

A Non-Iterative Transformation Method for Boundary-Layer with Power-Law Viscosity for non-Newtonian Fluids

Riccardo Fazio

Department of Mathematics, Computer Science,
Physical Sciences and Earth Sciences,

University of Messina

Viale F. Stagno D'Alcontres, 31

98166 Messina, Italy

E-mails: rfazio@unime.it

Home-page: <http://mat521.unime.it/fazio>

October 20, 2022

Abstract

In this paper, we have defined and applied a non-iterative transformation method to an extended Blasius problem describing a 2D laminar boundary-layer with power-law viscosity for non-Newtonian fluids. Let us notice that by using our method we are able to solve the boundary value problem defined on a semi-infinite interval, for each chosen value of the parameter involved, by solving a related initial value problem once and then rescaling the obtained numerical solution. This is, of course, much more convenient than using an iterative method because it reduces greatly the computational cost of the solution. Furthermore, by using Richardson's extrapolation we define

a posteriori error estimator and show how to deal with the accuracy question. For a particular value of the parameter involved, our problem reduces to the celebrated Blasius problem and in this particular case, our method reduces to the Töpfer non-iterative algorithm. In this case, we are able to compare favourably the obtained numerical result for the so-called missing initial condition with those available in the literature. Moreover, we have listed the computed values of the missing initial condition for a large range of the parameter involved, and for illustrative purposes, we have plotted, for two values of the related parameter, the numerical solution computed rescaling the computed solution. Finally, we have indicated the limitations of the proposed method as it seems not be suitable, for values of $n > 1$, to compute the values of the independent variable where the second derivative of the solution becomes zero or goes to infinity.

Key Words. Boundary-Layer with Power-Law Viscosity for non-Newtonian Fluids; BVPs on infinite intervals; scaling invariance properties; non-iterative transformation method.

AMS Subject Classifications. 65L10, 34B15, 65L08.

1 Introduction

In this study, we are going to consider an extended version of the classical Blasius problem of boundary layer theory. As well known, the basis of boundary layer theory were given by Prandtl at the beginning of the last century, see [28]. In this context, Töpfer [34] in 1912 published a paper where he reduced the solution of the Blasius problem to the solution of a related initial value problem (IVP). This was the first definition and application of a non-iterative transformation method (ITM). As a consequence, non-ITMs have been applied to several problems of practical interest within the applied sciences. In fact, a non-ITM was applied to the

Blasius equation with slip boundary condition, arising within the study of gas and liquid flows at the micro-scale regime [5, 27], see [14]. Moreover, a non-ITM was applied also to the Blasius equation with moving wall considered by Ishak et al. [24] or surface gasification studied by Emmons [6] and recently by Lu and Law [26] or slip boundary conditions investigated by Gad-el-Hak [5] or Martin and Boyd [27], see Fazio [16] for details. In particular, within these applications, we found a way to solve non-iteratively the Sakiadis problem [31, 32]. The application of a non-ITM to an extended Blasius problem has been the subject of a recent manuscript [20]. The interested reader can find in [18] a recent review dealing with the non-ITM and its applications.

Let us notice that, by using our method, we are able to solve the considered boundary value problem (BVP) defined on a semi-infinite interval, for each chosen value of the parameter involved, by solving a related IVP once and the rescaling the obtained numerical solution. This is, of course, much more convenient than using an iterative method because it reduces greatly the computational cost of the solution. Furthermore, by using Richardson's extrapolation we define a posteriori error estimator and show how to deal with the accuracy question.

