Applied Mathematics and Computation 215 (2009) 1513–1521

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical transformation methods: Blasius problem and its variants

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ARTICLE INFO

Keywords:
Boundary-layer theory
Scaling invariance
Transformation methods
Initial value methods

ABSTRACT

Blasius problem is the simplest nonlinear boundary-layer problem. We hope that any approach developed for this epitome can be extended to more difficult hydrodynamics problems. With this motivation we review the so called Töpfer transformation, which allows us to find a non-iterative numerical solution of the Blasius problem by solving a related initial value problem and applying a scaling transformation. The applicability of a non-iterative transformation method to the Blasius problem is a consequence of its partial invariance with respect to a scaling group. Several problems in boundary-layer theory lack this kind of invariance and cannot be solved by non-iterative transformation methods. To overcome this drawback, we can modify the problem under study by introducing a numerical parameter, and require the invariance of the modified problem with respect to an extended scaling group involving this parameter. Then we apply initial value methods to the most recent developments involving variants and extensions of the Blasius problem.

1. Introduction

At the beginning of the last century Prandtl [34] put the foundations of boundary-layer theory providing the basis for the unification of two, at that time seemingly incompatible, sciences: namely, theoretical hydrodynamics and hydraulics. Boundary-layer theory has found its main application in calculating the skin-friction drag which acts on a body as it is moved through a fluid: for example the drag of an airplane wing, of a turbine blade, or a complete ship [36]. With the turning of this new century, as the number of applications of microelectronics devices increases, boundary-layer theory has found a renewal of interest within the study of gas and liquid flows at the micro-scale regime [12,31].

Blasius problem is the simplest nonlinear boundary-layer problem. Recent publications on the Blasius problem are those by Yu and Chen [43], He [24,25], Liao [29], Lin [30], Boyd [5], Belhachmi et al. [3], Wang [41], Allan and Syam [2], Cortell [9], and Fang et al. [14]. In particular, a study by Boyd pointed out how this particular problem of boundary-layer theory has arose the interest of prominent scientist, like H. Weyl, J. von Neumann, M. Van Dyke, etc., see Table 1 in [5]. The main reason for this interest is due to the hope that any approach developed for this epitome can be extended to more difficult hydrodynamics problems. Last year, at the centenary of Blasius paper [4], further studies were developed by Brighi et al. [7] or Boyd [6].

Our main goal here is to show how to solve numerically the Blasius problem, and its variants and extensions, by initial value methods derived within scaling invariance theory. The Blasius equation and the two boundary conditions at the plate are invariant with respect to the scaling transformation

$$f^* = \lambda^{-\alpha} f, \quad \eta^* = \lambda^{\alpha} \eta,$$

where α is a non-zero parameter. This scaling invariance has both analytical and numerical interest. From a numerical viewpoint, as we will see shortly, a non-iterative transformation method (ITM) was defined by Töpfer [39] by transforming the

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doi:10.1016/j.amc.2009.07.019

boundary conditions to initial conditions. Owing to that transformation, a simple existence and uniqueness theorem was given by J. Serrin as reported by Meyer [32, pp. 104–105]. Let us note here that the mentioned invariance property is essential also to the error analysis, of the truncated boundary formulation of the Blasius problem, developed by Rubel [35]. Furthermore, it is possible to prove that the unique solution of the Blasius problem has a positive second order derivative, which is monotone decreasing on $[0,\infty)$ and approaches to zero as η goes to infinity (see Weyl [42]).

Here, we will consider two problems of interest in boundary-layer theory: the flow on a moving surface and slip boundary condition. In both cases the scaling invariance of the prescribed initial conditions will be lost and we have to apply extensions of Töpfer algorithm.

Preliminary results on the topic of this paper were presented at the World Congress on Engineering held in London (July 2–4, 2008) [22].

