A Non-Iterative Transformation Method for an Extended Blasius Problem

Riccardo Fazio

Department of Mathematics, Computer Science

Physical Sciences and Earth Sciences,

University of Messina

Viale F. Stagno D'Alcontres, 31

98166 Messina, Italy

E-mail: rfazio@unime.it

Home-page: http://mat521.unime.it/fazio

March 22, 2020

Abstract

In this paper, we define a non-iterative transformation method for an Extended Blasius Problem. The original non-iterative transformation method, which is based on scaling invariance properties, was defined for the classical Blasius problem by Töpfer in 1912. This method allows us to solve numerically a boundary value problem by solving a related initial value problem and then rescaling the obtained numerical solution. In recent years, we have seen applications of the non-iterative transformation method to several problems of interest.

The obtained numerical results are improved by both a mesh refinement strategy and Richardson's extrapolation technique. In this way, we can be confident that the computed six decimal places are correct.

1

Key Words. Extended Blasius problem; scaling invariance properties; non-iterative transformation method; BVPs on infinite intervals.

AMS Subject Classifications. 65L10, 34B15, 65L08.

1 Introduction.

The problem of determining the steady two-dimensional motion of a fluid past a flat plate placed edge-ways to the stream was formulated in general terms, according to the boundary layer theory, by Prandtl [32], and was investigated in detail by Blasius [2]. The engineering interest was to calculate the shear at the plate (skin friction), which leads to the determination of the viscous drag on the plate, see for instance Schlichting [38]). In this contest the celebrated Blasius problem is given by

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0$$

$$f(0) = \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta) \to 1 \quad \text{as} \quad \eta \to \infty .$$

(1)

This is a boundary value problem (BVP) defined on the semi-infinite interval $[0, \infty)$. According to Weyl [41], the unique solution of (1) has a positive second order derivative, which is monotone decreasing on $[0, \infty)$ and approaches to zero as η goes to infinity. The governing differential equation and the two boundary conditions at the origin in (1) are invariant with respect to the scaling group of transformations

$$\eta^* = \lambda^{-\alpha}, \qquad f^* = \lambda^\alpha f$$

(2)

where α is a nonzero constant, classically $\alpha = 1/3$; but here we have used also $\alpha = 1$ in order to simplify the analysis. The mentioned invariance property has both analytical and numerical interest. From a numerical viewpoint a non-iterative transformation method (ITM) reducing the solution of (1) to the solution of a related initial value problem (IVP) was defined by Töpfer [40]. Owing to that

transformation, a simple existence and uniqueness Theorem was given by J. Serrin as reported by Meyer [29, pp. 104-105]. Let us note here that the mentioned invariance property is essential to the error analysis of the truncated boundary solution due to Rubel [35], see Fazio [13]. Blasius problem was used, recently, by Boyd [4] as an example were some good analysis, before the computer invention, allowed researchers of the past to solve problems, governed by partial differential equations, that might be otherwise impossible to face.

The application of a non-ITM to the Blasius equation with slip boundary condition, arising within the study of gas and liquid flows at the micro-scale regime [5, 28], was considered already in [14]. We applied a non-ITM to the Blasius equation with moving wall considered by Ishak et al. [21] or surface gasification studied by Emmons [6] and recently by Lu and Law [27] or slip boundary conditions investigated by Gad-el-Hak [5] or Martin and Boyd [28], see Fazio [16] for details. In particular, we find a way to solve non-iteratively the Sakiadis problem [36, 37]. As far as the non-ITM is concerned, a recent review dealing with all the cited problems can be be found in [18].

Moreover, Töpfer's method has been extended to classes of problems in boundary layer theory involving a physical parameter. This kind of extension was considered first by Na [30], see also NA [31, Chapters 8-9].

Finally, an iterative extension of the transformation method has been introduced, for the numerical solution of free BVPs, by Fazio [19]. This iterative extension has been applied to several problems of interest: free boundary problems [19, 10, 11], a moving boundary hyperbolic problem [8], Homann and Hiemenz problems governed by the Falkner-Skan equation in [9], one-dimensional parabolic moving boundary problems [12], two variants of the Blasius problem [14], namely: a boundary layer problem over moving surfaces, studied first by Klemp and Acrivos [23], and a boundary layer problem with slip boundary condition, that has found application in the study of gas and liquid flows at the micro-scale regime [5, 28], parabolic problems on unbounded domains [20] and, recently, see [15], a further variant of the Blasius problem in boundary layer theory: the so-called Sakiadis problem [36, 37]. A recent review dealing with, the derivation and application of,

3

ITM can be be found by the interested reader in [17].