The non-ITM is based on scaling invariance theory. For its application, the governing differential equation as well as the prescribed initial conditions have to be invariant with respect to a scaling group of point transformations. Of course, there are several problems in the applied sciences that lack this kind of invariance and consequently cannot be solved by a non-ITM. However, in all those cases we can use an iterative extension of our method. In fact, an iterative extension of Töpfer's algorithm has been introduced, for the numerical solution of free BVPs, by Fazio [22]. This iterative extension has been applied to several problems of interest: free boundary problems [22, 11, 12], a moving boundary hyperbolic problem [8], Homann and Hiemenz problems governed by the Falkner-Skan equation and a mathematical model describing the study of the flow of an

incompressible fluid around a slender parabola of revolution [9, 10], one-dimensional parabolic moving boundary problems [13], two variants of the Blasius problem [14], namely: a boundary layer problem over moving surfaces, studied first by Klemp and Acrivos [25], and a boundary layer problem with slip boundary condition, that has found application in the study of gas and liquid flows at the micro-scale regime [5, 27], parabolic problems on unbounded domains [23] and, recently, see [15], a further variant of the Blasius problem in boundary layer theory: the so-called Sakiadis problem [31, 32]. Moreover, this iterative extension can be used to investigate the existence and uniqueness question for different classes of problems, as shown for free BVPs in [11], and for problems in boundary layer theory in [19]. A recent review dealing with the derivation and applications of the ITM can be found, by the interested reader, in [17]. A unifying framework, providing proof that the non-ITM is a special instance of the ITM and consequently can be derived from it, has been the argument of the paper [21].

2 The mathematical model

We study, here, the mathematical model arising in the study of a 2D laminar boundary-layer with power-law viscosity related to non-Newtonian fluids as studied by Schlichting and Gersten [33] or Benlahsen et al. [1]. Consider the two-dimensional steady flow of a non-Newtonian fluid of density ρ modelled by a power law fluid due to Ostwald-de Waele over a flat plate moving continuously with a velocity U_w in the opposite direction to the free stream U_∞ . The x -axis extends parallel to the plate, while the y -axis extends upwards, normal to it. The continuity and momentum

equations after making the necessary boundary layer approximations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = \frac{\partial \tau_{xy}}{\partial y},$$

where u and v are the velocity components along the x and y directions, respectively. We use, here, the power-law relation

$$\tau_{xy} = \mu_0 \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \quad (2)$$

between the shear stress and the shear rate, where μ_0 and n are material constants. Here, n is called power-law index, that is $n < 1$ for pseudo-plastic, $n = 1$ for Newtonian, and $n > 1$ for dilatant fluids. Then, the momentum equation becomes

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu_c \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right), \quad \mu_c = \frac{\mu_0}{\rho}. \quad (3)$$

The related boundary conditions can be set as

$$u|_{y=0} = U_w, \quad v|_{y=0} = 0, \quad u|_{y=+\infty} = U_\infty. \quad (4)$$

The continuity equation is satisfied by introducing a stream function ψ such that

$$u = \frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial y}. \quad (5)$$

The momentum equation can be transformed into the corresponding ordinary differential equation by the following transformations

$$\psi(x, y) = \gamma^{\frac{1}{n+1}} (U_\infty)^{\frac{2n-1}{n+1}} x^{\frac{1}{n}} f(\eta), \quad \eta = \gamma^{-\frac{1}{n+1}} (U_\infty)^{\frac{2-n}{n+1}} \frac{y}{x^{\frac{1}{n+1}}} \quad (6)$$

where η is the similarity variable and $f(\eta)$ is the dimensionless stream function. Equation (3) with the transformed boundary conditions can be

written as

$$\frac{d}{d\eta} \left(\left| \frac{d^2 f}{d\eta^2} \right|^{n-1} \frac{d^2 f}{d\eta^2} \right) + \frac{1}{n+1} f \frac{d^2 f}{d\eta^2} = 0$$

$$f(0) = \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty.$$
(7)

Let us remark here, that when $n = 1$ the BVP (7) reduces to the celebrated Blasius problem [2].

3 The non-ITM

In order to define our non-ITM we need to require the invariance of the governing differential equation and of the prescribed initial conditions in (7) with respect to the scaling group of point transformation

$$f^* = \lambda f, \quad \eta^* = \lambda^\delta \eta. \quad (8)$$

It is easily seen, that the governing differential equation and the prescribed initial conditions are invariant on condition that $\delta = (2-n)/(1-2n)$. Therefore, we have to require that $n \neq 1/2$ and $n \neq 2$ in fact for these two special values of n the scaling group (8) does not exist. Now, we can integrate the governing equation in (7) written in the star variables on $[0, \eta_\infty^*]$, where η_∞^* is a suitable truncated boundary, with the attached initial conditions

