infinite pile embedded in soft soil at a distance x below the surface of the soil. The governing differential equation for the movement of the pile, in dimensionless form, is given by

$$\begin{aligned} &\frac{d^4u}{dx^4} + P_1\left(1 - e^{-P_2 u}\right) = 0 \,, \qquad 0 < x < \infty \,, \\ &\frac{d^2u}{dx^2}(0) = 0 \,, \qquad \frac{d^3u}{dx^3}(0) = P_3 \,. \\ &u(\infty) = \frac{du}{dx}(\infty) = 0 \,, \end{aligned} \tag{18}$$

where P_1 and P_2 are positive material constants. At the origin, a zero moment and a positive shear are assumed. Moreover, from physical considerations it follows that $u(x)$ and all its derivatives go to zero at infinity, that is as $x \to \infty$, so that, the asymptotic boundary conditions are obtained. This problem is of interest in foundation engineering: for instance, in the design of drilling rigs above the ocean floor. We consider now, a free boundary formulation for the BVP (18), namely,

$$\begin{aligned} &\frac{d^4u}{dx^4} + P_1\left(1 - e^{-P_2 u}\right) = 0 \,, \qquad x \in [0, \infty) \,, \\ &\frac{d^2u}{dx^2}(0) = 0 \,, \qquad \frac{d^3u}{dx^3}(0) = P_3 \,. \\ &u(\infty) = \frac{du}{dx}(x_\epsilon) = 0 \,, \qquad \left|\frac{d^2u}{dx^2}(x_\epsilon)\right| + \left|\frac{d^3u}{dx^3}(x_\epsilon)\right| = \epsilon \,, \end{aligned} \tag{19}$$

where P_3 is a physical parameter.

For comparative purposes we used the same parameter values employed by [8]:

$$P_1 = 1, \quad P_2 = \frac{1}{2} \qquad \text{and} \qquad P_3 = \frac{1}{2} \tag{20}$$

Moreover, we choose to consider the values of the missing initial conditions $u(0)$ and $\frac{du}{dx}(0)$ as representative results. A direct way to proceed is to fix a suitable fine grid and to perform a convergence test for decreasing values of ϵ, note that we should set $\epsilon \ll 1$ (see Table 3). Here, the E notation indicates a single precision arithmetic.

Figure 6 displays the numerical results related to different values of ϵ obtained by setting 2001 grid-points. As it is easily seen none of the solution components is monotone on the interval of interest.

To verify the numerical accuracy, we applied a mesh refinement obtained by fixing a value of positive M and setting the number of mesh-points equal to 2^kM for $k = 0, 1, 2, \dots$. For the results reported in Table 4 we fixed $M = 125$ and the number of grid-points given by $2^kM + 1$ for $k = 0, 1, 2, \dots, 7$. Here, we fixed $\epsilon = 1E-4$. When the number of grid-points $2^kM + 1$ was 2001 we found $\frac{d^2u}{dx^2}(x_\epsilon) = -7E-5$, $\frac{d^3u}{dx^3}(x_\epsilon) = -3E-5$ and $x_\epsilon = 17.747988$.

6 | Conclusions

In this paper, we propose a review on the free boundary formulation for BVPs defined on semi-infinite intervals. The main idea and result is illustrated by using a class second order of problems. Moreover, we are able to show the effectiveness of the proposed approach using two examples where we can use the exact solution both for the BVPs and their free boundary formulation s. Then, we describe the free boundary formulation for a general class of BVPs governed by an *n* order differential equation. In this context, we report three problems solved by using the free boundary formulation. The reported numerical results, obtained by the iterative transformation method or the Keller's second-order finite difference method, are found to be in very good agreement with those available in the literature. It is evident that, being our method an initial value method, the order of convergence of it is the same of the numerical initial value method used. Of course, we may expect to see further and more important applications of the free boundary formulation for BVPs defined on semi-infinite intervals in the future. In this context, as far as this author know-ledges are concerned, several problems of boundary layer theory are defined on a semi-infinite interval and then can take advantages and solved by using our free boundary formulation.

Author Contributions

Riccardo Fazio: conceptualization, investigation, funding acquisition, writing – original draft, methodology, validation, visualization, writing – review and editing, software, formal analysis, project administration, data curation, supervision, resources.

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References

1. L. Horwarth, "On the Solution of the Laminar Boundary Layer Equations," *Proceedings of the Royal Society A* 164 (1938): 547–579.

2. S. Goldstein, *Modern Developments in Fluid Dynamics* (Clarendon Press, 1938).

3. H. Blasius, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zeitschrift für Mathematik und Physik* 56 (1908): 1–37.

4. L. Fox, *Numerical Solution of Two-Point Boundary Value Problems in Ordinary Differential Equations* (Clarendon Press, 1957).

5. L. Collatz, *The Numerical Treatment of Differential Equations*, 3rd ed. (Springer, 1960).

6. M. Lentini and H. B. Keller, "The von Karman Swirling Flows," *SIAM Journal on Applied Mathematics* 38 (1980): 52–64.

7. F. R. de Hoog and R. Weiss, "An Approximation Theory for Boundary Value Problems on Infinite Intervals," *Computing* 24 (1980): 227–239.

8. M. Lentini and H. B. Keller, "Boundary Value Problems on Semi-Infinite Intervals and Their Numerical Solutions," *SIAM Journal on Numerical Analysis* 17 (1980): 577–604.

9. P. A. Markowich, "A Theory for the Approximation of Solution of Boundary Value Problems on Infinite Intervals," *SIAM Journal on Mathematical Analysis* 13 (1982): 484–513.

