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TWO FINITE DIFFERENCE METHODS FOR A NONLINEAR BVP ARISING IN PHYSICAL OCEANOGRAPHY

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ABSTRACT. In this paper we define two finite difference methods for a nonlinear boundary value problem on infinite interval. In particular, we report and compare the numerical results for an ocean circulation model obtained by the free boundary approach and a treatment of the problem on the original semi-infinite domain by introducing a quasi-uniform grid. In the first case we apply finite difference formulae on a uniform grid and in the second case we use non-standard finite differences on a quasi-uniform grid. We point out how both approaches represent reliable ways to solve boundary value problems defined on semi-infinite intervals. In fact, both approaches overcome the need to define *a priori*, or find by trials, a suitable truncated boundary used by the classical numerical treatment of boundary value problems defined on a semi-infinite interval. Finally, the reported numerical results allow to point out how the finite difference method with a quasi-uniform grid is the least demanding approach between the two and that the free boundary approach provides a more reliable formulation than the classical truncated boundary one.

1. Introduction

Boundary value problems (BVPs) on infinite intervals arise in several branches of science. The classical numerical treatment of these problems consists in replacing the original problem by one defined on a finite interval, say $[0, x_\infty]$, where x_∞ is a truncated boundary. The oldest and simplest approach is to replace the boundary conditions at infinity by the same conditions at the value chosen as the truncated boundary. This approach was used, for instance, by Howarth (1938) to get and tabulate the numerical solution of the Blasius problem, see also Goldstein (1938, p. 136). However, in order to achieve an accurate solution, a comparison of numerical results obtained for several values of the truncated boundaries is necessary as suggested by Fox (1957, p. 92), or by Collatz (1960, pp.150-151). Moreover, in some cases accurate solutions can be found only by using very large values of the truncated boundary. This is, for instance, the case of the fourth branch of the von Karman swirling flows, where values of x_∞ up to 200 were used by Lentini and Keller (1980b).

To overcome the mentioned difficulties of the classical approach described above, Lentini and Keller (1980a) and de Hoog and Weiss (1980) suggested to apply asymptotic boundary conditions (ABCs) at the truncated boundaries; see also the theoretical works of Markowich (1982, 1983). Those ABCs have to be derived by a preliminary asymptotic analysis involving the Jacobian matrix of the right-hand side of the governing differential equations evaluated at infinity. The main idea of this ABCs approach is to project the solution into the manifold of bounded solutions. By this approach more accurate numerical solutions can be found than those obtained by the classical approach with the same values of the truncated boundaries, because the imposed conditions are obtained from the asymptotic behavior of the solution. However, we should note that for nonlinear problems highly nonlinear ABCs may result. Moreover, it has been noticed by Ockendon that "Unfortunately the analysis is heavy and relies on much previous work, . . . " see *Math. Rev.* 84c:34201. On the other hand, starting with the work by Beyn (1990a,b, 1991), the ABCs approach has been applied successfully to "connecting orbits" problems. Connecting orbits are of interest in the study of dynamical systems as well as of traveling wave solutions of partial differential equations of parabolic type. However, a truncated boundary allowing for a satisfactory accuracy of the numerical solution has to be determined by trial, and this seems to be the weakest point of the classical approach. Hence, *a priori* definition of the truncated boundary was indicated by Lentini and Keller (1980a) as an important area of research.

For the numerical solution of BVPs on unbounded domains it is also possible to consider spectral methods that use mapped Jacobi, Laguerre and Hermite functions, see the book by Boyd (2001), or the review by Shen and Wang (2009), or the paper by Liu and Zhu (2015) for more details on this topic.

