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On the equivalence of non-iterative transformation methods based on scaling and spiral groups

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The non-iterative numerical solution of nonlinear boundary value problems is a subject of great interest. The present paper is concerned with the theory of non-iterative transformation methods (TMs). These methods are defined within group invariance theory. Here we prove the equivalence between two non-iterative TMs defined by the scaling group and the spiral group, respectively. Then, we report on numerical results concerning the steady state temperature space distribution in a non-linear heat generation model. These results improve the ones, available in the literature, obtained by using the invariance with respect to a spiral group. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

The non-iterative numerical solution of nonlinear boundary value problems (BVPs) is a topic of great interest. In this context, transformation methods (TMs) are founded on group invariance theory, see Klamkin [1] or [2]. These methods are initial value methods because they transform BVPs to initial value problems (IVPs).

The first application of a non-iterative TM was given by Töpfer in [3] for the numerical solution of the classical Blasius problem in boundary layer theory. This result is quoted in several books on fluid dynamics, see, for instance, Goldstein [4, pp. 135–136]. Acrivos *et al*. [5] extended Töpfer's method to a more general problem. Klamkin [1] showed the relation between the invariance properties, with respect to a linear group of transformation: the scaling group, and the applicability of a non-iterative TM. Na [6, 7] considered BVPs defined on a finite domain and the invariance with respect to a nonlinear group of transformations: the spiral group.

Belford [8], Ames and Adams [9, 10] proposed non-iterative TMs for eigenvalue problems. A review paper on this topic was written by Klamkin [2]. The invariance of one and of two or more physical parameters, if they are involved in the mathematical model, were, respectively, proposed by Na [11] and by Scott *et al*. [12]. The book [13, Chapters 7–9] written by Na, on the numerical solution of BVP, devoted three chapters to numerical TMs. Fazio and Evans [14] introduced non-iterative TMs for free BVPs. Within the last years the non-iterative numerical solution of BVPs has been used to solve connecting orbits problems [15], and parabolic problems defined on an unbounded domain [16].

Because of the utilization of the spiral group, the paper of Na [6] has been quoted by several authors. As a consequence of this work, it has been thought that TMs ought to be generalized by introducing groups different from the scaling or spiral ones, see Belford [8], Ames and Ibragimov [17], Ames and Adams [9, 10], Na [13, p. 5] or Seshadry and Na [18, pp. 157–168]. However, we shall prove in the following that the utilization of scaling or of spiral groups are equivalent. In fact, we can extend the applicability of non-iterative TMs by some variable transformations linking different invariant groups, see Fazio [19, 20]. Through a transformation of variables, we can sometimes state a given problem in an alternative form which, though equivalent, may benefit of established

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resolution (non-iterative) methods. A preliminary version of this work was presented recently at The 2008 International Conference of Applied and Engineering Mathematics within the World Congress on Engineering [21].

A remark is in order here: non-iterative TMs are applicable only to particular classes of BVPs so that they have been considered as an *ad hoc* methods, see Meyer [22, pp. 35–36], Na [13, p. 137] or Sachdev [23, p. 218]. In fact, it is a simple matter to show a differential equation not admitting a group of transformations, see for instance, Hill [24, pp. 81–82] who reported a classical example due to Bianchi [25, pp. 470–475]. Consequently, it is easy to realize that non-iterative TMs cannot be extended to every BVPs. However, BVPs governed by the most general second-order differential equation in normal form can be solved iteratively by extending a scaling group via the introduction of a numerical parameter so as to recover the original problem as this parameter goes to one, see Fazio [26–28]. The extension of this iterative TM to moving boundary problems governed by parabolic equations have been considered in [29].

This paper is organized as follows. In the next two sections we consider two-point BVPs and we prove that two apparently different non-iterative TMs, defined by a spiral and a scaling group are equivalent. It is worth mentioning that the quoted example due to Na [6]—see Na and Tang [30], Klamkin [2], Ames [31, p. 140], Na [13, pp. 155–158], Seshadri and Na [18, pp. 157–168] or Ames [32]—belongs to the class of problems characterized by the invariance with respect to the spiral group. Then, the equivalence proved from a theoretical point of view is verified from a numerical viewpoint.