2. Fluid flow on a flat plate

The model describing the steady plane flow of a fluid past a thin plate, provided the boundary-layer assumptions are verified (the flow has a very thin layer attached to the plate and $v \gg w$), is given by

$$\begin{aligned} &\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ &v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2}, \\ &v(y,0) = w(y,0) = 0, \\ &v(y,z) \rightarrow V_\infty \quad \text{as} \quad z \rightarrow \infty, \end{aligned} \tag{1}$$

where the governing differential equations, namely conservation of mass and momentum, are the steady-state 2D Navier–Stokes equations under the boundary-layer approximations, v and w are the velocity components of the fluid in the y and z direction, V_∞ represents the main-stream velocity, see the draft in Fig. 1, and ν is the viscosity of the fluid. The boundary conditions at $z = 0$ are based on the assumption that neither slip nor mass transfer are permitted at the plate whereas the remaining boundary condition means that the velocity v tends to the main-stream velocity V_∞ asymptotically.

In order to study this problem it is convenient to introduce a potential (stream function) $\psi(y,z)$ defined by

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y}.$$

The physical motivation for introducing this function is that constant ψ lines are steam-lines. The mathematical motivation for introducing such a new variable is that the equation of continuity is satisfied identically, and we have to deal only with the transformed momentum equation. In fact, introducing the stream function the problem can be rewritten as follows

$$\begin{aligned} &\nu\frac{\partial^3 \psi}{\partial z^3} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial y \partial z} = 0, \\ &\frac{\partial \psi}{\partial y}(y,0) = \frac{\partial \psi}{\partial z}(y,0) = 0, \\ &\frac{\partial \psi}{\partial z}(y,z) \rightarrow V_\infty \quad \text{as} \quad z \rightarrow \infty. \end{aligned} \tag{2}$$

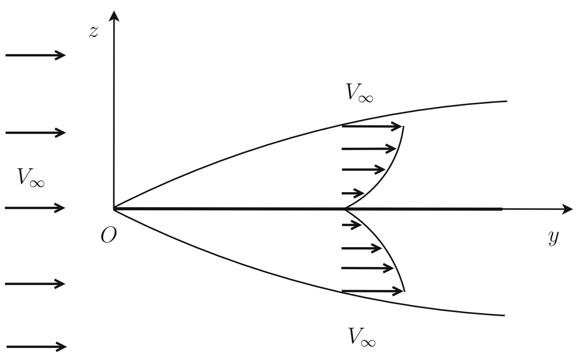

Fig. 1. Boundary layer over a thin plate.

2.1. Blasius problem

Blasius [4] used the following similarity transformation

$$\eta = z\left(\frac{V_\infty}{\nu y}\right)^{1/2}, \quad f(\eta) = \psi(y,z)(\nu y V_\infty)^{-1/2},$$

that reduces the partial differential model (2) to

$$\begin{aligned} & \frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \\ & f(0) = \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta}(\eta) \to 1 \text{ as } \eta \to \infty \end{aligned} \tag{3}$$

i.e., a boundary value problem (BVP) defined on a semi-infinite interval. Blasius solved this BVP by patching a power series to an asymptotic approximation at some finite value of η.

2.2. Töpfer transformation

By considering the derivation of the series expansion solution of the Blasius problem, Töpfer [39] defined a transformation of variables that reduces the Blasius problem into an initial value problem (IVP). However, it is much simpler to consider directly the scaling transformation

$$f^* = \lambda^{-1/3} f, \quad \eta^* = \lambda^{1/3}\eta, \tag{4}$$

and to define a non-iterative transformation method. We notice that the governing differential equation and the initial conditions at the plate in (3) are left invariant by the new variables defined in (4). Moreover, Töpfer used the missed initial condition

$$\frac{d^2 f^*}{d\eta^{*2}}(0) = 1.$$

Under (4) the first and second order derivatives transform in the following way

$$\frac{df^*}{d\eta^*} = \lambda^{-2/3}\frac{df}{d\eta}, \quad \frac{d^2 f^*}{d\eta^{*2}} = \lambda^{-1}\frac{d^2 f}{d\eta^2},$$

and the value of λ can be found on condition that we have an approximation for $\frac{df^*}{d\eta^*}(\infty)$. In fact, by the above relations we get

$$\lambda = \frac{d^2 f}{d\eta^2}(0) = \left[\frac{df^*}{d\eta^*}(\infty)\right]^{-3/2}. \tag{5}$$