2   Extended Blasius problem

Our extended Blasius problem is given by,

$$
\frac{d^3 f}{d\eta^3} \frac{d^2 f}{d\eta^2}^{(P-1)} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0
$$

$$
f(0) = \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta) \to 1 \quad \text{as} \quad \eta \to \infty \,,
$$

(3)

where P verifies the conditions $1 \le P < 2$, see Schowalter [39], Lee and Ames [24], Lin and Chern [26], Kim et al. [22], or Akcay and Yükselen [1]. Liao [25] has found analytically that the extended Blasius problem for $P = 2$ admit an infinite number of solutions anf therefore in his opinion can be considered as a challenge problem for numerical techniques.

2.1   The non-ITM

The applicability of a non-ITM to the Blasius problem (1) is a consequence of both: the invariance of the governing differential equation and the two boundary conditions at $\eta = 0$, and the non invariance of the asymptotic boundary condition under the scaling transformation (2). In order to apply a non-ITM to the BVP (3) we investigate its invariance with respect to the scaling group

$$
f^* = \lambda f \,, \qquad \eta^* = \lambda^\delta \eta \,.
$$

(4)

We find that the extended Blasius problem (3) is invariant under (4) iff

$$
\delta = \frac{2 - P}{1 - 2P} \,.
$$

(5)

Now, we can integrate the extended Blasius equation in (3) written in the star variables on $[0, \eta_\infty^*]$, where η_∞^* is a suitable truncated boundary, with initial conditions

$$
f^*(0) = \frac{df^*}{d\eta^*}(0) = 0 \,, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1 \,,
$$

(6)

4

in order to compute an approximation $\frac{df^*}{d\eta^*}(\eta^*_\infty)$ for $\frac{df^*}{d\eta^*}(\infty)$ and the corresponding value of λ according to the equation

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta^*_\infty)\right]^{1/(1-\delta)} . \tag{7}$$

Once the value of λ has been computed by equation (7), we can find the missed initial condition by the equation

$$\frac{d^2 f}{d\eta^2}(0) = \lambda^{2\delta-1}\frac{d^2 f^*}{d\eta^{*2}}(0) . \tag{8}$$

Moreover, the numerical solution of the original BVP (3) can be computed by rescaling the solution of the IVP. In this way we get the solution of a given BVP by solving a related IVP.

3   Numerical results

In this section we report the numerical results computed with our non-ITM. To compute the numerical solution, we used the classical fourth order Runge-Kutta method with constant step size. For the results shown in this figure we used $\Delta\eta = 0.001$ and $\eta^*_\infty = 5$. Figure 1 shows the solution of the extended Blasius problem, describing the behaviour of a boundary layer flow due to a moving flat surface immersed in an otherwise quiescent fluid, corresponding to $P = 3/2$. Let us notice here that, by rescaling, we get $\eta^*_\infty < \eta_\infty$.

In table 1 we report the missing initial condition $\frac{d^2 f}{d\eta^2}(0)$ obtained by a mesh refinement. As it is easily seen the decimal digits, starting to the left of the decimal point, tend to be confirmed, in particular, here we have confirmed the first four decimal values.

In order to improve the obtained missing initial condition value, we can apply a Richardson's extrapolation technique, see [33, 34]. Let us consider that we would like to compute the value $U = \frac{d^2 f}{d\eta^2}(0)$ as $\Delta\eta$ goes to zero, then we can apply the extrapolation formula