2. Scaling invariance

As far as the numerical solution of BVPs is concerned, we report here the definition within scaling invariance theory of an initial value method, referred in the literature also as a non-iterative TM [33]. To this end we consider the class of two-point BVPs

$$
\begin{aligned}
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} &= u^{1-2\delta}\Phi\left(xu^{-\delta}, \frac{\mathrm{d}u}{\mathrm{d}x}u^{\delta-1}\right), \quad x\in(0,b)\\
\frac{\mathrm{d}u}{\mathrm{d}x}(0) &= Au(0)^{1-\delta}\\
u(b)^{\zeta}\Psi\left(\frac{\mathrm{d}u}{\mathrm{d}x}(b)u(b)^{\delta-1}\right) &= B
\end{aligned}
\tag{1}
$$

where A and δ are arbitrary constants, $\Phi(\cdot,\cdot)$ and $\Psi(\cdot)$ are arbitrary functions of their arguments, $b>0, \zeta\neq 0$ and $B\neq 0$. Here the governing differential equation and the boundary condition at $x=0$ are invariant with respect to the following scaling group of transformations:

$$
x^* = \lambda^{\delta}x, \quad u^* = \lambda u \tag{2}
$$

where λ is the group parameter. The non-iterative numerical solution of (1) can be obtained by the following steps:

- we fix a value of $u^*(0)$, this defines a value of $(\mathrm{d}u^*/\mathrm{d}x^*)(0)$ according to the first boundary condition;
- next we integrate numerically, with initial data $u^*(0)$ and $(\mathrm{d}u^*/\mathrm{d}x^*)(0)$, forwards in $[0, b^*]$ where b^* is defined by

$$
b^* = b\left\{\left[\frac{u^*(b^*)^{\zeta}\ \Psi\left(\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(b^*)u^*(b^*)^{\delta-1}\right)}{B}\right]^{1/\zeta}\right\}^{\delta} \tag{3}
$$

- finally, group properties can be used to get

$$
\begin{aligned}
\lambda &= \left[\frac{u^*(b^*)^{\zeta}\Psi\left(\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(b^*)u^*(b^*)^{\delta-1}\right)}{B}\right]^{1/\zeta}\\
u(0) &= \lambda^{-1}u^*(0), \quad \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \lambda^{\delta-1}\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(0)\\
u(b) &= \lambda^{-1}u^*(b^*), \quad \frac{\mathrm{d}u}{\mathrm{d}x}(b) = \lambda^{\delta-1}\frac{\mathrm{d}u^*}{\mathrm{d}x^*}(b^*)
\end{aligned}
$$

We note that b^* is defined implicitly in (3), and it can be easily found by using the event locator option available within modern ODE solvers.

3. On the transformation via spiral group

Let us consider now the class of BVPs

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \mathrm{e}^{-2\delta v}\Omega\left(x\mathrm{e}^{-\delta v}, \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{e}^{\delta v}\right)$$
$$\frac{\mathrm{d}v}{\mathrm{d}x}(0) = A\mathrm{e}^{-\delta v(0)} \tag{4}$$
$$\mathrm{e}^{\zeta\, v(b)}\,\Psi\left(\frac{\mathrm{d}v}{\mathrm{d}x}(b)\mathrm{e}^{\delta\, v(b)}\right) = B$$

where A and δ are arbitrary constants, $\Omega(\cdot,\cdot)$ and $\Psi(\cdot)$ are arbitrary functions of their arguments, $b>0, \zeta\neq 0$ and $B\neq 0$. Here the governing differential equation and the boundary condition at $x=0$ are invariant with respect to the following spiral group:

$$x^* = \mathrm{e}^{\delta\mu}x, \quad v^* = v + \mu \tag{5}$$

where μ is the group parameter. Via the simple transformation

$$u(x) = \mathrm{e}^{v(x)} \tag{6}$$

and setting $\lambda = \mathrm{e}^{\mu}$, the spiral group (5) is transformed to the scaling group (2). The class of BVPs (4), under (6), transforms to the class of BVPs (1) with

$$\Phi(\cdot,\cdot) = \Omega(\cdot,\cdot) + \left(\frac{\mathrm{d}u}{\mathrm{d}x}u^{\delta-1}\right)^2$$

