Quasi-uniform Grids and Ad Hoc Finite Difference Schemes for BVPs on Infinite Intervals

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- BVPs on infinite intervals
- Finite differences on quasi-uniform grids
- Numerical Tests
- Concluding Remarks

BVPs on infinite intervals

Let us consider the class of BVPs defined on an infinite interval

$$\begin{aligned} \frac{d\mathbf{u}}{dx} &= \mathbf{f}\left(x, \mathbf{u}\right) \quad x \in [0, \infty) \\ \mathbf{g}\left(\mathbf{u}(0), \mathbf{u}(\infty)\right) &= \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{u}(x)$ is a $d-$dimensional vector with $u_\ell(x)$ for $\ell = 1, \ldots, d$ as components. Moreover,

$$\begin{aligned} \mathbf{f} &: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \\ \mathbf{g} &: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \end{aligned}$$

Quasi-uniform grids

Let us consider the smooth strict monotone quasi-uniform maps $x = x(\xi)$, the so-called grid generating functions,

$$x = -c \cdot \ln(1 - \xi) \ , \tag{2}$$

and

$$x = c\frac{\xi}{1 - \xi} \ , \tag{3}$$

where $\xi \in [0, 1]$, $x \in [0, \infty)$, and $c > 0$ is a control parameter.

A family of uniform grids $\xi_n = n/N$ defined on interval $[0, 1]$ generates one parameter family of quasi-uniform grids $x_n = x(\xi_n)$ on the interval $[0, \infty)$.

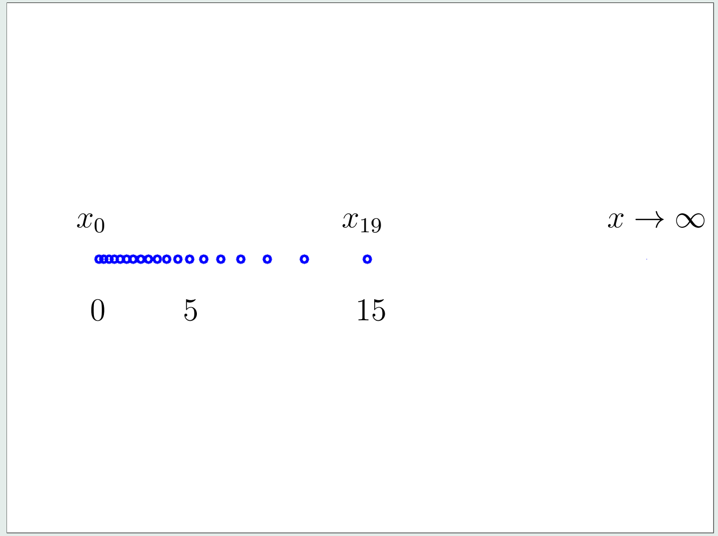

Figure 1 *Quasi-uniform mesh for (2) with $c = 5$ and $N = 20$. We notice that the last mesh-point is $x_N = \infty$.*

Half of the intervals are in the domain with length approximately equal to c and $x_{N-1} \approx c \ln N$ for (2).

The problem (1) can be discretized by introducing a uniform grid ξ_n of $N+1$ nodes in $[0,1]$ with $\xi_0 = 0$ and $\xi_{n+1} = \xi_n + h$ with $h = 1/N$, so that x_n is a quasi-uniform grid in $[0, \infty)$.

The last interval in (2) and (3), namely $[x_{N-1}, x_N]$, is infinite but the point $x_{N-1/2}$ is finite, because the non integer nodes are defined by

$$x_{n+\alpha} = x\left(\xi = \frac{n+\alpha}{N}\right) \quad , \tag{4}$$

with $n = 0, 1, \ldots, N-1$ and $0 < \alpha < 1$.

These maps describe the infinite domain by a finite number of intervals. The last node is placed on infinity so right boundary condition is taken into account correctly.

