Free Boundary Formulations for two Extended Blasius Problems

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Abstract

In this paper we have defined the free boundary formulation for two extended Blasius problems. These problems are of interest in boundary layer theory and are deduced from the governing partial differential equations by using appropriate similarity variables. The computed results, for the so-called missing initial condition, are favourably compared with recent results available in the literature.

Key Words. Boundary-layer theory; BVPs on infinite intervals; free boundary formulation.

AMS Subject Classifications. 65L10, 34B15, 65L08.

1 Introduction

In this paper, we define a free boundary formulation for two extended Blasius problems available in literature. These two boundary value problems (BVPs), which will be recalled in the next section, are both defined on a semi-infinite interval. From a numerical viewpoint the application of the asymptotic boundary condition cannot be enforced in a simple way. To overcome this drawback several approaches have been studied.

The classical approach for such a condition is to replace it with the same condition prescribed at a (finite) truncated boundary, as described by Fox [14, p.92] or Collatz [7, pp. 150-151]. In many cases, this simple approach, used by trial and errors, result to be sufficiently accurate, although sometimes it provides good results only for very large values of the truncated boundary. Seldom a simple estimate of the error due to the introduced truncated boundary is available, and for instance, in the case of the Blasius problem an analysis based on the scaling properties of the mathematical model is developed by Rubel [22].

A better approach, provided an asymptotic analysis to find the the appropriate boundary conditions to be imposed at the truncated boundary can be developed, was proposed by de Hoog and Weiss [8], Lentini and Keller [17] and Marcowich [20, 21]. Since the imposed conditions are related to the asymptotic behaviour of the solution, then, usually, the obtained numerical results are more accurate that those from the previous approach. That is, smaller values of the truncated boundary are necessary in this approach compared with the values required by the classical approach.

The free boundary formulation main idea is simple to explain: we replace the asymptotic boundary conditions with two boundary conditions given at an unknown free boundary that has to be determined as part of the solution. This idea was formulated for the first time for the numerical solution of the Blasius problem by Fazio [10]. Moreover, its application has been proposed for the problems in boundary layer theory, see Fazio

[11].

2 Two extended Blasius problems

The first extended Blasius problem was already considered by Schowalter [24], Lee and Ames [16], Lin and Chern [19], Kim et al. [15], or Akcay and Yükselen [1].

$$
\begin{gathered}
\frac{d^3 f}{d\eta^3}\frac{d^2 f}{d\eta^2}^{(P-1)} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \\
f(0) = \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta) \to 1 \quad \text{as} \quad \eta \to \infty \,,
\end{gathered}
\tag{1}
$$

where f and η are appropriate similarity variables and P verifies the conditions $1 \leq P < 2$. Liao [18] has found analytically that the extended Blasius problem (1) for $P = 2$ admit an infinite number of solutions and, therefore, in his opinion can be considered as a challenging problem for numerical techniques. Let us remark here, that the case $P = 1$ is the classical Blasius problem, see Blasius [5].

As far as the second extended Blasius problem is concerned, the mathematical model arises in the study of a 2D laminar boundary-layer with power-law viscosity for Newtonian fluids and is given by, see Schlichting and Gersten [23] or Benlahsen et al. [4],

$$
\begin{gathered}
\frac{d}{d\eta}\left(\left|\frac{d^2 f}{d\eta^2}\right|^{P-1} \frac{d^2 f}{d\eta^2}\right) + \frac{1}{P+1} f \frac{d^2 f}{d\eta^2} = 0 \\
f(0) = \frac{df}{d\eta}(0) = 0 \,, \qquad \frac{df}{d\eta}(\eta) \to 1 \quad \text{as} \quad \eta \to \infty \,,
\end{gathered}
\tag{2}
$$

where $f(\eta)$ is the non-dimensional stream function and P is a given positive value bigger than zero. Let us remark here, that when $P = 1$ the BVP (2) reduces to the celebrated Blasius problem, see Blasius [5].

3 The free boundary formulation idea

As mentioned before, in a free boundary formulation we replace the asymptotic boundary conditions with two boundary conditions fixed at an unknown free boundary that has to be determined as part of the solution. For both the extended Blasius problems considered in this paper, a free boundary formulation can be defined in the same way. We replace the asymptotic boundary condition with the two conditions

$$\frac{df}{d\eta}(\eta_\epsilon) \, , \qquad \frac{d^2 f}{d\eta^2} = \epsilon \, , \tag{3}$$

where η_ϵ is an unknown free boundary and ϵ is an assigned small value. Of course, we can verify if η_ϵ goes to infinity as ϵ goes to zero. In order to have a valid formulation, this should always be true.