A free boundary formulation for the numerical solution of BVPs on infinite intervals was proposed by Fazio (1996). In this approach the truncated boundary can be identified as an unknown free boundary that has to be determined as part of the solution. As a consequence, the free boundary approach overcomes the need for *a priori* definition of the truncated boundary. This new approach has been applied to: the Blasius problem by Fazio (1992), a two-dimensional stagnation point flow by Ariel (1993), the Falkner-Skan equation with relevant boundary conditions by Fazio (1994), Zhang and Chen (2009) and Zhu *et al.* (2009), a model describing the flow of an incompressible fluid over a slender parabola of revolution by Fazio (1996), a model describing the deflection of a semi-infinite pile embedded in soft soil by Fazio (2003), and the Thomas-Fermi equation by Zhu *et al.* (2012). An application of the free boundary approach to a homoclinic orbit problem can be found in (Fazio 2002). Moreover, a possible way to extend the free boundary formulation to problems governed by parabolic partial differential equations is the main topic in (Fazio and Iacono 2010).

It might seem that in order to face numerically a BVP defined on an infinite interval, we have to reformulate it in a way or another. However, recently, we have found that it is also possible to apply directly to the given BVP a non-standard finite difference method defined on a quasi-uniform grid. To this end it is necessary to derive special finite difference formulae on the grid involving the given boundary conditions at infinity, but the last grid point value (infinity) is not required; see Fazio and Jannelli (2014, 2017) and Fazio *et al.* (2018) for details. The quasi-uniform grid can be defined by the coordinate transform approach used, for ordinary and partial differential equations, for instance by Grosch and Orszag (1977) and Koleva (2006), see also the books of Boyd (2001, pp. 325-326) or Canuto *et al.* (2006, p. 96).

In this paper, for an ocean circulation model, we report a comparison of numerical results obtained by the free boundary approach with a finite difference method, and the ones obtained by a non-standard finite difference method with a quasi-uniform grid.

2. The physical model

A steady-state wind-driven ocean circulation model can be introduced by considering the barotropic vorticity equation

$$J(\psi, y+\gamma\nabla^2\psi) = \kappa\gamma\nabla^4\psi - \cos\left(\frac{\pi y}{2}\right) , \tag{1}$$

in a region defined by $x \in [-1,1]$ and $y \in [-1,1]$ with the following boundary conditions

$$\psi(\pm 1, y) = 0 , \qquad \psi(x, \pm 1) = 0 , \tag{2}$$

and either

$$\frac{\partial \psi}{\partial x}(\pm 1, y) = 0 , \qquad \frac{\partial \psi}{\partial y}(x, \pm 1) = 0 , \tag{3}$$

known as "rigid" or no-slippery boundary conditions, or

$$\frac{\partial^2 \psi}{\partial x^2}(\pm 1, y) = 0 , \qquad \frac{\partial^2 \psi}{\partial y^2}(x, \pm 1) = 0 , \tag{4}$$

known as "slippery" or stress-free boundary conditions. Eq. (1) is written in a non-dimensional form; $\psi(x,y)$ is the stream function; a reference coordinate system is fixed with the x axis directed to the east and the y axis directed to the north; $J(a,b)$ is the Jacobian of the functions a and b with respect to x and y, ∇^2 is the Laplacian operator on the (x,y) plane; the square $[-1,1]\times[-1,1]$ models a region of the subtropical gyre formation (Ierley and Ruehr 1986). Here the Jacobian represents nonlinear advection and the Laplacian the viscous drag. We assume that the curl of the wind stress in the region can be approximated by $-\cos\left(\frac{\pi y}{2}\right)$; γ and κ are non-dimensional parameters characterizing the widths of inertial and viscous boundary layers, respectively. We use impermeability and no-slip conditions (3) at the coasts and impermeability and slippery conditions (4) at the fluid boundaries. We consider a particular solution to (1) of the form

$$\psi = \pi(y+1)u(x) . \tag{5}$$

Relation (5) represents the first term in the expansion of a solution of (1) with respect to y near boundaries of the region: at $y=-1$ and at $y=1$. Substituting (5) into (1), using a Taylor series expansion near $y=-1$ of the wind-stress term and assuming that a steady boundary-layer type solution exists, we obtain the equation for the boundary layer at the western coast, *i.e.*, at $x=-1$,

$$\kappa\gamma\frac{d^4u}{dx^4} = \pi\gamma\left(\frac{du}{dx}\frac{d^2u}{dx^2} - u\frac{d^3u}{dx^3}\right) + \frac{du}{dx} , \qquad x \in [0,\infty) . \tag{6}$$