We define the values of $u(x)$ on the mid-points of the grid

$$u_{n+1/2} \approx \frac{x_{n+1} - x_{n+1/2}}{x_{n+1} - x_n} u_n + \frac{x_{n+1/2} - x_n}{x_{n+1} - x_n} u_{n+1} \,. \tag{5}$$

As far as the derivative, by considering $u = u(\xi(x))$, we write

$$\left.\frac{du}{dx}\right|_{n+1/2} = \left.\frac{du}{d\xi}\right|_{n+1/2} \left.\frac{d\xi}{dx}\right|_{n+1/2} \approx \frac{(u_{n+1} - u_n)}{(\xi_{n+1} - \xi_n)} \frac{2\left(\xi_{n+3/4} - \xi_{n+1/4}\right)}{2\left(x_{n+3/4} - x_{n+1/4}\right)}$$

then

$$\left.\frac{du}{dx}\right|_{n+1/2} \approx \frac{u_{n+1} - u_n}{2\left(x_{n+3/4} - x_{n+1/4}\right)} \,. \tag{6}$$

because $2\left(\xi_{n+3/4} - \xi_{n+1/4}\right) = \xi_{n+1} - \xi_n$.

This formulae uses the value $u_N = u_\infty$, but not $x_N = \infty$.

The two finite difference approximations (5) and (6) have order of accuracy $O(N^{-2})$.

A non-standard finite difference scheme on quasi-uniform grids

A second order finite difference scheme can be written as follows

$$\mathbf{U}_{n+1} - \mathbf{U}_n - a_{n+1/2}\mathbf{f}\left(x_{n+1/2}, b_{n+1/2}\mathbf{U}_{n+1} + c_{n+1/2}\mathbf{U}_n\right) = \mathbf{0} \ , \tag{7}$$

$$\mathbf{g}\left(\mathbf{U}_0, \mathbf{U}_N\right) = \mathbf{0} \ ,$$

where the $d-$dimensional vector $\mathbf{U}_n$ is the numerical approximation to the solution $\mathbf{u}(x_n)$ at the points of the mesh, and

$$a_{n+1/2} = 2\left(x_{n+3/4} - x_{n+1/4}\right) \qquad b_{n+1/2} = \frac{x_{n+1/2} - x_n}{x_{n+1} - x_n}$$

$$c_{n+1/2} = \frac{x_{n+1} - x_{n+1/2}}{x_{n+1} - x_n} \qquad \text{for} \quad n = 0, 1, \ldots, N-1 \ .$$

We notice that

$$b_{n+1/2} \approx c_{n+1/2} \approx 1/2 \text{ for all } n = 0, 1, \ldots, N-2,$$

but when

$$n = N-1, \text{ then } b_{N-1/2} = 0 \text{ and } c_{N-1/2} = 1.$$

On the contrary, we choose to set

$$b_{N-1/2} = b_{N-3/2} \text{ and } c_{N-1/2} = c_{N-3/2}$$

in order to avoid a suddenly jump for the coefficients of the finite difference scheme.

As it will be clear from the results reported in the next section this produces a much smaller error in the numerical solution of the system at x_N.

The method (7) is a nonlinear system of $d \cdot (N+1)$ equations in the $d \cdot (N+1)$ unknowns $\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_N)^T$.
For the solution of (7) we can apply the classical Newton's method along with the simple termination criterion

$$\frac{1}{d(N+1)} \sum_{\ell=1}^{d} \sum_{n=0}^{N} |\Delta U_{n\ell}| \leq \text{TOL} \ ,$$

where $\Delta U_{n\ell}$, for $n = 0, 1, \ldots, N$ and $\ell = 1, 2, \ldots, d$, is the difference between two successive iterate components and TOL is a fixed tolerance.
The results listed in the next sections were computed by setting $\text{TOL} = 1\text{E} - 6$.

The Falkner-Skan model

The Falkner-Skan model of boundary layer theory is given by

$$\frac{d^3u}{dx^3} + u\frac{d^2u}{dx^2} + P\left[1 - \left(\frac{du}{dx}\right)^2\right] = 0$$
$$u(0) = \frac{du}{dx}(0) = 0, \qquad \frac{du}{dx}(\infty) = 1\ . \tag{8}$$

It is a BVP defined on a semi-infinite interval.