4 Numerical results

In this section, we report the computed numerical results for the free boundary formulations of the two extended Blasius problem.

4.1 First extended Blasius

Before considering to solve numerically the free boundary formulation of the first extended Blasius problem it may be convenient to reformulate it in normal form, sse Asher and Russel [2]. To this end, we introduce the new variables $\theta = \eta/\eta_\epsilon$, $u_1 = f(\eta)$, $u_2 = \frac{df}{d\eta}(\eta)$,$u_3 = \frac{d^2f}{d\eta^2}(\eta)$ and $u_4 = \eta_\epsilon$. So that, the free boundary formulation for the problem (1) is given by:

$$
\begin{aligned}
&\frac{du_1}{d\theta} = \eta_\epsilon u_2 \\
&\frac{du_2}{d\theta} = \eta_\epsilon u_3 \\
&\frac{du_3}{d\theta} = -\eta_\epsilon \frac{1}{2} u_1 u_3^{2-P} \\
&\frac{du_4}{d\theta} = 0 \\
&f(0) = \frac{df}{d\theta}(0) = 0, \qquad \frac{df}{d\theta}(1) = 1 \quad \frac{d^2 f}{d\theta^2}(1) = \epsilon \ ,
\end{aligned}
\tag{4}
$$

For the numerical solution of the BVP (4) we used the *bvp4c.m* MATLAB routine with initial iterate given by

$$
u_1 = \theta \ , \quad u_2 = 2 + \theta \ , \quad u_3 = \theta \ , \quad u_4 = 1 \ . \tag{5}
$$

As it is easily seen, these are coarse approximations of the actual solution components.

As far as the first extended Blasius problem is concerned, in table 1 we list the chosen values of ϵ, the corresponding free boundary values η_ϵ, and the related missing initial conditions $\frac{d^2 f}{d\eta^2}(0)$ for the problem (1) with $P = 3/2$. The obtained value of the missing initial condition can be compared with the one 0.46905520505 obtained by Fazio, using an iterative transformation method and reported in a recent preprint [13]. As it is easily seen, the two values agree up to the first six decimal places. Figure 1 shows the solution of the extended Blasius problem (1) with $P = 3/2$

4.2 Second extended Blasius

Once again, we rewrite the free boundary formulation of the second extended Blasius problem in standard form. We define the same variables as before: $\theta = \eta/\eta_\epsilon$, $u_1 = f(\eta)$, $u_2 = \frac{df}{d\eta}(\eta)$,$u_3 = \frac{d^2 f}{d\eta^2}(\eta)$ and $u_4 = \eta_\epsilon$, so that, the

Table 1: Numerical data and results.

ϵ	η_ϵ	$\frac{d^2 f}{d\eta^2}(0)$
0.1	2.708708	0.482527634
0.01	3.193357	0.469356138
0.0001	3.323660	0.469098357
0.00001	3.364091	0.469055438
0.000001	3.376487	0.469055050
0.0000001	3.380402	0.469055086
0.00000001	3.381636	0.469055082
0.000000001	3.382027	0.469055080
0.0000000001	3.382150	0.469055080

free boundary formulation for the problem (2) is

$$
\begin{aligned}
&\frac{du_1}{d\theta} = \eta_\epsilon u_2 \\
&\frac{du_2}{d\theta} = \eta_\epsilon u_3 \\
&\frac{du_3}{d\theta} = -\eta_\epsilon \frac{u_1 u_3}{(P+1) * ((P-1) * |u_3|^{P-2} + |u_3|^{P-1})} \\
&\frac{du_4}{d\theta} = 0 \\
&f(0) = \frac{df}{d\theta}(0) = 0, \qquad \frac{df}{d\theta}(1) = 1 \quad \frac{d^2 f}{d\theta^2}(1) = \epsilon \,,
\end{aligned}
\tag{6}
$$

For the numerical solution of the BVP (6) we used the *bvp4c.m* MATLAB routine with initial iterate again provided by the relations (5).