Let us list the steps necessary to solve the Blasius problem by the considered approach, we have to:

1. Solve the IVP
$$\begin{aligned} & \frac{d^3 f*}{d\eta^{*3}} + \frac{1}{2} f^* \frac{d^2 f^*}{d\eta^{*2}} = 0, \\ & f^*(0) = \frac{df^*}{d\eta^*}(0) = 0, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1 \end{aligned} \tag{6}$$
   and, in particular, get an approximation for $\frac{df^*}{d\eta^*}(\infty)$;
2. Compute λ by Eq. (5);
3. Obtain $f(\eta)$, and its derivatives, by the inverse transformation of (4).

In this way, we have defined an initial value method for the Blasius problem. In literature such a method is also known as a non-ITM.

2.3. Truncated boundary approximation

From a numerical point of view the request to get $\frac{df^*}{d\eta^*}(\infty)$ is not a simple one. Several strategies have been proposed in order to provide an approximation of this value. Töpfer solved the IVP for the Blasius equation once. At large but finite η^*_j, ordered so that $\eta^*_j < \eta^*_{j+1}$, he computed, by Eq. (5), the corresponding λ_j. The main idea is simple: if two subsequent values

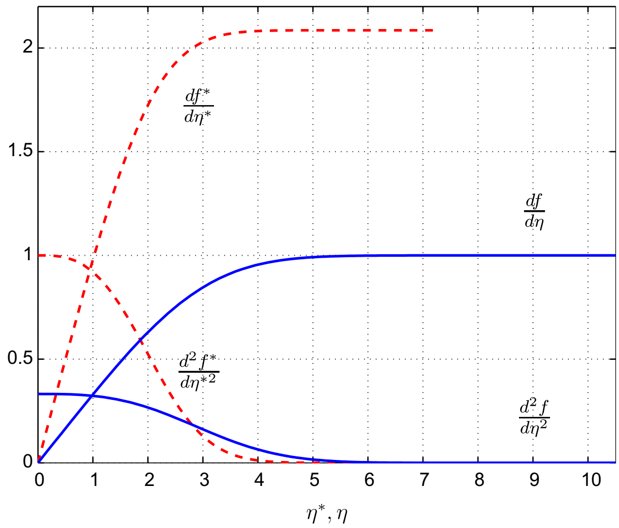

Fig. 2. Blasius solution by a non-ITM.

of λ_j agree within a specified accuracy, then λ is approximately equal to the common value of the λ_j, otherwise, march to a larger value of η and try again.

In a more general setting, the simplest and widely used numerical approach to boundary value problems defined on infinite domains is to introduce a suitable truncated boundary η_∞ instead of infinity. The error analysis of the truncated boundary solution for the Blasius problem is due to Rubel [35], see also [20]. The question on how to set a satisfactory value of η_∞ is not addressed in this work. A recent successful way to deal with such a question is to reformulate the considered problem as a free BVP [15–17]. For instance, as far as the Blasius problem is concerned, we can replace the asymptotic condition with the free boundary conditions

$$\frac{df}{d\eta}(\eta_\epsilon) = 1, \quad \frac{d^2f}{d\eta^2}(\eta_\epsilon) = \epsilon, \tag{7}$$

where η_ϵ is the unknown free boundary and $0 \leqslant \epsilon \ll 1$ is a continuation parameter, see [15] for details. For a recent survey on this topic see [21].

For the sake of simplicity we will not use the free boundary approach here, but following Töpfer, we perform some computational tests in order to find a suitable value for the truncated boundary.