We rewrite the (8) as a first order system with

$$\frac{du_1}{dx} = u_2$$
$$\frac{du_2}{dx} = u_3$$
$$\frac{du_3}{dx} = -u_1u_3 - P(1 - {u_2}^2)\ .$$

Then, we have

$$\mathbf{u} = (u_1, u_2, u_3)^T$$
$$\mathbf{f}(x, \mathbf{u}) = \left(u_2, u_3, -u_1 u_3 - P(1 - {u_2}^2)\right)^T$$
$$\mathbf{g}\left(\mathbf{u}(0), \mathbf{u}(\infty)\right) = \left(u_1(0), u_2(0), u_2(\infty) - 1\right)^T \ ,$$

with the boundary conditions

$$u_1(0) = u_2(0) = 0 \ , \qquad u_2(\infty) = 1 \ .$$

For all values of N we used the initial iterate

$$u_1(x) = u_2(x) = 1/2 \ , \quad u_3(x) = 10^{-2} \ .$$

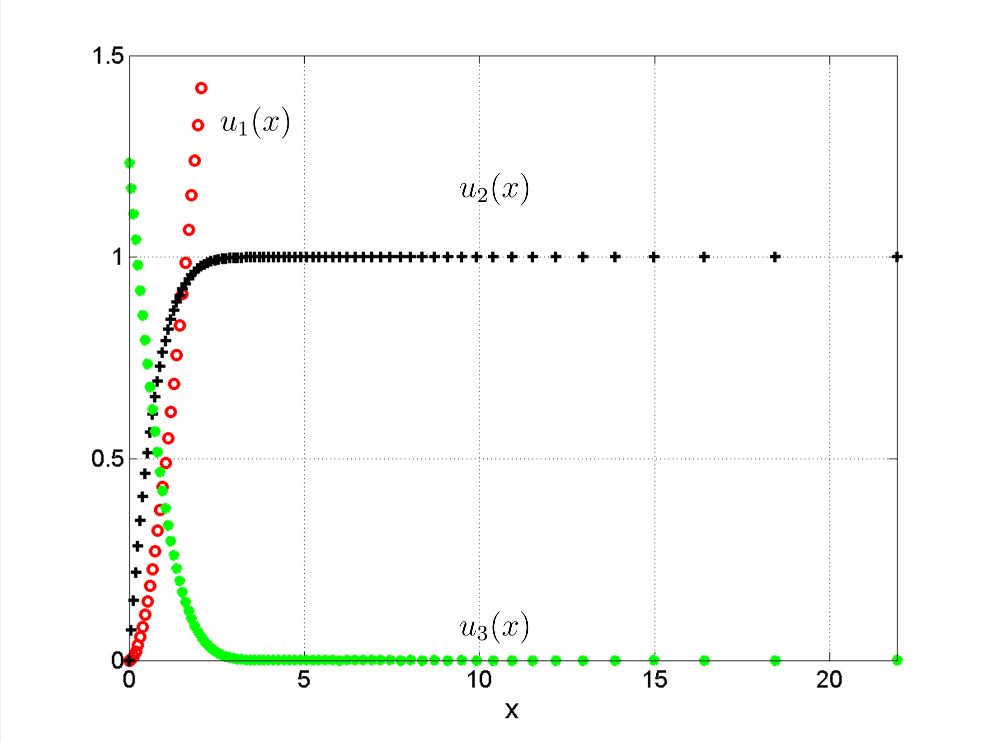

Figure 2 *Numerical solution of Falkner-Skan model with* $P = 1$ *obtained with the map* $x = x(\xi)$ *defined by* $x = -c \cdot \ln(1-\xi)$ *with* $c = 5$ *for* $N = 80$.

Table 1: *Numerical approximation of* $\frac{d^2u}{dx^2}(0)$ *and* $\frac{d^2u}{dx^2}(\infty)$.