$$\frac{d}{d\eta^*} \left(\left| \frac{d^2 f^*}{d\eta^{*2}} \right|^{n-1} \frac{d^2 f^*}{d\eta^{*2}} \right) + \frac{1}{n+1} f^* \frac{d^2 f^*}{d\eta^{*2}} = 0$$

$$f^*(0) = \frac{df^*}{d\eta^*}(0) = 0, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1,$$
(9)

in order to compute an approximation $\frac{df^*}{d\eta^*}(\eta_\infty^*)$ for $\frac{df^*}{d\eta^*}(\infty)$ and the corresponding value of λ by the equation

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta_\infty^*) \right]^{1/(1-\delta)}. \quad (10)$$

Once the value of λ has been computed by equation (10), we can find the missed initial condition by the relation

$$\frac{d^2 f}{d\eta^2}(0) = \lambda^{2\delta-1} \frac{d^2 f^*}{d\eta^{*2}}(0), \quad (11)$$

and rescale the solution components according to

$$f(\eta) = \lambda^{-1} f^*(\eta^*), \quad \frac{df}{d\eta}(\eta) = \lambda^{\delta-1} \frac{df^*}{d\eta^*}(\eta^*), \quad \frac{d^2 f}{d\eta^2}(\eta) = \lambda^{2\delta-1} \frac{d^2 f^*}{d\eta^{*2}}(\eta^*). \quad (12)$$

As suggested by a referee, in choosing η_∞^* we must take into account the sensibility of the related numerical results on this value. For instance, for small values of n he found that the obtained numerical results are really sensible. For example, for $n = 0.1$, taking $\eta_\infty^* = 10$ and then doubling the interval of integration it is possible to find a value for the missing initial condition of about 0.813576 that can be compared with the one obtained with $\eta_\infty^* = 10$, that is 0.826474, providing a difference of 1.5% .

In the non-ITM we proceed as follows: we set the values of η_∞^* and integrate the IVP (9) on $[0, \eta_\infty^*]$.

4 A posteriori error estimator

In this section, we would like to indicate a simple way to define a posteriori error estimator related to the so-called truncation error. To this end, we will apply the so-called Richardson's extrapolation, introduced by Richardson in [29, 30]. We must be alert the reader that in no way this is an evaluation of the round-off error which is related instead to the machine precision and can be only reduced by using double or quadruple precision. Let us consider a numerical value of interest U and let us suppose to have computed it with two related grids. So we have computed the values $U_{g+1,k}$ and $U_{g,k}$, now in order to find a more accurate approximation we can apply a Richardson's extrapolations on the used grids

$$U_{g+1,k+1} = U_{g+1,k} + \frac{U_{g+1,k} - U_{g,k}}{q^{p_k} - 1}, \quad (13)$$

where the constant q appearing in the denominator is the grid refinement ratio and p_k is the true order of the discretization error. This formula is asymptotically exact in the limit as the number of grid points goes to infinity if we use uniform or quasi-uniform grids. Therefore, Richardson's extrapolation can be easily reiterated. We notice that to obtain each value of $U_{g+1,k+1}$ requires having computed two solution U in two embedded grids, namely $g + 1$ and g at the extrapolation level k . For any g , the level $k = 0$ represents the numerical solution of U without any extrapolation, which is obtained as described in the previous section.

Of course this is a possible way to deal with problems that could be sensible to the chosen value of the involved parameter, n , and might lose accuracy in some ranges of it. For instance, let us consider the case $n = 0.3$ and apply a 8-order explicit Runge-Kutta method, as reported by Butcher [4, p. 180], with 101 grid point, that is $\Delta\eta = 0.1$, we obtain for the corresponding missing initial condition the value 0.391515346640558, while we get the value 0.391515346639927 when $\Delta\eta = 0.05$ and these values lead to a Richardson's extrapolation, according with the above formula (13) of 0.391515346639924. Which means that the value computed with the finer grid has a relative error smaller than $7.66 * 10^{-15}$, while the first computed value has a relative error less than $1.62 * 10^{-12}$. Since this computations are easy to implement we will not pursue this path here for the sake of simplicity.