2.4. Numerical results

Fig. 2 shows a sample numerical computation for the ITM defined above. We used a variable step-size classical order-four Runge-Kutta method, implemented in order to maintain a local error of the order of 10^{-6}. Moreover, the calculation were performed in the starred variables with a first time step equal to 0.1 and $\eta^*_\infty = 7.25$. The asymptotic value of interest was found to be

$$\frac{df^*}{d\eta^*}(\infty) \approx 2.085409.$$

This value can be used in Eq. (5) to get

$$\frac{d^2f}{d\eta^2}(0) \approx 0.332057.$$

Blasius solution, displayed on Fig. 2, was found by rescaling.

3. The iterative transformation method

The applicability of a non-ITM to the Blasius problem is a consequence of its partial invariance with respect to the transformation (4); the asymptotic boundary condition is not invariant. Several problems in boundary-layer theory lack this kind of invariance and cannot be solved by non-ITMs. To overcome this drawback, we can modify the problem under study by introducing a numerical parameter h, and require the invariance of the modified problem with respect to an extended scaling group involving h, see [18] for details.

An ITM can be defined as follows:

1. The original BVP is embedded into a modified problem involving the numerical parameter h, so that it is ensured the invariance of the modified problem with respect to an extended scaling group involving h.
2. By starting with suitable values of h_0^* and h_1^* a root-finder method is used to define a sequence h_j^*, for $j = 2, 3, \ldots$. At each iteration the group parameter λ is obtained by solving an IVP numerically. A related sequence $\Gamma(h_j^*)$, for $j = 0, 1, 2, \ldots$, can be defined by the equation

$$\Gamma(h^*) = h - 1, \tag{8}$$

   where $\Gamma(\cdot)$ is defined implicitly by the solution of an IVP written in the starred variables and as a consequence $h = h(h^*)$.
3. Suitable termination criteria have to be used to verify whether $\Gamma(h_j^*) \to 0$ as $.j \to \infty$.
4. The solution of the original problem can be obtained by rescaling to $h = 1$.

By defining an ITM the existence and uniqueness question can be reduced to finding the number of real zeros of the transformation function $\Gamma(\cdot)$. This result can be stated as follows.

Theorem 1. *Let us assume that IVPs used to define the transformation function are well posed. Then, the considered BVP has a unique solution if and only if the transformation function has a unique real zero; nonexistence (nonuniqueness) of the solution is equivalent to nonexistence of real zeros (existence of more than one real zero) of* $\Gamma(\cdot)$.

The underlying idea of the proof of this theorem is that there exists a one-to-one and onto correspondence between the set of solutions of the BVP and the set of real zeros of the transformation function, see [18]. This theorem is applied in the next section.

4. Variants of the Blasius problem

In this section we report on extensions of the Blasius problem and the related numerical approximation. The results reported in this section were found by the **ODE113** solver, from the MATLAB ODE suite written by Samphine and Reichelt [37], with the accuracy and adaptivity parameters defined by default.

4.1. Moving surfaces

Klemp and Acrivos [28] were the first to define the similarity model of a boundary-layer problem over moving surfaces. For this model the Blasius equation has to be considered along with the usual asymptotic boundary condition at infinity, and the following non-homogeneous boundary conditions at $\eta = 0$

$$f(0) = 0, \quad \frac{df}{d\eta}(0) = -P, \tag{9}$$

where P is the ratio of the plate velocity to the free stream velocity. Klemp and Acrivos studied the effect of the parameter P on the boundary-layer thickness. For $P > 0$, two solutions exist only for P less than a critical value P_c, as shown numerically by Hussaini and Lakin [26]. These authors found a numerical value of P_c equal to 0.3541. Hussaini et al. [27] proved the non-uniqueness and analyticity of solutions for $P \leqslant P_c$, and derived the upper bound 0.46824 for P_c.

More recently, a modified Blasius equation, taking into account the effect of P on the boundary-layer thickness, has been introduced by Allan [1]. Moreover, Allan and Syam [2], using an homotopy analysis method, defined an implicit relation between the wall shear stress and the moving wall parameter. The study of these relation shows that two solutions exist when $P \leqslant P_c \approx 0.354\ldots$, one solution exists for $P = P_c$ and no solution exists for $P > P_c$.