N	iter	$\frac{d^2u}{dx^2}(0)$	$\frac{d^2u}{dx^2}(\infty)$
20	6	1.238724	$-0.21 \cdot 10^{-7}$
40	5	1.234124	$0.24 \cdot 10^{-7}$
80	5	1.232972	$-0.33 \cdot 10^{-7}$
160	5	1.232684	$0.14 \cdot 10^{-7}$
320	5	1.232612	$-0.25 \cdot 10^{-7}$
640	5	1.232594	$0.39 \cdot 10^{-7}$
1280	5	1.232589	$0.33 \cdot 10^{-7}$

Table 2: *Comparison of* $\dfrac{d^2u}{dx^2}(0)$ *and free or truncated boundary* (x_ϵ *and* x_∞ *respectively) for* $P = 0.5$ *and* $P = 1$.

	Nasr et al. [1] Chebyshev method		Fazio [2] Free BF		Asaithambi [3] Finite difference		This paper Quasi-uniform	
P	x_∞	$\dfrac{d^2u}{dx^2}(0)$	x_ϵ	$\dfrac{d^2u}{dx^2}(0)$	x_∞	$\dfrac{d^2u}{dx^2}(0)$	x_N	$\dfrac{d^2u}{dx^2}(0)$
0.5	3.7	0.927805						
0.5	7.4	0.927680	5.09	0.927680	5.67	0.927682	∞	0.927681
1	3.5	1.232617						
1	7.	1.232588	5.19	1.232588	5.14	1.232589	∞	1.232589

[1] H. Nasr et al. Int. J. Computer Math., 33 : 127 – 132, 1990.
[2] R. Fazio. Calcolo, 31 : 115 – 124, 1994.
[3] A. Asaithambi. Appl. Math. Comput., 92 : 135 – 141, 1998.

A pile in soil

Let $u(x)$ be the deflection of a semi-infinite pile embedded in soft soil at a distance x below the surface of the soil. The differential equation for the movement of the pile is given by

$$\begin{gathered}\frac{d^4u}{dx^4} = -P_1\left(1 - e^{-P_2 u}\right) \ , \qquad x \in [0, \infty) \ , \\ \frac{d^2u}{dx^2}(0) = 0 \ , \qquad \frac{d^3u}{dx^3}(0) = P_3 \ , \qquad u(\infty) = \frac{du}{dx}(\infty) = 0 \ ,\end{gathered} \tag{9}$$

where P_1 and P_2 are positive material constants. We rewrite the (9) as a first order system

$$\begin{aligned}\frac{du_1}{dx} &= u_2 \ , & \frac{du_2}{dx} &= u_3 \ , \\ \frac{du_3}{dx} &= u_4 \ , & \frac{du_4}{dx} &= -P_1\left(1 - e^{-P_2 u_1}\right) \ .\end{aligned}$$

Then, we have

$$\mathbf{u} = (u_1, u_2, u_3, u_4)^T$$
$$\mathbf{f}(x, \mathbf{u}) = \left(u_2, u_3, u_4, -P_1\left(1 - e^{-P_2 u_1}\right)\right)^T$$
$$\mathbf{g}\left(\mathbf{u}(0), \mathbf{u}(\infty)\right) = \left(u_3(0), u_4(0) - P_3, u_1(\infty), u_2(\infty)\right)^T$$

with the boundary conditions

$$u_3(0) = 0 \ , \qquad u_4(0) = P_3 \ , \qquad u_1(\infty) = 0 \ , \qquad u_2(\infty) = 0 \ .$$

For all values of N we used the initial iterate

$$u_1(x) = u_2(x) = u_3(x) = u_4(x) = 1$$

and

$$P_1 = 1, \quad P_2 = \frac{1}{2} \quad \text{and} \quad P_3 = \frac{1}{2} \ .$$