The parameters involved can be reduced to one if we define

$$b = \pi\left(\frac{\gamma}{\kappa^2}\right)^{1/3} ,$$

and introduce the new independent variable

$$\xi = \frac{x}{(\kappa\gamma)^{1/3}} \ .$$

In the above physical context, the limit of vanishing viscosity (small values of κ) is of particular interest. Indeed, the parameter γ is also small, of the order of 10^{-3}. Therefore, in terms of the new independent variable ξ, far from the boundaries for asymptotically matching the interior solution ψ_I, taken of the following form

$$\psi_I \approx (1-x)\cos\left(\frac{\pi y}{2}\right) \ ,$$

we have to require that

$$u(x) \to 1 \quad \text{as} \quad x \to \infty \ .$$

The fourth order ordinary differential equation (6) can be integrated once, using zero boundary conditions at infinity for the second and third derivative of $u(\xi)$, to give

$$\frac{d^3u}{d\xi^3} = b\left[\left(\frac{du}{d\xi}\right)^2 - u\frac{d^2u}{d\xi^2}\right] + u - 1 \ , \qquad \xi \in [0,\infty) \ . \tag{7}$$

The boundary conditions follow from (2)-(4). In particular, we can have no-slip (or rigid) boundary data

$$u(0) = \frac{du}{d\xi}(0) = 0, \qquad u(\xi) \to 1 \quad \text{as} \quad \xi \to \infty \ , \tag{8}$$

or stress-free (or slippery) boundary conditions

$$u(0) = \frac{d^2u}{d\xi^2}(0) = 0, \qquad u(\xi) \to 1 \quad \text{as} \quad \xi \to \infty \ . \tag{9}$$

Therefore, we get the two point BVP defined on an unbounded domain that has been investigated by Ierley and Ruehr (1986), Mallier (1994), and Sheremet *et al.* (1997).

The parameter b in (7) can be used as a measure of the strength of the nonlinearity. In fact, for $b = 0$ we get the simple linear model formulated by Munk (1950). Ierley and Ruehr (1986) discovered an analytical approximation for the relation between the missing initial condition and the parameter b. In particular, for rigid conditions they found the relation

$$\left(\frac{d^2u}{d\xi^2}(0)\right)^2 \approx \frac{2}{1 \pm \left(1+\frac{4}{3}b\right)^{1/2}} \ , \tag{10}$$

where we are advised to take the positive root for $b > 0$, and for slippery conditions they reported the approximation

$$\frac{du}{d\xi}(0) \approx \frac{2}{1 \pm \left(1+\frac{10}{3}b\right)^{1/2}} \ . \tag{11}$$

Following the results (10) and (11), Ierley and Ruehr discussed the existence and uniqueness question for the problem (7)-(8) or (7)-(9). For the problem (7)-(8) we have a multiplicity of solutions similar to the one of the Falkner-Skan model (Fazio 2013), and, in particular: for $0 < b$ there is exactly one solution, two solutions exits for $b_c < b < 0$ where $b_c \approx -0.79130$

is a negative critical value, whereas for $b \leq b_c$ no solutions exist at all. On the other hand, for the problem (7)-(9) two solutions can be found for b greater than a negative value $b_c \approx -0.29657$ and no solutions exist for $b \leq b_c$. Of course, the approximations provided by (10) and (11) can be used for comparison with the corresponding numerical results.