We have used the ITM in order to investigate the existence and uniqueness question for the Blasius model on a moving plate. For the modified problem we defined the boundary condition

$$\frac{df}{d\eta}(0) = -h\,P$$

and used the extended scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^{-1}\eta, \quad h^* = \lambda^2 h. \tag{10}$$

so that λ is defined by

$$\lambda = \left[\frac{df^*}{d\eta^*}(\infty)\right]^{1/2}. \tag{11}$$

Let us discuss here three specific test cases. First, we consider the case $P = 0.25$, and we report, in Table 1, the related numerical results found by the ITM. In this case $\Gamma(\cdot)$ has two different zeros. Fig. 3 shows the two corresponding solutions.

Table 1
Fluid flow on a moving plate: numerical results by the ITM.

h^*	$\Gamma(h^*)$	h^*	$\Gamma(h^*)$
1	−0.528765	10	−0.008382
3	0.198514	3	0.198514
2.454091	0.045354	9.716410	0.016266
2.339221	0.008091	9.903562	-1.52×10^{-4}
2.319037	0.001371	9.901831	-2.96×10^{-6}
2.315626	2.21×10^{-4}	9.901798	-5.79×10^{-8}
2.315076	3.87×10^{-5}	–	–
2.314979	6.78×10^{-6}	–	–
2.314963	1.19×10^{-6}	–	–
2.314960	2.08×10^{-7}	–	–
$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$
−0.25	0.283928	−0.25	0.032094

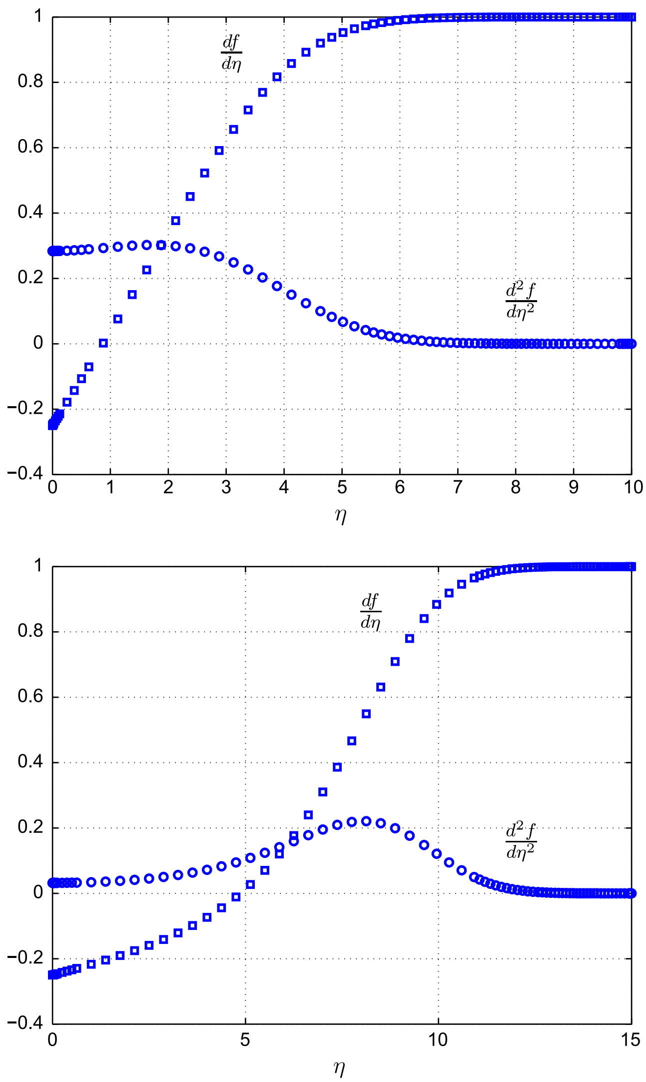

Fig. 3. Blasius problem with moving plate boundary conditions. The two different solutions for $P = 0.25$.