$$U_{g+1,k+1} = U_{g+1,k} + \frac{U_{g+1,k} - U_{g,k}}{4 - 1} , \tag{9}$$
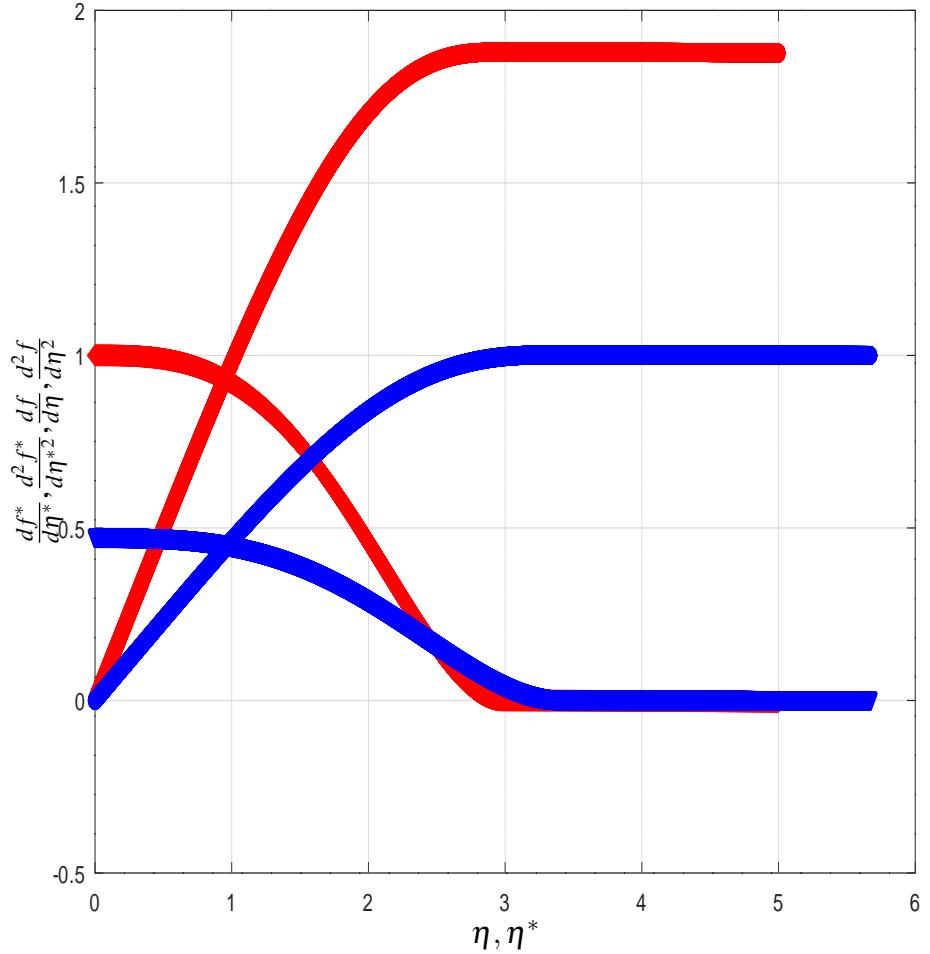
5

Figure 1: Numerical results of the non-ITM for (3) with $P = 3/2$. The starred variables problem and the original problem solution components found after rescaling.

where $g \in \{0, 1, 2, \ldots, G-1\}$, $k \in \{0, 1, 2, \ldots, G-1\}$, here g indicates the g-esima computational grid, k the level of extrapolation, and 4 is the true order of the discretization error for the classical Runge-Kutta method. The formula (9) is asymptotically exact in the limit as $\Delta\eta$ goes to zero if we use uniform grids. We notice that to obtain each value of $U_{g+1,k+1}$ requires two computed solutions U in two adjacent grids, namely $g + 1$ and g at the extrapolation level k. For any g, the

Table 1: Missing initial condition by mesh refinement.

$\Delta\eta$	$\frac{d^2 f}{d\eta^2}(0)$
0.001	0.47001559210
0.0005	0.46953524356
0.00025	0.46929513793
0.000125	0.46917509666
0.0000625	0.46911507775
0.00003125	0.46908506833
0.000015625	0.46907006359
0.0000078125	0.46906256120

level $k = 0$ represents the numerical solution of U without any extrapolation. The numerical results obtained applying formula (9) to the values reported in table 1 are reported in table 2.

Naturally, we can stop the mesh refinement as soon as $U_{g+1,k} = U_{g,k}$ or Richardon's extrapolation when $U_{g,k+1} = U_{g,k}$. It is evident, from the data reported in table 2, that we have achieved six decimal places of agreement.