5 Numerical results

As mentioned before, the case $n = 1$ is the Blasius problem and in this case, our non-ITM reduces to the classical Töpfer algorithm, see Töpfer [34]. If we set, for the sake of simplicity, $\eta_\infty^* = 10$ then the numerical value computed for the missing initial condition, namely 0.332057268052, can be compared with the value 0.332057336215 obtained by Fazio [7] by a free boundary formulation of the Blasius problem or the value 0.33205733621519630

computed by Boyd [3] who believes that all the decimal digits are correct.

Let us notice first that, for small values of n the obtained numerical results are very sensitive to the chosen value of the truncated boundary. Therefore, for values of n less than one we proceed as follows. We start by fixing a small value of the truncated boundary value, say equal to two point five, and then we make our computations both for this value and by doubling it. Our strategy will be to stop at a suitable value for our truncated boundary when the results don't differ by a suitable fixed tolerance. For the following results this tolerance was set equal to 10^{-6} . After some sample computations, we set the truncated boundary value equal to $\eta_{\infty}^* = 40960$, that is for all values of n smaller than one. On the other hand, for n greater or equal to one we use $\eta_{\infty}^* = 10$.

In table 1 we report the chosen parameter values and the related missing initial conditions $\frac{d^2f}{d\eta^2}(0)$. The values listed in table 1 for the two values of $n = 1/2$ and $n = 2$ are second order approximations. To verify the last value reported in this table we have also considered the case $n = 1.999$ and found the missing initial condition value 0.399700. Figure 1 shows the behaviour of the missing initial condition versus n .

Figure 2 shows the solution of the extended Blasius problem in the particular case when we set $n = 0.3$ and $n = 1.7$.

Let us notice here, that for $n > 1$ the solutions of our BVP are no longer regular: indeed, the third derivative of the solution has a jump discontinuity at a certain value of η , therefore the second derivative is only a C^0 function. Moreover, this second derivative becomes zero for a finite value of the independent variable, let us call it η_0 . In order to compute the correct value of η_0 we can add the boundary condition $d^2f/d\eta^2(\eta_0) = 0$ and define a free boundary value problem (FBVP). However, for the numerical solution of this resulting FBVP our non-ITM seems not be suitable and we need a more complex numerical method. Finally, it seems that again for $n > 1$ the second derivative diverges for a finite critical value of the independent variable, let say η_c . Again, to compute this critical value we need

Table 1: Numerical data and results. Missing initial condition versus the power-law index.

n	$\frac{d^2 f}{d\eta^2}(0)$	n	$\frac{d^2 f}{d\eta^2}(0)$
0.1	0.804704	1.1	0.337833
0.2	0.482101	1.2	0.344165
0.3	0.387964	1.3	0.350851
0.4	0.348927	1.4	0.357752
0.5	0.337170	1.5	0.364772
0.6	0.323802	1.6	0.371841
0.7	0.322009	1.7	0.378906
0.8	0.323542	1.8	0.385936
0.9	0.327150	1.9	0.392894
1	0.332057	2	0.399852

a different more suitable numerical method than our simple non-ITM. In both cases we have to define a suitable FBVP. In those FBVPs η_0 and η_c are the unknown free boundary which must be computed as part of the numerical solution.

6 Concluding remarks

In this paper we have defined and applied a non-ITM to an extended Blasius problem describing a 2D laminar boundary-layer with power-law viscosity for non-Newtonian fluids as described by Schlichting and Gersten [33] or Benlahsen et al. [1]. Let us notice that by using our method we are able to solve the BVP defined on a semi-infinite interval, for each cho-