It is evident, from the two frames of this figure, that the truncated boundary approach has to be supplemented by some numerical experiments, and this is more relevant in the case of nonuniqueness of solution. In fact, by setting $\eta_\infty = 10$, we miss the solution shown in the bottom frame of Fig. 3.

As a second test case, by setting $P = 0.4$ we find that $\Gamma(\cdot)$ has always the same negative sign, so that no solution is available for such a case. Finally, by considering the case $P = P_c = 0.3541$ the ITM provided $\frac{d^2f}{d\eta^2}(0) \approx 0.148850$. The results obtained by the ITM in all the above cases are in agreement with the results by Hussaini and Lakin [26] and Allan and Syam [2]. In particular, the behaviour of $\Gamma(\cdot)$ in the second test case shows that the upper bound for P_c found by Hussaini et al. [27] is quite inaccurate.

For the ITM we used the Regula Falsi method as a root-finder, bracketing out the zeros of the transformation function $\Gamma(\cdot)$, along with a convergence criterion given by the inequality $|\Gamma(\cdot)| < 10^{-6}$.

4.2. Slip flow condition

We consider now the case of a rarefied flow where the no-slip condition at the wall, considered in the previous section, must be replaced by a slip-flow condition, see for instance Gad-el-Hak [12]. For an isothermal wall, the slip condition can be defined as

$$v(y,0) = \frac{2-\sigma}{\sigma}\ell\frac{\partial v}{\partial z}(y,0),$$

where ℓ is the mean free path, and σ is the tangential momentum accommodation coefficient. Within a similarity transformation this slip boundary condition becomes

$$\frac{df}{d\eta}(0) = P\,\frac{d^2f}{d\eta^2}(0),$$

where P is a non-dimensional parameter, that takes into account the behaviour at the surface, defined by

$$P = \frac{2-\sigma}{\sigma} Kn\, Re\, y^{1/2},$$

where Kn and Re are the Knudsen and Reynolds numbers based on y.

For the Blasius problem with slip condition we implemented both an extension of the non-ITM and the ITM. In order to apply the non-ITM we consider P as a parameter involved in the scaling invariance, i.e., we defined the extended scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^{-1}\eta, \quad P^* = \lambda^{-1}P. \tag{12}$$

As far as the application of the ITM is concerned, we used a modified problem with the boundary condition

$$\frac{df}{d\eta}(0) = h\,P\,\frac{d^2f}{d\eta^2}(0),$$

and the extended scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^{-1}\eta, \quad h^* = \lambda^{-1}h. \tag{13}$$

Henceforth, in both cases λ is defined, once again, by Eq. (11).

Sample numerical results are reported in Tables 2 and 3. The results listed in the last two columns of Tables 2 and 3 can be compared with similar results, obtained via a shooting method, shown in Fig. 1 of the proceedings report by Martin and Boyd [31]. It is clear that our non-ITM would be faster and easier to implement than any iterative algorithm.

As far as the non-ITM is concerned, we set a value of P^* and get the numerical solution of the problem for a different value of P. As an example, Fig. 4 shows a sample numerical integration for $P = 1.562257$. Note that the solution of the Blasius prob-

Table 2
Slip boundary condition: non-iterative numerical results.

P^*	$\frac{df^*}{d\eta^*}(\infty)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	P
0	2.085393	0	0.332061	0
0.1	2.090453	0.047836	0.330856	0.144584
0.5	2.191907	0.228112	0.308153	0.740255
1	2.440648	0.409727	0.262266	1.562257
5	5.771518	0.866323	0.072122	12.011992
10	10.554805	0.947436	0.029162	32.488159
20	20.394883	0.980638	0.010857	90.321389
25	25.353618	0.986053	0.007833	125.880941

Table 3
Slip boundary condition: numerical results by the ITM.