As mentioned before, the case $P = 1$ is the Blasius problem (1). In this case our non-ITM reduces to the original method defined by Töpfer [40]. In that case, the computed skin friction coefficient value, namely 0.332057336215, obtained with $\Delta\eta = 0.001$ and $\eta_\infty^* = 10$, is in good agreement with the values available in literature, see for instance the value 0.332057336215 computed by Fazio [7] by a free boundary formulation of the Blasius problem or the value 0.33205733621519630 computed by Boyd [3] who believes all the decimal digits to be correct.

Table 2: Missing initial condition by Richardson's extrapolation.

	$U_{g,0}$	$U_{g,1}$	$U_{g,2}$	$U_{g,3}$	$U_{g,4}$	$U_{g,5}$	$U_{g,6}$	$U_{g,7}$
∞	0.47001559210							
	0.46953524356	0.46937512738						
	0.46929513793	0.46921510272	0.46916176117					
	0.46917509666	0.46913508290	0.46910840963	0.46909062578				
	0.46911507775	0.46909507145	0.46908173430	0.46907284252	0.46906691477			
	0.46908506833	0.46907506519	0.46906839644	0.46906395049	0.46906098648	0.46905901038		
	0.46907006359	0.46906506201	0.46906172762	0.46905950468	0.46905802274	0.46905703482	0.46905637630	
	0.46906256120	0.46906006040	0.46905839320	0.46905728173	0.46905654074	0.46905604674	0.46905571739	0.46905549774

4   Concluding remarks.

The main contribution of this paper is the extension of the non-ITM, proposed by Töpfer [40] and defined for the numerical solution of the celebrated Blasius problem [2], to an extended Blasius problem. This method allows us to solve numerically the extended Blasius problem by solving a related initial value problem and then rescaling the obtained numerical solution. The obtained numerical results, have been improved both by a mesh refinement and using the Richardson's extrapolation technique. In this way, we can be confident that the computed six decimal place are correct.

References

[1] M. Akcay and M. A. Yükselen. Drag reduction of a non-Newtonian fluid by fluid injection on a moving wall. *Arch. Appl. Mech.*, 69:215–225, 1999.

[2] H. Blasius. Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.*, 56:1–37, 1908.

[3] J. P. Boyd. The Blasius function in the complex plane. *Exp. Math.*, 8:381–394, 1999.

[4] J. P. Boyd. The Blasius function: computation before computers, the value of tricks, undergradute projects, and open research problems. *SIAM Rev.*, 50:791–804, 2008.

[5] M. Gad el Hak. The fluid mechanics of microdevices — the Freeman scholar lecture. *J. Fluids Eng.*, 121:5–33, 1999.

[6] H. W. Emmons. The film combustion of liquid fluid. *ZAMM - J. Appl. Math. Mech.*, 36:60–71, 1956.

[7] R. Fazio. The Blasius problem formulated as a free boundary value problem. *Acta Mech.*, 95:1–7, 1992.

[8] R. Fazio. A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material. *Wave Motion*, 16:299–305, 1992.

[9] R. Fazio. The Falkner-Skan equation: numerical solutions within group invariance theory. *Calcolo*, 31:115–124, 1994.

[10] R. Fazio. A numerical test for the existence and uniqueness of solution of free boundary problems. *Appl. Anal.*, 66:89–100, 1997.

[11] R. Fazio. A similarity approach to the numerical solution of free boundary problems. *SIAM Rev.*, 40:616–635, 1998.

[12] R. Fazio. The iterative transformation method: numerical solution of one-dimensional parabolic moving boundary problems. *Int. J. Computer Math.*, 78:213–223, 2001.

[13] R. Fazio. A survey on free boundary identification of the truncated boundary in numerical BVPs on infinite intervals. *J. Comput. Appl. Math.*, 140:331–344, 2002.

[14] R. Fazio. Numerical transformation methods: Blasius problem and its variants. *Appl. Math. Comput.*, 215:1513–1521, 2009.

[15] R. Fazio. The iterative transformation method for the Sakiadis problem. *Comput. & Fluids*, 106:196–200, 2015.