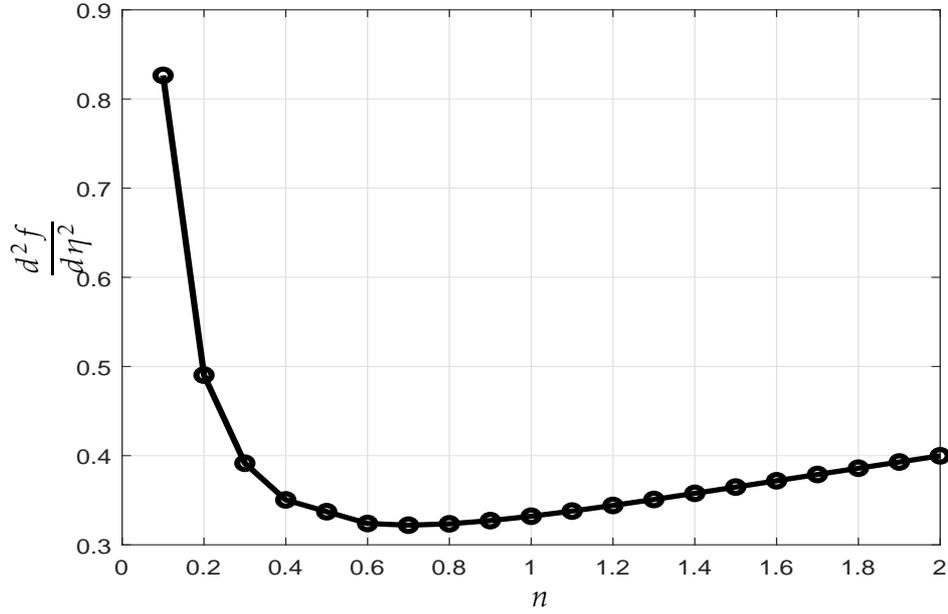


Figure 1: Behaviour of the missing initial condition versus n .

sen value of the parameter n involved, by solving a related IVP once and then rescaling the obtained numerical solution. This is, of course, much more convenient than using an iterative method because it reduces greatly the computational cost of the solution. To asses the the obtained numerical results we indicate a simple way to define a posteriori error estimator related to the so-called truncation error, and we report in a sample case, corresponding for $n = 0.3$, the obtained relative error for the missing initial condition. The problem under study reduces to the celebrated Blasius problem for $n = 1$ and in this particular case, our method reduces to the Töpfer non-iterative algorithm [34]. So, we are able to compare favourably the obtained numerical result for the so-called missing initial condition with those available in the literature.

Moreover, we have listed the computed values of the missing initial condition for a large range of the parameter involved, n and plotted the corresponding behaviour, and, for illustrative purposes, we have plotted, for two values of the related parameter, the numerical solution computed