10. P. A. Markowich, "Analysis of Boundary Value Problems on Infinite Intervals," *SIAM Journal on Mathematical Analysis* 14 (1983): 11–37.

11. J. R. Ockendon, *Mathematical Reviews* 84c: 34021.

12. W. J. Beyn, "Global Bifurcation and Their Numerical Computation," in *Bifurcation: Numerical Techniques and Applications*, eds. D. Rossed, B. D. Dier, and A. Spence (Kluwer, 1990), 169–181.

13. W. J. Beyn, "The Numerical Computation of Connecting Orbits in Dynamical Systems," *IMA Journal of Numerical Analysis* 9 (1990): 379–405.

14. W. J. Beyn, "Numerical Methods for Dynamical Systems," in *Advances in Numerical Analysis*, ed. W. Light (Clarendon Press, 1992), 175–236.

15. M. J. Friedman and E. J. Doedel, "Numerical Computation and Continuation of Invariant Manifolds Connecting Fixed Points," *SIAM Journal on Numerical Analysis* 28 (1991): 789–808.

16. F. Bai, A. Spence, and A. M. Stuart, "The Numerical Computation of Heteroclinic Connections in Systems of Gradient Partial Differential Equations," *SIAM Journal on Applied Mathematics* 53 (1993): 743–769.

17. L. Liu, G. Moore, and R. D. Russell, "Computation and Continuation of Homoclinic and Heteroclinic Orbits With Arclength Parameterization," *SIAM Journal on Scientific Computing* 18 (1997): 69–93.

18. R. Fazio, "The Blasius Problem Formulated as a Free Boundary Value Problem," *Acta Mechanica* 95 (1992): 1–7.

19. H. G. Landau, "Heat Conduction in Melting Solid," *Quarterly of Applied Mathematics* 8 (1950): 81–94.

20. P. D. Ariel, "Stagnation Point Flow - A Free Boundary Value Problem Formulation," *International Journal of Computer Mathematics* 49 (1993): 123–131.

21. R. Fazio, "The Falkner-Skan Equation: Numerical Solutions within group invariance Theory," *Calcolo* 31 (1994): 115–124.

22. J. Zhang and B. Chen, "An Iterative Method for Solving the Falkner-Skan Equation," *Applied Mathematics and Computation* 210 (2009): 215–222.

23. S. Zhu, Q. Wu, and X. Cheng, "Numerical Solution of the Falkner-Skan Equation Based on Quasilinearization," *Applied Mathematics and Computation* 215 (2009): 2472–2485.

24. R. Fazio, "A Novel Approach to the Numerical Solution of Boundary Value Problems on Infinite Intervals," *SIAM Journal on Numerical Analysis* 33 (1996): 1473–1483.

25. R. Fazio, "A Survey on Free Boundary Identification of the Truncated Boundary in Numerical BVPs on Infinite Intervals," *Journal of Computational and Applied Mathematics* 140 (2002): 331–344.

26. R. Fazio, "A Free Boundary Approach and Keller's Box Scheme for BVPs on Infinite Intervals," *International Journal of Computer Mathematics* 80 (2003): 1549–1560.

27. C. E. Grosch and S. A. Orszag, "Numerical Solution of Problems in Unbounded Regions: Coordinate Transforms," *Journal of Computational Physics* 25 (1977): 273–296.

28. M. N. Koleva, "Numerical Solution of the Heat Equation in Unbounded Domains Using Quasi-Uniform Grids," in *Large-scale Scientific Computing*. *Lecture Notes in Comput. Sci*., vol. 3743, eds. I. Lirkov, S. Margenov, and J. Wasniewski (Springer, 2006), 509–517.

29. R. Fazio and A. Jannelli, "Finite Difference Schemes on Quasi-Uniform Grids for BVPs on Infinite Intervals," *Journal of Computational and Applied Mathematics* 269 (2014): 14–23.

30. R. Fazio, "A Similarity Approach to the Numerical Solution of Free Boundary Problems," *SIAM Review* 40 (1998): 616–635.

31. G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer, 1974).

32. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, 1989).

33. G. I. Barenblatt, *Scaling, Self-Similarity and Intermediate Asymptotics* (Cambridge University Press, 1996).

34. L. Dresner, *Applications of Lie's Theory of Ordinary and Partial Differential Equations* (Institute of Physics Publishing, 1999).

35. M. Countryman and R. Kannan, "A Class of Nonlinear Boundary Value Problems on the Half Line," *Computers & Mathematcs with Applications* 28 (1994): 121–130.

36. T. Y. Na, *Computational Methods in Engineering Boundary Value Problems* (Academic Press, 1979).

37. B. C. Sakiadis, "Boundary-Layer Behaviour on Continuous Solid Surfaces: I. Boundary-Layer Equations for Two-Dimensional and Axisymmetric Flow," *AICHE Journal* 7 (1961): 26–28.

38. B. C. Sakiadis, "Boundary-Layer Behaviour on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface," *AICHE Journal* 7 (1961): 221–225.

39. A. Ishak, R. Nazar, and I. Pop, "Boundary Layer on a Moving Wall With Suction and Injection," *Chinese Physics Letters* 24 (2007): 2274–2276.

40. R. Cortell Bataller, "Numerical Comparisons of Blasius and Sakiadis Flows," *Matematika* 26 (2010): 187–196.

41. H. B. Keller, "Accurate Difference Methods for Nonlinear Two-Point Boundary Value Problems," *SIAM Journal on Numerical Analysis* 11 (1974): 305–320.