Here, we report the numerical results obtained for the free boundary formulation of the second extension (2) of the Blasius problem for

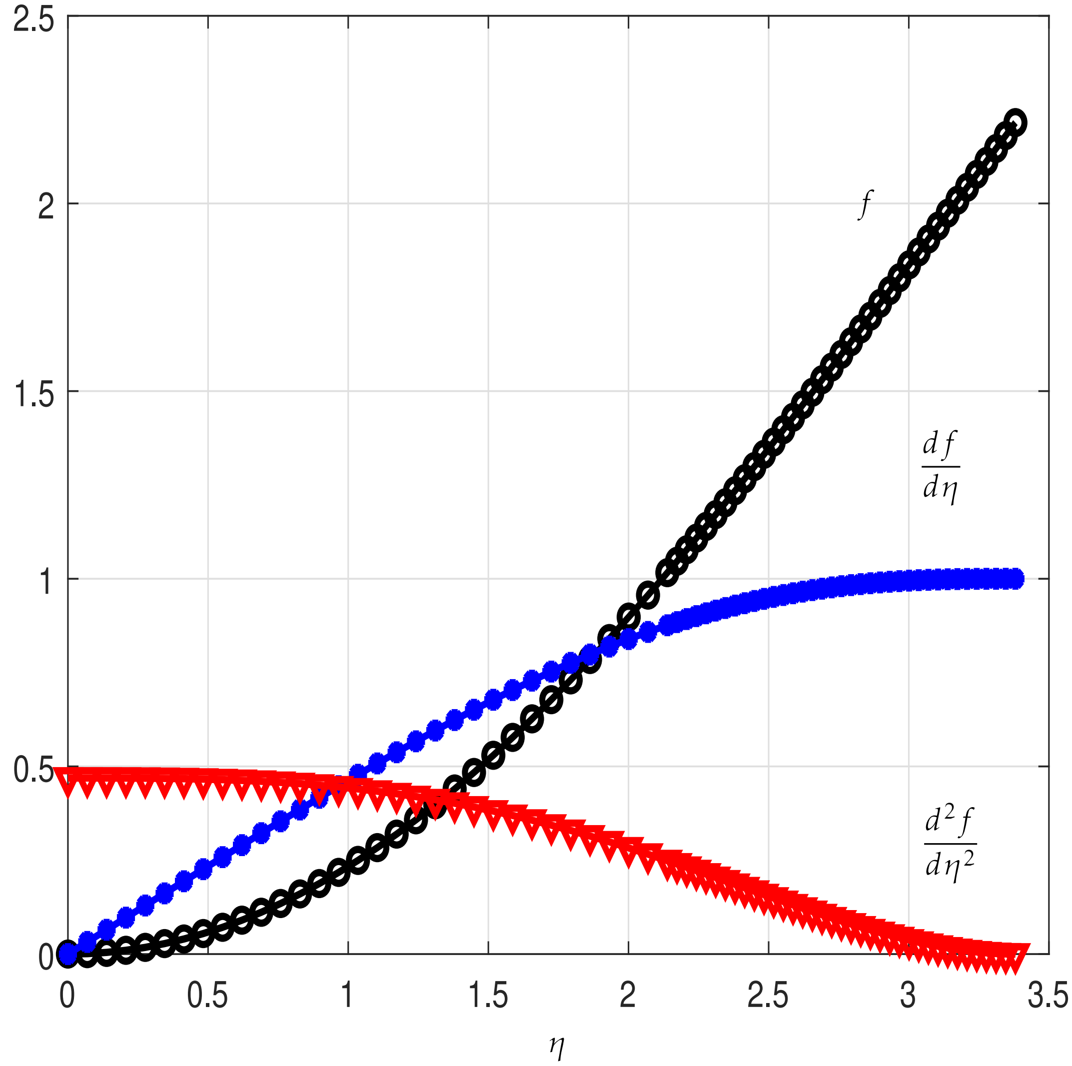

Figure 1: Numerical result obtained using the free boundary formulation for (1) with $P = 3/2$ and $\epsilon = 1D-06$.

two values of the involved parameter. The first value of the parameter is $P = 1/2$, and in this case for $\epsilon = 1D-06$ we find $\eta_\epsilon = 56.654480$ and $\frac{d^2f}{d\eta^2}(0) = 0.331237479$. The second considered value of the parameter is $P = 2$, and in this case for $\epsilon = 1D-06$ we find $\eta_\epsilon = 4.346478$ and $\frac{d^2f}{d\eta^2}(0) = 0.364773537$. These values of the missing initial condition can be compared with those found by Fazio [12], using a non-iterative transformation method. Those values are 0.337170 for $P = 1/2$ and 0.364772 for $P = 3/2$. In figure 2, for the reader convenience, we plot the two numerical solution components $f(\eta)$ and $\frac{df}{d\eta}(\eta)$.