3. Numerical methods

In this section we present the numerical methods used in order to solve the ocean model (7). As a first step we rewrite the ocean equation in (7) as a first order system

$$\begin{aligned} &\frac{d\mathbf{u}}{d\xi} = \mathbf{f}(\xi, \mathbf{u}) \ , \quad \xi \in [0, \infty) \ , \\ &\mathbf{g}(\mathbf{u}(0), \mathbf{u}(\infty)) = \mathbf{0} \ , \end{aligned} \tag{12}$$

by setting

$$u_{i+1}(\xi) = \frac{d^i u}{d\xi^i}(\xi) \ , \quad \text{for } i = 0, 1, 2 \ .$$

In this way the original BVP (7) specializes into

$$\begin{aligned} \frac{du_1}{d\xi} &= u_2 \\ \frac{du_2}{d\xi} &= u_3 \\ \frac{du_3}{d\xi} &= b(u_2^2 - u_1 u_3) + u_1 - 1 \ , \end{aligned} \tag{13}$$

that is,

$$\begin{gathered} \mathbf{u} = (u_1, u_2, u_3)^T \\ \mathbf{f}(\xi, \mathbf{u}) = \left(u_2, u_3, b(u_2^2 - u_1 u_3) + u_1 - 1\right)^T \end{gathered}$$

with

$$\mathbf{g}(\mathbf{u}(0), \mathbf{u}(\infty)) = (u_1(0), u_2(0), u_1(\infty) - 1)^T$$

or

$$\mathbf{g}(\mathbf{u}(0), \mathbf{u}(\infty)) = (u_1(0), u_3(0), u_1(\infty) - 1)^T$$

in (12). In the following, in order to set a specific test problem, we consider the ocean model with $b = 2$.

3.1. The free boundary formulation and a relaxation method. In order to introduce a free boundary formulation for our problem, we replace the far boundary condition by two boundary conditions at the free boundary ξ_ε

$$u(\xi_\varepsilon) = 1 \ , \qquad \frac{du}{dx}(\xi_\varepsilon) = \varepsilon \ , \tag{14}$$

where ξ_ε can be considered as a truncated boundary. Then we rewrite the resulting free BVP in standard form (Ascher and Russell 1981), defining $u_4 = \xi_\varepsilon$ and using the new independent variable

$$z = \frac{\xi}{u_4} \; . \tag{15}$$

In general, we end up with a BVP belonging to the general class:

$$\begin{aligned} &\frac{d\mathbf{U}}{dz} = \mathbf{F}\left(z, \mathbf{U}\right) \; , \quad z \in [0,1] \; , \\ &\mathbf{G}\left(\mathbf{U}(0), \mathbf{U}(1)\right) = \mathbf{0} \; , \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbf{U}(z) &\equiv (\mathbf{u}(z), u_4)^T \; , \\ \mathbf{F}(z,\mathbf{U}) &\equiv (u_4 \mathbf{f}(u_4 z, \mathbf{u}), 0)^T \; , \\ \mathbf{G}(\mathbf{U}(0), \mathbf{U}(1)) &\equiv (\mathbf{g}(\mathbf{u}(0), \mathbf{u}(1)), h(\mathbf{u}(1)))^T \; , \end{aligned} \tag{17}$$

where, in our case, $h(\mathbf{u}(1)) = u_2(1) - \varepsilon$. In order to simplify notation in (16), (17) and in the following, we omitted the dependence of $\mathbf{u}$ and $\mathbf{U}$ on ε.

In order to solve the resulting problem we apply a relaxation method. Let us introduce a mesh of points $z_0 = 0$, $z_j = j\Delta z$, for $j = 1, 2, \ldots, J$, of uniform spacing Δz and naturally $z_J = 1$. We denote by the $4-$dimensional vector $\mathbf{V}_j$ the numerical approximation to the solution $\mathbf{U}(z_j)$ of (16) at the points of the mesh, that is for $j = 0, 1, \ldots, J$. Keller's box scheme for (16) can be written as follows:

$$\begin{aligned} \mathbf{V}_j - \mathbf{V}_{j-1} - \Delta z \mathbf{F}\left(z_{j-1/2}, \frac{\mathbf{V}_j + \mathbf{V}_{j-1}}{2}\right) = \mathbf{0} \; , \quad \text{for} \quad j = 1, 2, \ldots, J \\ \mathbf{G}\left(\mathbf{V}_0, \mathbf{V}_J\right) = \mathbf{0} \; , \end{aligned} \tag{18}$$

where $z_{j-1/2} = (z_j + z_{j-1})/2$. It is evident that (18) is a nonlinear system with respect to the unknown $4(J+1)-$dimensional vector $\mathbf{V} = (\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_J)^T$. Following Keller, the classical Newton's method, along with a suitable termination criterion, is applied to solve (18).