$\frac{df^*}{d\eta^*}(\infty)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	P
2.085393	0	0.332061	0
2.087710	0.033151	0.331509	0.1
2.262516	0.293841	0.293841	1
3.644351	0.718686	0.143737	5
5.203210	0.842545	0.084255	10
13.894469	0.965399	0.019308	50

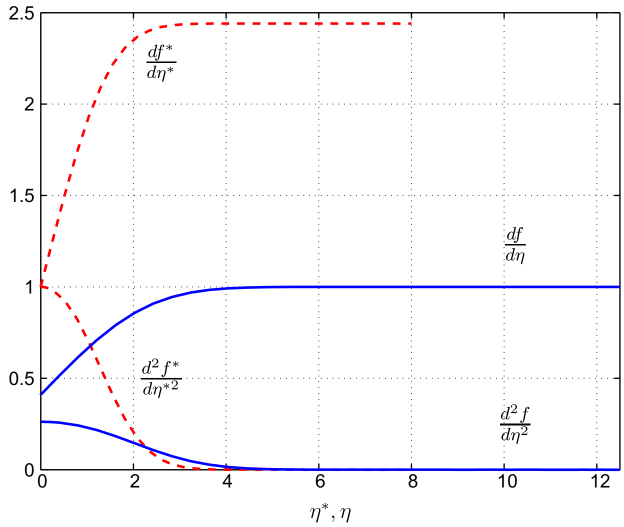

Fig. 4. Blasius problem with slip condition. Numerical solution by a non-ITM with $P^* = 1$ and $P = 1.562257$.

lem with slip boundary condition was computed by rescaling. If we need the solution for a specific value of P, then we can apply interpolation techniques to the results of Table 2 or we can use the ITM as was done for all the values of P listed in Table 3. For the ITM, we always used $h_0^* = 0.1$ and $h_1^* = 1$ but for the case $P = 50$ where, in order to speed up the convergence, we set $h_1^* = 0.5$. For the sake of brevity, we omit to report the iterations related to the results listed in Table 3. However, by setting again $|\Gamma(\cdot)| < 10^{-6}$, as a convergence criterion, the Regula Falsi method converged within 8 iterations in all cases.

5. Conclusions

In this paper, we have shown how the original treatment of the Blasius problem due to Töpfer can be extended to more complex problems of boundary-layer theory. As pointed out by NA [33, Chapters 7–9], usually a given, even simple, extension of the Blasius problem cannot be solved by Töpfer algorithm. Therefore, in order to extend the applicability of this non-ITM an iterative version has been developed in [16–19]. Here, we focused our attention to two problems of relevant interest: moving surfaces and slip boundary condition. For moving surfaces the ITM was able to deal with multiple solutions, whereas in the case of slip condition both an extended non-ITM and the ITM provided reliable numerical results, cf. Fig. 1 in [31].

The ideas outlined in this paper can be applied to other problems of boundary-layer theory as well. As an example, let us consider the Falkner–Skan equation with relevant boundary conditions:

$$\begin{aligned} &\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0 \\ &f(0) = \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta}(\infty) = 1, \end{aligned} \tag{14}$$

where f and η are appropriate similarity variables and β is a parameter. This problem describes the flow of a fluid past a wedge, see Falkner and Skan [13]. The application of the ITM to (14) has been reported in [16] but only in the simple cases where $\beta = 1/2$ or $\beta = 1$. It is well known that the case $\beta > 1$ is more interesting, because the Falkner–Skan model loses the uniqueness property and a hierarchy of solution with reversed flow exists. In fact, for $\beta > 1$ Craven and Peletier [11] have calculated solutions for which $\frac{df}{d\eta} < 0$ for some value of η.

The existence and uniqueness question for the problem (14) is really a complex matter. Assuming that $\beta > 0$, under the restriction $0 < \frac{df}{d\eta} < 1$, it has been proved by Hartree [23] and Stewartson [38] that the problem (14) has an unique solution, whose first derivative tends to one exponentially. Coppel [8] and Craven and Peletier [10] have proved that the above restriction on the first derivative can be omitted when $0 \leqslant \beta \leqslant 1$. The case when $\beta < 0$ is more complicated, but its treatment here will be out of scope; the interested reader may have a look at the paper of Veldman and van de Vooren [40].

Acknowledgements

This work was supported by the University of Messina and partially by the Italian MUR.

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