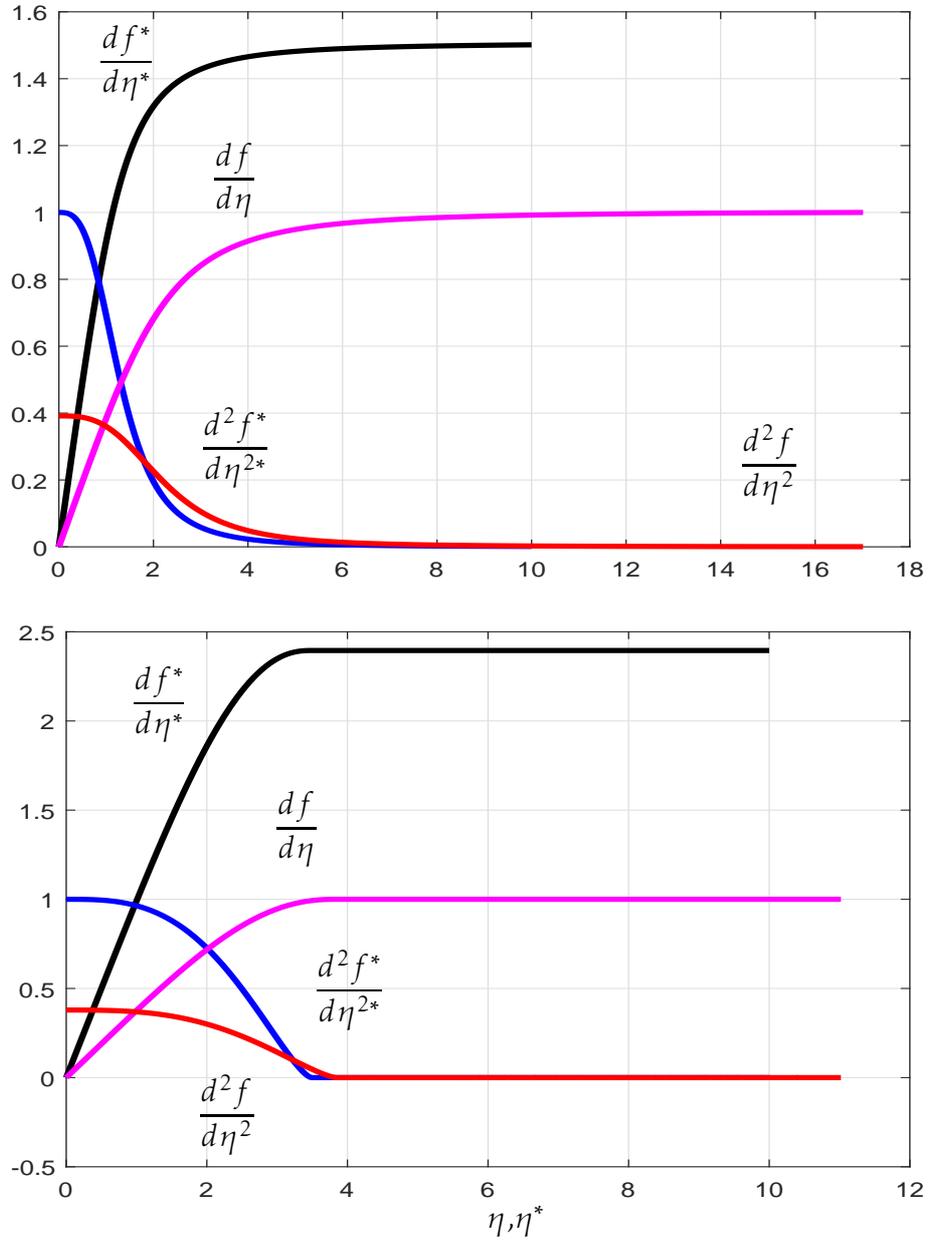


Figure 2: Numerical results of the non-ITM for (7) with: top frame $n = 0.3$ and bottom frame $n = 1.7$. Numerical solution of the IVP and solution of the BVP obtained after rescaling.

rescaling a computed solution.

Finally, we have indicated the limitations of the proposed method as

it seems not be suitable, for values of $n > 1$, to compute the values of the independent variable where the second derivative of the solution becomes zero or goes to infinity. In both cases we have to define a suitable FBVP where η_0 and η_c are the unknown free boundaries that must be computed as part of the solution.

Acknowledgement. The research of this work was partially supported by the FFABR grant of the University of Messina and by the GNCS of INDAM.

References

- [1] M. Benlahsen, M. Guedda, and R. Kersner. The generalized Blasius equation revisited. *Math. Computer Model.*, 47:1063–1076, 2008.
- [2] H. Blasius. Grenzsichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.*, 56:1–37, 1908.
- [3] J. P. Boyd. The Blasius function in the complex plane. *Exp. Math.*, 8:381–394, 1999.
- [4] J. C. Butcher. *Numerical Methods for Ordinary Differential Equations*. Wiley, Chichester, 2003.
- [5] M. Gad el Hak. The fluid mechanics of microdevices — the Freeman scholar lecture. *J. Fluids Eng.*, 121:5–33, 1999.
- [6] H. W. Emmons. The film combustion of liquid fluid. *ZAMM - J. Appl. Math. Mech.*, 36:60–71, 1956.
- [7] R. Fazio. The Blasius problem formulated as a free boundary value problem. *Acta Mech.*, 95:1–7, 1992.
- [8] R. Fazio. A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material. *Wave Motion*, 16:299–305, 1992.