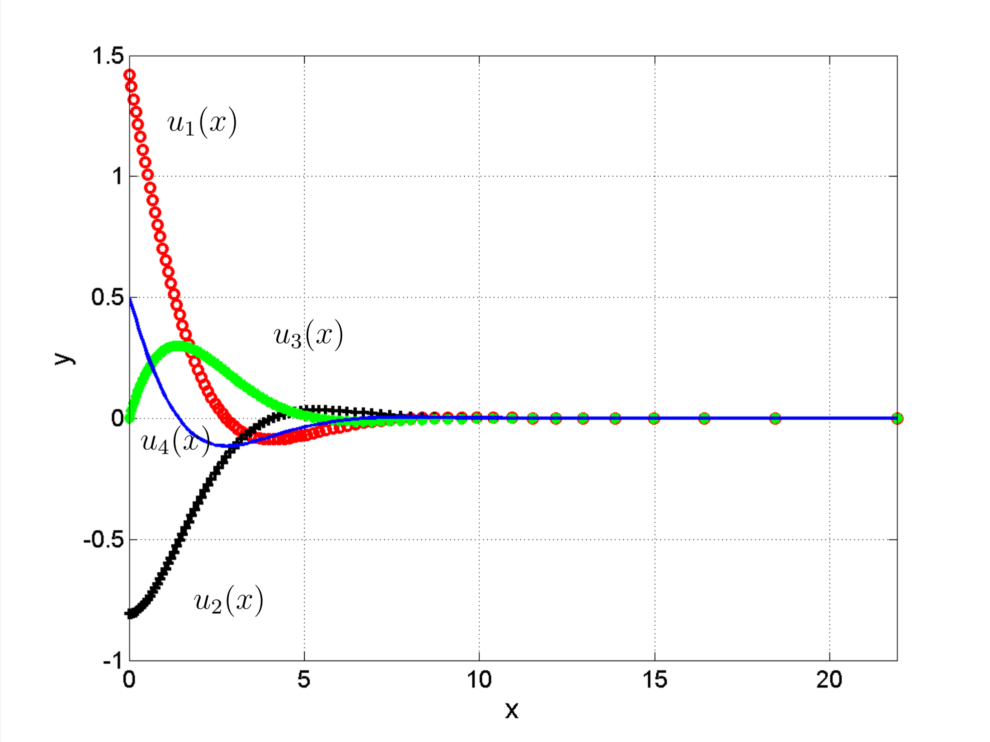

Figure 3 *Numerical solution of pile model (9) obtained with the map* $x = x(\xi)$ *defined by* $x = -c \cdot \ln(1-\xi)$ *with* $c = 5$ *for* $N = 80$.

Table 3: *Numerical approximation of* $u(0)$ *and* $\frac{du}{dx}(0)$.

N	iter	$u(0)$	$-\frac{du}{dx}(0)$
20	5	1.420337	0.807289
40	5	1.421243	0.807934
80	5	1.421469	0.808094
160	5	1.421526	0.808135
320	5	1.421540	0.808145
640	5	1.421544	0.808150
1280	5	1.421544	0.808150

Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

Table 4: *Comparison of* $u(0)$, $\dfrac{du}{dx}(0)$ *and free or truncated boundary* (x_ϵ *and* x_∞ *respectively) for the pile problem.*

Lentini and Keller [1]		Fazio [2]		This paper	
Asymptotic BCs $x_\infty = 10$		Free BF $x_\epsilon = 17.75$		Quasi-uniform $x_N = \infty$	
$u(0)$	$\dfrac{du}{dx}(0)$	$u(0)$	$\dfrac{du}{dx}(0)$	$u(0)$	$\dfrac{du}{dx}(0)$
1.4215	−0.80814	1.42154	−0.808144	1.421544	−0.808145

[1] M. Lentini and H. B. Keller. SIAM J. Numer. Anal., 17 : 577−604, 1980.
[2] R. Fazio. Calcolo, 31 : 115 − 124, 1994.

Concluding Remarks

Let us discuss, at the end of this work, a possible way to extend the finite difference schemes on quasi-uniform grids to the numerical solutions of problems defined on the whole real line. For these problems, all boundary conditions are imposed at $\pm\infty$. In such a case it is possible to use the tangential quasi-uniform grid

$$x_n = c \cdot \tan\left(\frac{n\pi}{2N}\right) ,$$

where $c > 0$ is a control parameter.
In fact, if $n = -N, -N+1, \cdots - 1, 0, 1, \cdots, N-1, N$, then this grid covers the whole infinite line, and in particular we have that $x_{-N} = -\infty$.