Therefore, $u(x)>0$ for every $x \in (0,b)$. It is interesting to note that the example

$$\frac{\mathrm{d}^2 T}{\mathrm{d}r^2} + \frac{p+1}{r}\frac{\mathrm{d}T}{\mathrm{d}r} + q\mathrm{e}^{\mathrm{T}} = 0$$
$$\frac{\mathrm{d}T}{\mathrm{d}r}(0) = 0 \tag{7}$$
$$T(1) = 0$$

due to Na [6] (where $p=-1$, see Na and Tang [30] for $p=0$ and 1), is a particular case of (4). To see this, let us set $v=T$, $x=r$, $\delta=-\frac{1}{2}$, $\Omega(\cdot,\cdot) = -((p+1)/r)(\mathrm{d}T/\mathrm{d}r)\mathrm{e}^{-T} - q, \Psi(\cdot)=1, A=0, b=1, \zeta=1$ and $B=1$. Now, by using (6), problem (7) takes the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{p+1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} + qu^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 u^{-1} = 0$$
$$\frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0 \tag{8}$$
$$u(1) = 1$$

Therefore, the non-iterative numerical solution of (7) can be obtained by solving (8) with the TM defined in the previous section. This will be the topic of the next subsection.

3.1. An application to non-linear heat generation

According to the model used in the paper by Na and Tang [30], the dimensionless one governing the transient distribution in the radial direction with heat generation e^{T}, in a solid cylinder ($p=0$) or in a sphere ($p=1$) of radius r with $0<r<1$, is given by

$$\frac{\partial T}{\partial t} = \frac{1}{r^{p+1}}\frac{\partial}{\partial r}\left(r^{p+1}\frac{\partial T}{\partial r}\right) + q\mathrm{e}^{\mathrm{T}} \tag{9}$$

$$T(r,0) = 0 \tag{10}$$

$$\frac{\partial T}{\partial r}(0,t) = 0 \tag{11}$$

$$T(1,t) = 0 \tag{12}$$

where q is a dimensionless parameter, whereas (10) stands for the initial temperature of the cylinder (or the sphere), (11) stands for the initial temperature radial gradient, and (12) stands for the temperature on the surface.

By requiring the invariance of this model with respect to the following spiral group

$$T^* = T + \mu, \quad t^* = \mathrm{e}^{\alpha\mu}t, \quad r^* = \mathrm{e}^{\beta\mu}r$$

we end up with the conditions

$$\begin{cases} \alpha-2\beta=0 \\ 1+2\beta=0 \end{cases} \Rightarrow \begin{cases} \beta=-\dfrac{1}{2} \\ \alpha=-1 \end{cases}$$

As a result the spiral group specializes to

$$T^*=T+\mu, \quad r^*=\mathrm{e}^{-\mu/2}r, \quad t^*=\mathrm{e}^{-\mu}t \tag{13}$$

Form a practical viewpoint it is more interesting to study the asymptotic behavior of the spacial distribution temperature. In fact, the dynamics of heat propagation in most of the application is mainly performed over long time scale. The steady state temperature space distribution is simply obtained by setting time derivative of temperature T to zero. As a consequence instead of the partial differential model (9)–(12), we can consider the following ordinary differential model

$$\frac{1}{r^{p+1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{p+1}\frac{\mathrm{d}T}{\mathrm{d}r}\right)+q\mathrm{e}^{\mathrm{T}}=0 \tag{14}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r}(0)=0 \tag{15}$$

$$T(1)=0 \tag{16}$$

Obviously (14) is still invariant with regard to the finite point transformations of the spiral group (13) relative to T and r. Besides it is easy to see that the boundary condition (15) is invariant, whereas the other one (16) is not.

According to what said above, by using the equivalence between the spiral group and the scaling one, we can apply a simple change of variable $W=\mathrm{e}^{\mathrm{T}}$. The first and second derivative transform as follows:

$$\frac{\mathrm{d}T}{\mathrm{d}r}=\frac{1}{W}\frac{\mathrm{d}W}{\mathrm{d}r}, \quad \frac{\mathrm{d}^2T}{\mathrm{d}r^2}=-\frac{1}{W^2}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2+\frac{1}{W}\frac{\mathrm{d}^2W}{\mathrm{d}r^2}$$