- [9] R. Fazio. The Falkner-Skan equation: numerical solutions within group invariance theory. *Calcolo*, 31:115–124, 1994.
- [10] R. Fazio. A novel approach to the numerical solution of boundary value problems on infinite intervals. *SIAM J. Numer. Anal.*, 33:1473–1483, 1996.
- [11] R. Fazio. A numerical test for the existence and uniqueness of solution of free boundary problems. *Appl. Anal.*, 66:89–100, 1997.
- [12] R. Fazio. A similarity approach to the numerical solution of free boundary problems. *SIAM Rev.*, 40:616–635, 1998.
- [13] R. Fazio. The iterative transformation method: numerical solution of one-dimensional parabolic moving boundary problems. *Int. J. Computer Math.*, 78:213–223, 2001.
- [14] R. Fazio. Numerical transformation methods: Blasius problem and its variants. *Appl. Math. Comput.*, 215:1513–1521, 2009.
- [15] R. Fazio. The iterative transformation method for the Sakiadis problem. *Comput. & Fluids*, 106:196–200, 2015.
- [16] R. Fazio. A non-iterative transformation method for Blasius equation with moving wall or surface gasification. *Int. J. Non-Linear Mech.*, 78:156–159, 2016.
- [17] R. Fazio. The iterative transformation method. *Int. J. Non-Linear Mech.*, 116:181–194, 2019.
- [18] R. Fazio. The non-iterative transformation method. *Int. J. Non-Linear Mech.*, 114:41–48, 2019.
- [19] R. Fazio. Existence and uniqueness of BVPs defined on infinite intervals: Insight from the iterative transformation method, 2020. Preprint available at the URL: <http://mat521.unime.it/~fazio/preprints/NTest2020.pdf>.

- [20] R. Fazio. A non-iterative transformation method for an extended Blasius problem, 2020. Preprint available at the URL: <http://mat521.unime.it/~fazio/preprints/ExBlasius2020.pdf>.
- [21] R. Fazio. Scaling invariance theory and numerical transformation methods: A unifying framework, 2020. Preprint available at the URL: <http://mat521.unime.it/~fazio/preprints/SINTMUF2020.pdf>.
- [22] R. Fazio and D. J. Evans. Similarity and numerical analysis for free boundary value problems. *Int. J. Computer Math.*, 31:215–220, 1990. 39 : 249, 1991.
- [23] R. Fazio and S. Iacono. On the moving boundary formulation for parabolic problems on unbounded domains. *Int. J. Computer Math.*, 87:186–198, 2010.
- [24] A. Ishak, R. Nazar, and I. Pop. Boundary layer on a moving wall with suction and injection. *Chin. Phys. Lett.*, 24:2274–2276, 2007.
- [25] J. P. Klemp and A. Acrivos. A moving-wall boundary layer with reverse flow. *J. Fluid Mech.*, 53:177–191, 1972.
- [26] Z. Lu and C. K. Law. An iterative solution of the Blasius flow with surface gasification. *Int. J. Heat and Mass Transfer*, 69:223–229, 2014.
- [27] M. J. Martin and I. D. Boyd. Blasius boundary layer solution with slip flow conditions. In *Rarefied Gas Dynamics: 22nd International Symposium*, volume 585 of *American Institute of Physics Conference Proceedings*, pages 518–523, 2001, DOI: 10.1063/1.1407604.
- [28] L. Prandtl. Über Flüssigkeiten mit kleiner Reibung. In *Proc. Third Inter. Math. Congr.*, pages 484–494, 1904. Engl. transl. in NACA Tech. Memo. 452.

- [29] L. F. Richardson. The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. *Proc. R. Soc. London Ser. A*, 210:307–357, 1910.
- [30] L. F. Richardson and J. A. Gaunt. The deferred approach to the limit. *Proc. R. Soc. London Ser. A*, 226:299–349, 1927.
- [31] B. C. Sakiadis. Boundary-layer behaviour on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow. *AIChE J.*, 7:26–28, 1961.
- [32] B. C. Sakiadis. Boundary-layer behaviour on continuous solid surfaces: II. The boundary layer on a continuous flat surface. *AIChE J.*, 7:221–225, 1961.
- [33] H. Schlichting and K. Gersten. *Boundary Layer Theory*. Springer, Berlin, 8th edition, 2000.
- [34] K. Töpfer. Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.*, 60:397–398, 1912.