5 Conclusions

In this paper we have defined the free boundary formulation for two extended Blasius problems. These problems are of interest in boundary layer theory and are deduced from the governing partial differential equations by using appropriate similarity variables. As far as the scaling invariance theory is concerned, we refer the interested reader to the books by Bluman and Cole [6], Barenblatt [3], or Dresner [9].

For both the extended Blasius problems, the computed results, for the so-called missing initial condition, are favourably compared with recent results available in the literature.

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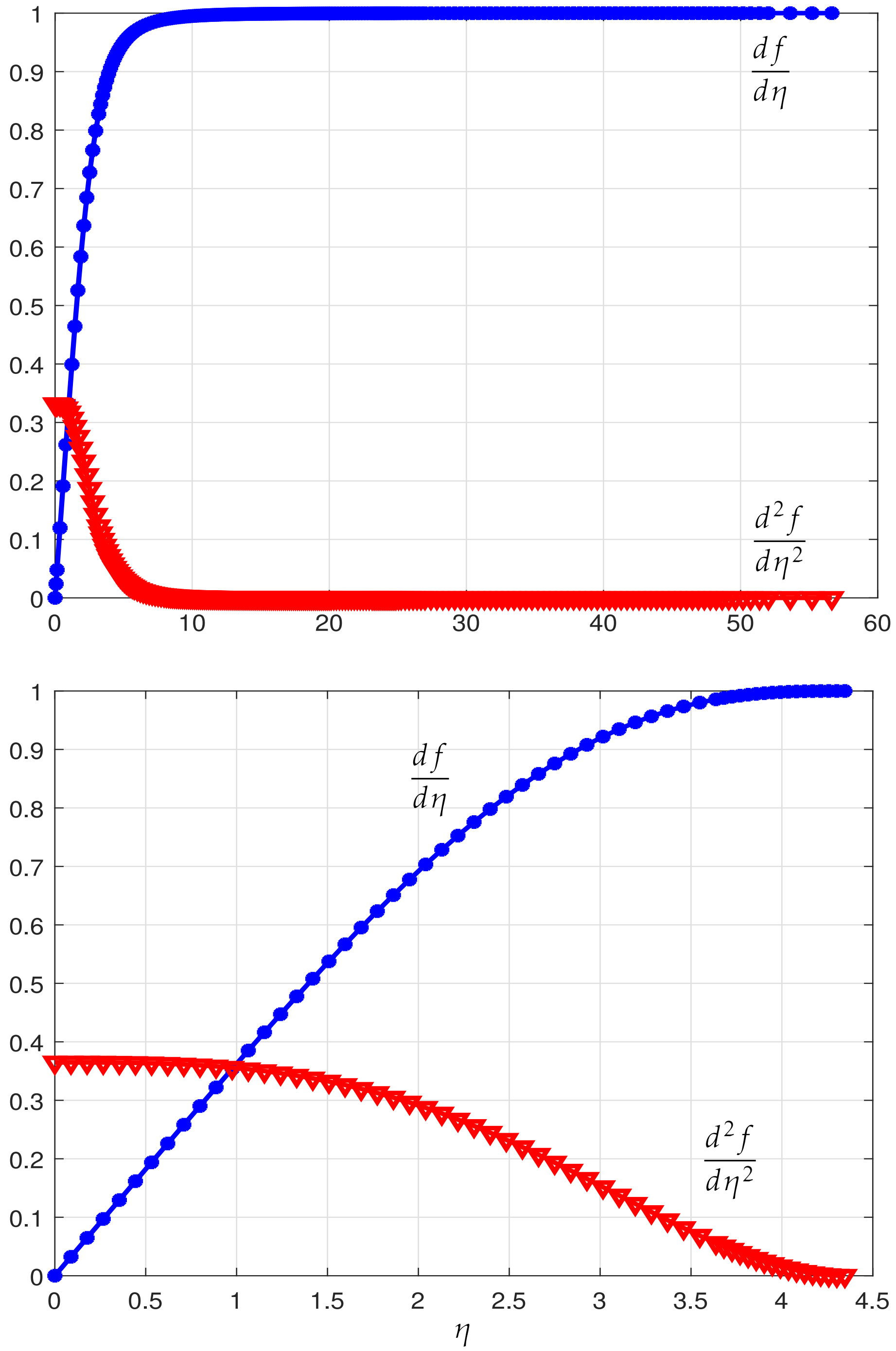

Figure 2: Numerical result obtained using the free boundary formulation with $\epsilon = 1D-06$ for (2) with: top frame $P = 1/2$ and bottom $P = 3/2$.

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