If we replace them in (14)–(16), we obtain

$$\left[(p+1)r^{-1}\frac{\mathrm{d}W}{\mathrm{d}r}-\frac{1}{W}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2+\frac{\mathrm{d}^2W}{\mathrm{d}r^2}\right]+qW^2=0 \tag{17}$$

$$\frac{\mathrm{d}W}{\mathrm{d}r}(0)=0 \tag{18}$$

$$W(1)=1 \tag{19}$$

By replacing the new variable W in the temperature point transformation of the spiral group (13), we get just the scaling group

$$r^*=\lambda^{-1/2}r, \quad W^*=\lambda W \tag{20}$$

where $\lambda=\mathrm{e}^{\mu}$. Such a BVP can be solved by defining an auxiliary IVP (written in starred variables) consisting in the governing equation (17), the invariant initial condition on the space derivative (18), and, as missing initial condition for the temperature, the value $W^*(0)=1$. Some care has to be used for the numerical solution of this auxiliary IVP. In fact, because of the initial condition on the first derivative, the right-hand side of the governing differential equation has an indeterminate term of $\frac{0}{0}$ at the initial point $r^*=0$. The correct value for the above term can be found by a limiting argument using l'Hospital's rule

$$\lim_{r^*\to 0}\frac{\dfrac{\mathrm{d}W^*}{\mathrm{d}r^*}(r^*)}{r^*}=\lim_{r^*\to 0}\frac{\dfrac{\mathrm{d}^2W^*}{\mathrm{d}r^{*2}}(r^*)}{1}=\frac{\mathrm{d}^2W^*}{\mathrm{d}r^{*2}}(0)$$

so that, by using the governing differential equation we get

$$\frac{\mathrm{d}^2W^*}{\mathrm{d}r^{*2}}(0)=\frac{\dfrac{1}{W^*(0)}\left(\dfrac{\mathrm{d}W^*}{\mathrm{d}r^*}(0)\right)^2-qW^*(0)}{p+2}$$

This value can be used to start the integration by setting a very small initial step.

As far as the solution to the original problem is concerned, we have that at $r=1$ is $W(1)=1$, and therefore the point transformations (20) provide

$$W^*(r_0^*)=\lambda, \quad r_0^*=\lambda^{-1/2} \tag{21}$$

where r_0^* is the image under (20) of $r=1$. Now, by eliminating the parameter μ from (21) we get

$$r_0^*{W^*}^{1/2}(r_0^*)-1=0 \tag{22}$$

Table I. Nonlinear heat generation: numerical results.

	Cylinder $p=0$		Sphere $p=1$	
q	$\log(W(0))$	$T(0)$(Na-Tang [30])	$\log(W(0))$	$T(0)$(Na-Tang [30])
0.1	0.025578	0.0252	0.016864	0.0168
0.2	0.052024	0.0519	0.034136	0.0343
0.3	0.079605	0.0796	0.051842	0.0522
0.4	0.108532	0.1090	0.070010	0.0706
0.5	0.138769	0.1380	0.088649	0.0894
0.6	0.170440	0.1690	0.107799	0.1070
0.7	0.203795	0.2035	0.127513	0.1270
0.8	0.239104	0.2380	0.147804	0.1470
1.0	0.316630	0.650	0.190203	
1.5	0.575076	1.10	0.309517	
1.7	0.730435	2.0	0.364734	
1.9	0.983252		0.424694	
1.99	1.253611			
2.5			0.651629	
3.1			1.056357	
3.3			1.424619	