Let us recall now the main properties of the box scheme proved by Keller in the main theorem in (Keller 1974). Under the assumption that $\mathbf{U}(z)$ and $\mathbf{F}(z, \mathbf{U})$ are sufficiently smooth, each isolated solution of (16) is approximated by a difference solution of (18) which can be computed by Newton's method, provided that a sufficiently fine mesh and an accurate initial guess for the Newton's method are used. As far as the accuracy issue is concerned, the truncation error has an asymptotic expansion in powers of $(\Delta z)^2$.

For the Newton's method the simple termination criterion

$$\frac{1}{4(J+1)} \sum_{\ell=1}^{4} \sum_{j=0}^{J} |\Delta V_{j\ell}| \leq \text{TOL} \; , \tag{19}$$

where $\Delta V_{j\ell}$, $j = 0, 1, \ldots, J$ and $\ell = 1, 2, 3, 4$, is the difference between two successive iterate components and TOL is a fixed tolerance. The key point for the numerical solution of the nonlinear system is that Newton's method converges only locally. Therefore, some preliminary numerical experiments may be helpful and worth of consideration. However, for the results reported, the initial guess to start the iterations is as follows

$$u_1(z) = 1\ , \quad u_2(z) = 0.1\ , \quad u_3(z) = 0.1\ , \quad u_4(z) = 1 \tag{20}$$

for the BVP (7) with no-slip conditions (8) and with slippery conditions (9),

$$u_1(z) = -1\ , \quad u_2(z) = -0.5\ , \quad u_3(z) = 1\ , \quad u_4(z) = 2 \tag{21}$$

for the BVP (7) with slippery conditions (9). In Tables 1, 2 and 3 and in Figures 1, 2 and 3, we report some of the numerical results, obtained with the free boundary approach, related to different values of ε and obtained by setting $J = 2000$ and $\mathrm{TOL} = 1\mathrm{E}-6$. Here and in the following $1\mathrm{E}-k$ is the standard notation for 10^{-k} in simple precision arithmetic.

TABLE 1. Free boundary formulation for the BVP (7)-(8) and initial guess (20). The positive root of Eq. (10) gives $\frac{d^2u}{d\xi^2}(0) = 0.828336$.

ε	iter	ξ_ε	$\dfrac{d^2u}{d\xi^2}(0)$
1E−2	6	6.485761	0.826184
1E−3	8	8.792991	0.826141
1E−4	9	11.098635	0.826141
1E−5	10	13.402219	0.826142

TABLE 2. Free boundary formulation for the BVP (7)-(9) and initial guess (20). Equation (11) gives $\frac{du}{d\xi}(0) = 0.530662$.

ε	iter	ξ_ε	$\dfrac{du}{d\xi}(0)$
1E−2	6	5.828307	0.528970
1E−3	7	8.132813	0.528922
1E−4	9	10.437875	0.528921
1E−5	10	12.741323	0.528921

We consider the free boundary formulation more effective than the simple truncated boundary approach. In fact, under suitable hypothesis, it is possible to prove the convergence of the free boundary solution to the solution of the original BVP. Fazio (1996) proves that as ε goes to zero the solution of the free boundary formulation converges to the solution of the original problem and the free boundary ξ_ε goes to infinity. The obtained approximation becomes the more accurate the more ε is near zero. Then, since we know that the free boundary increases as ε goes to zero, we can consider ε as a continuation parameter. This means that, in order to reduce the iterations number of the Newton's method, the results obtained for a value of ε can be used as the initial guess for the next value of ε. Adopting this criterion, we start by setting as initial guesses given by (20) and (21) for $\varepsilon = 1E-2$

