On the moving boundary formulation for parabolic problems on unbounded domains

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A parabolic problem defined on unbounded domains and admitting invariance property to Lie group of scalings can be transformed into an equivalent boundary value problem governed by an ODE defined on an infinite interval. A free boundary formulation and a convergence theorem for this kind of transformed problems are available in [Fazio, SIAM J. Numer. Anal., 33 (1996), pp. 1473-1484]. Depending on its scaling invariance properties, the free boundary problem is then solved numerically using either an iterative or a non-iterative method. Finally, the solution of the parabolic problem is retrieved applying the inverse map of similarity.

1 The moving boundary formulation approach

For the sake of brevity we describe our approach for the following model

$$
\begin{gathered}
\frac{\partial^2 u}{\partial x^2} = u\frac{\partial u}{\partial t} \\
u(x,0) = 0 \qquad x > 0 \\
u(x,t) \to 0 \qquad x \to \infty \qquad t > 0 \\
\frac{\partial u}{\partial x}(0,t) = -c \qquad t > 0 \ ,
\end{gathered}
\tag{1}
$$

describing thermal expulsion of fluid from a long slender heated tube where u is flow velocity induced in the fluid by the heating of the tube wall (see Dresner [1, pp. 35-40]). With regard to scaling group

$$x^* = \mu x \ , \qquad t^* = \mu^\rho t \ , \qquad u^* = \mu^{\rho\sigma} u \ ,$$

the governing equation results to be invariant if $\rho = \frac{2}{1-\sigma}$. By imposing the invariance of the boundary condition at $x = 0$, we get the value $\sigma = 1/3$. Therefore, the two invariants for this problem are given by

$$\eta = x t^{-1/3} \ , \qquad F(\eta) = u(x,t) t^{-1/3} \ .$$

Expressing the function $u(x,t)$ and its derivatives involved in the model (1) in terms of the similarity variables we obtain the following boundary value problem

$$
\begin{gathered}
\frac{d^2 F}{d\eta^2} = \frac{F}{3}\left(F - \eta\frac{dF}{d\eta}\right) \\
\frac{dF}{d\eta}(0) = -c \ , \qquad F(\infty) = 0 \ .
\end{gathered}
\tag{2}
$$

The ODE in (2), called the principal ODE, is still invariant with regard to the scaling group (the Dresner associated group)

$$\eta^* = \lambda\eta, \qquad F^* = \lambda^{-2} F \ .$$

As one of the boundary conditions is defined at infinity, we can reformulate such a BVP problem (2) as a free boundary one, in this context see Fazio [2] for a general theory. The resulting free boundary problem is given by

$$
\begin{gathered}
\frac{d^2 F}{d\eta^2} + \frac{\eta}{3} F \frac{dF}{d\eta} - \frac{F^2}{3} = 0 \\
\frac{dF}{d\eta}(0) = -c \ , \qquad F(\eta_\epsilon) = 0 \ , \qquad \frac{dF}{d\eta}(\eta_\epsilon) = \epsilon \ ,
\end{gathered}
\tag{3}
$$

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where η_ϵ is the introduced unknown free boundary.

For the numerical solution of (3) we can use the non-iterative method introduced by Fazio and Evans [3]. Firstly, we fix $F^*(\eta_\epsilon^*) = 0$, $dF^*/d\eta^*(\eta_\epsilon^*) = -1$, that is $\epsilon^* = -1$. Choosing a value for η_ϵ^* and integrating the problem (3) (written in the starred variables) backward from this value up to zero, we compute the value of $dF^*/d\eta^*(0)$. From the similarity relations we can deduce the following relations

$$\frac{dF^*}{d\eta^*}(0) = \lambda^{-3}\frac{dF}{d\eta}(0) \ , \qquad \eta_\epsilon^* = \lambda\eta_\epsilon \ ,$$

so that we can work out the value for the similarity parameter and the free boundary

$$\lambda = \left(\frac{-c}{\dfrac{dF^*}{d\eta^*}(0)} \right)^{1/3} \ , \qquad \eta_\epsilon = \lambda^{-1}\eta_\epsilon^* \ .$$

In a similar way, we can obtain the corresponding original (not-starred) value at the origin assumed by the function as well as the value of ϵ

$$F(0) = \lambda^2 F^*(0) \ , \qquad \epsilon = \lambda^3\epsilon^* \ .$$

A numerical experiment has been carried out and its result is summarized in Table 1 for the value of $c = 0.1$. The Table caption shows the used value of c and the conditions imposed at the free boundary. It is evident the convergence of the process

η_ϵ^*	λ	η_ϵ	ϵ	$F(0)$
2.5	0.193356	12.929284	$-7.2\text{E}{-03}$	0.324044
2.6	0.167000	15.568807	$-4.6\text{E}{-03}$	0.324796
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
2.95	0.052287	56.418804	$-1.4\text{E}{-04}$	0.325569
3.	0.028296	106.022609	$-2.2\text{E}{-05}$	0.325573
3.025	0.012804	236.248810	$-2.1\text{E}{-06}$	0.325573

Table 1 Conditions at the free boundary and parameter used: $F^*(\eta_\epsilon^*) = 0$, $\dfrac{dF^*}{dz^*}(\eta_\epsilon^*) = -1$, and $\dfrac{dF}{dz}(0) = -c = -0.1$.

because as the parameter $\epsilon \to 0$ it follows that the free boundary $\eta_\epsilon \to \infty$, proving the validity of the theoretical assumptions taken.

By recalling the scaling relation involving the function $F(\eta)$ and its derivative, it is easy to realize that the quantity

$$I = \frac{dF}{d\eta}(0)/[F(0)]^{3/2} \ ,$$

is an invariant that can be numerically deduced from the above Table. As a result, for any other value of the parameter c, the missing initial condition is given by

$$F(0) = \left(\frac{-c}{I}\right)^{2/3} \ .$$

All computations were performed by the ODE45 Runge-Kutta's routine available with MATLAB (TM), with a relative local error tolerance set equal to 10^{-6}.

As a final remark we point out that sometimes the principal ODE does not admit a scaling invariance, this is for instance the case for the ODE derived from the linear heat equation. However, in such a case by introducing an *ad hoc* dimensionless parameter it is possible to impose the invariance under an extended scaling group for the activation of an iterative method. For further details and examples on this topics see Fazio [4, 2].

References

[1] L. Dresner, Similarity Solutions of Non-linear Partial Differential Equations, Research Notes in Math., Vol. 88 (Pitman, London, 1983).

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[3] R. Fazio and D. J. Evans, Int. J. Computer Math. **31**, 215–220 (1990), 39:249, 1991.

[4] R. Fazio, SIAM Rev. **40**, 616–635 (1998).