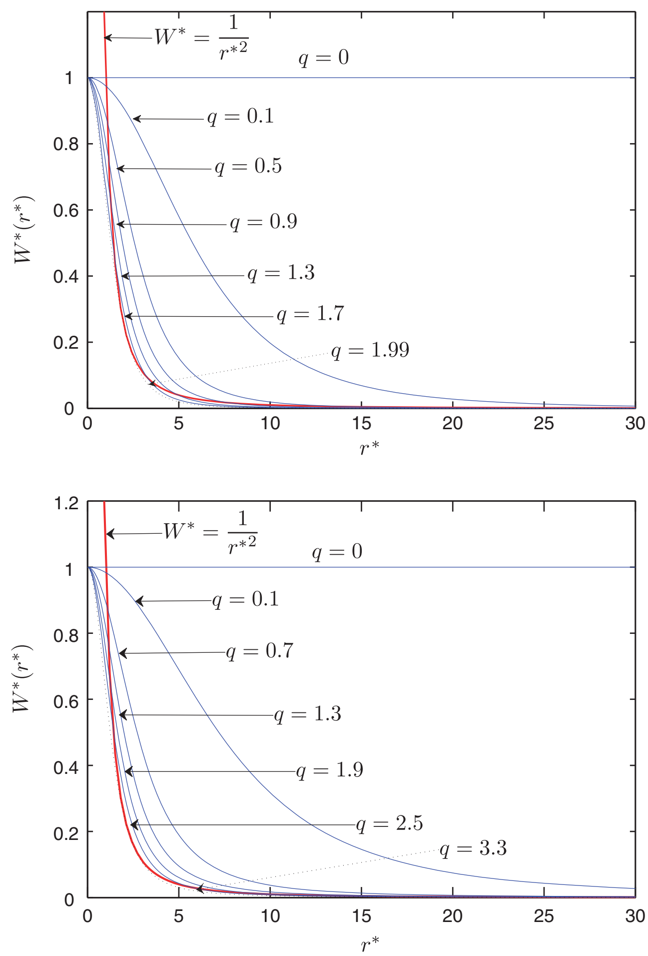

Figure 1. Numerical results for the problem (17)–(19): top $p=0$ and bottom $p=1$ (dotted lines correspond to the upper values of q for the existence of solutions).

whose zero(s) is (are) just the sought value(s) of r_0^*. We have only to be cautious, both in integrating up to a value of r^* greater than r_0^* and in finding a range for the parameter q, so that Equation (22) can have at least one root. Once found the zero, it is immediate to deduce the value of the sought similarity parameter λ from (21) and the right value for the missing initial condition $W(0)=\lambda^{-1}$.

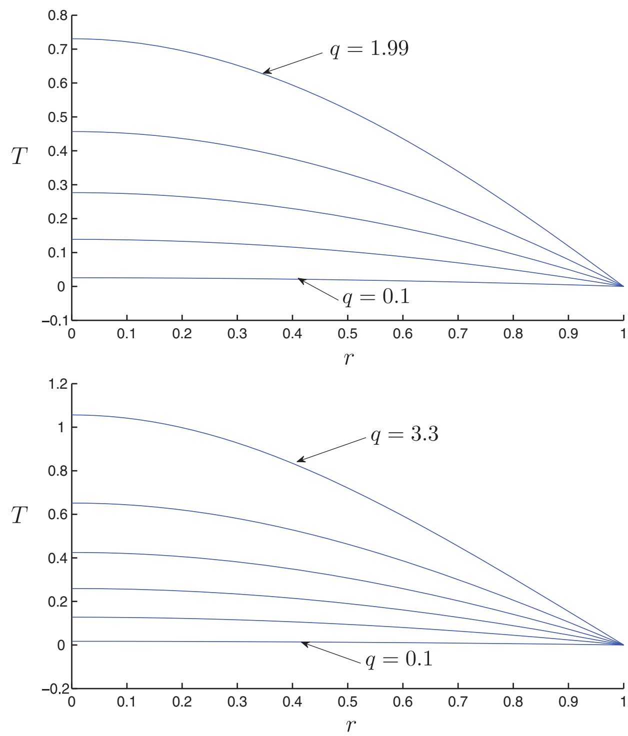

Figure 2. Top for $p=0$, bottom for $p=1$: solutions to the problem (14)–(16) obtained by solving an IVP with the missing initial condition reported in Table I for each considered q.

A numerical test, for several values of the parameter q, has been carried out by means of the MATLAB ODE variable step integrator *ODE*45 [34], with the help of the event locator set at picking those values of the solution where Equation (22) is verified. The values of the missing initial condition are reported in Table I, by using the same values of q chosen in the paper [30] by Na and Tang where they found the solution to the same problem in (14)–(16) by means of a spiral group. Besides it can be observed from Figure 1, that for values of the parameter $q>1.99$, in the cylindric case ($p=0$), and $q>3.3$, in the spherical case ($p=1$), no solution exists, whereas for lower values there exist two distinct solutions. Na and Tang proved that only the first one out of them has physical meaning for the hypothesis of asymptotic analysis to be valid. Just for the sake of validation of our results, we decided to verify numerically the equivalence between a spiral group and a scaling group, by integrating the problem (14) and (15), with the final condition (16) replaced by the calculated missing initial condition reported in Table I, for several values of the parameter q. The obtained results are shown in Figure 2.

From the comparison between the present values and the ones obtained by Na and Tang, it can be pointed out that up to a value for q of about 0.9 there is a difference of about $\pm 1\times 10^{-4}$. Unfortunately for values of the parameter q greater than 0.9, the values of the missing initial condition provided by Na and Tang are clearly incorrect or misreported. This can be likely due to the fact that our values have been obtained by means of the adaptive step integrator whereas the others nearly for sure by using a constant step routine. Correctness of our results is also confirmed by the numerical validation depicted in Figure 2.

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