[16] R. Fazio. A non-iterative transformation method for Blasius equation with moving wall or surface gasification. *Int. J. Non-Linear Mech.*, 78:156–159, 2016.

[17] R. Fazio. The iterative transformation method. *Int. J. Non-Linear Mech.*, 116:181–194, 2019.

[18] R. Fazio. The non-iterative transformation method. *Int. J. Non-Linear Mech.*, 114:41–48, 2019.

[19] R. Fazio and D. J. Evans. Similarity and numerical analysis for free boundary value problems. *Int. J. Computer Math.*, 31:215–220, 1990. 39 : 249, 1991.

[20] R. Fazio and S. Iacono. On the moving boundary formulation for parabolic problems on unbounded domains. *Int. J. Computer Math.*, 87:186–198, 2010.

[21] A. Ishak, R. Nazar, and I. Pop. Boundary layer on a moving wall with suction and injection. *Chin. Phys. Lett.*, 24:2274–2276, 2007.

[22] H. W. Kim, D. R. Jeng, and K. J. DeWitt. Momentum and heat transfer in power-law fluid flow over two-dimensional or axisymmetic bodies. *Inter. J. Heat Mass Transfer*, 26:245–259, 1983.

[23] J. P. Klemp and A. Acrivos. A moving-wall boundary layer with reverse flow. *J. Fluid Mech.*, 53:177–191, 1972.

[24] S. Y. Lee and W. F. Ames. Similarity solutions for non-Newtonian fluids. *A.I.Ch.E. J.*, 12:700–708, 1966.

[25] S.-J. Liao. A challenging nonlinear problem for numerical techniques. *J. Comput. Appl. Math.*, 181:467–472, 1997.

[26] F. N. Lin and S. Y. Chern. Laminar boundary-layer flow of non-Newtonian fluid. *Inter. J. Heat Mass Transfer*, 22:1323–1329, 1979.

[27] Z. Lu and C. K. Law. An iterative solution of the Blasius flow with surface gasification. *Int. J. Heat and Mass Transfer*, 69:223–229, 2014.

[28] M. J. Martin and I. D. Boyd. Blasius boundary layer solution with slip flow conditions. In *Rarefied Gas Dynamics: 22nd International Symposium*, volume 585 of *American Institute of Physics Conference Proceedings*, pages 518–523, 2001, DOI: 10.1063/1.1407604.

[29] R. E. Meyer. *Introduction to Mathematical Fluid Dynamics*. Wiley, New York, 1971.

[30] T. Y. Na. An initial value method for the solution of a class of nonlinear equations in fluid mechanics. *J. Basic Engrg. Trans. ASME*, 92:503–509, 1970.

[31] T. Y. Na. *Computational Methods in Engineering Boundary Value Problems*. Academic Press, New York, 1979.

[32] L. Prandtl. Über Flüssigkeiten mit kleiner Reibung. In *Proceedings Third Internernatinal Math. Congress*, pages 484–494, 1904. Engl. transl. in NACA Tech. Memo. 452.

[33] L. F. Richardson. The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. *Proc. R. Soc. London Ser. A*, 210:307–357, 1910.

[34] L. F. Richardson and J. A. Gaunt. The deferred approach to the limit. *Proc. R. Soc. London Ser. A*, 226:299–349, 1927.

[35] L. A. Rubel. An estimation of the error due to the truncated boundary in the numerical solution of the Blasius equation. *Quart. Appl. Math.*, 13:203–206, 1955.

[36] B. C. Sakiadis. Boundary-layer behaviour on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow. *AIChE J.*, 7:26–28, 1961.

[37] B. C. Sakiadis. Boundary-layer behaviour on continuous solid surfaces: II. The boundary layer on a continuous flat surface. *AIChE J.*, 7:221–225, 1961.

[38] H. Schlichting and K. Gersten. *Boundary Layer Theory*. Springer, Berlin, 8th edition, 2000.

[39] W. R. Schowalter. The application of boundary-layer theory to power-law fluids: similar solutions. *A.I.Ch.E. j.*, 6:24–28, 1960.

[40] K. Töpfer. Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.*, 60:397–398, 1912.

[41] H. Weyl. On the differential equation of the simplest boundary-layer problems. *Ann. Math.*, 43:381–407, 1942.