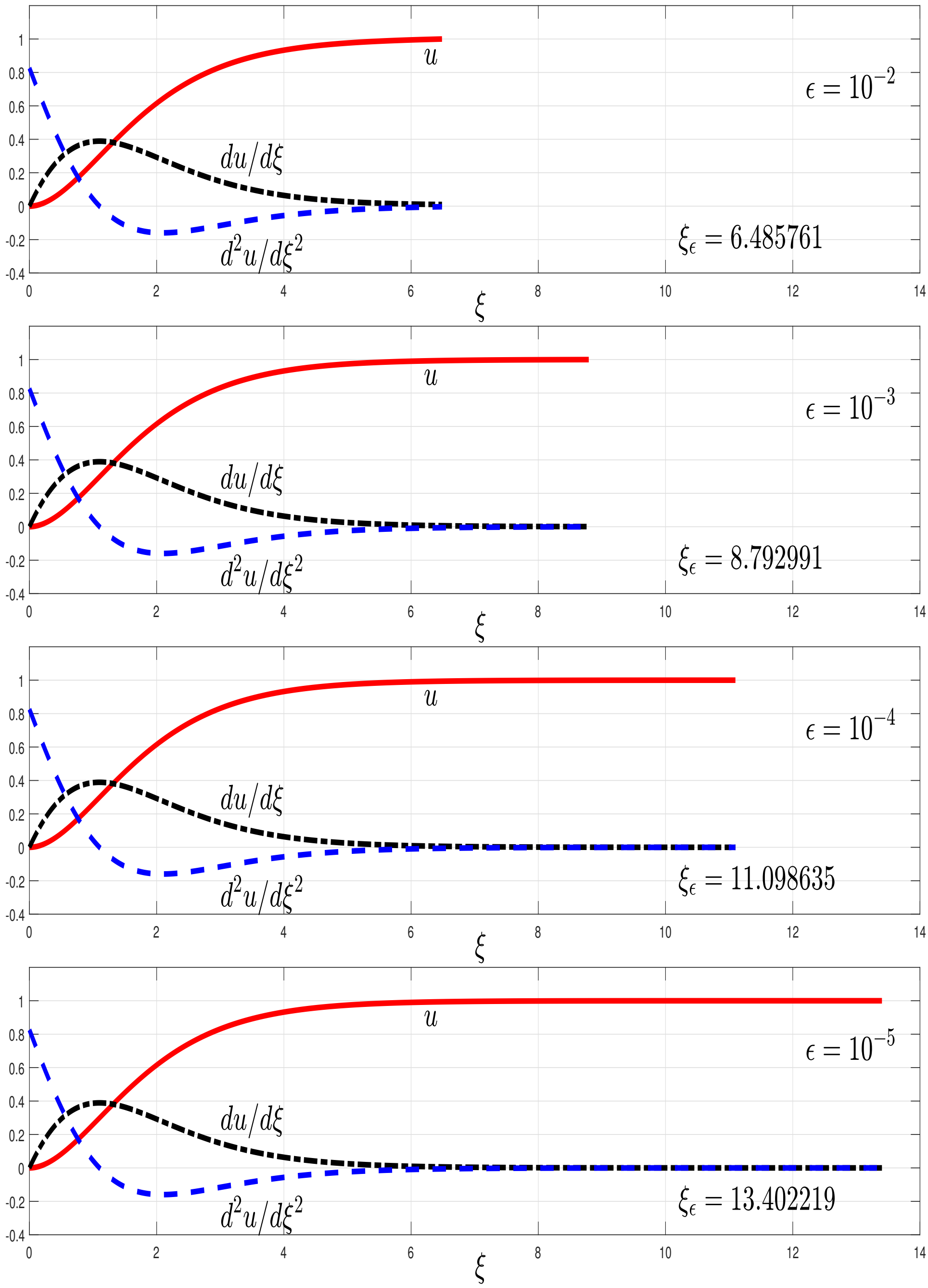

FIGURE 1. Numerical solution of the BVP (7) with no-slip conditions (8) by the free boundary approach and initial guess (20).

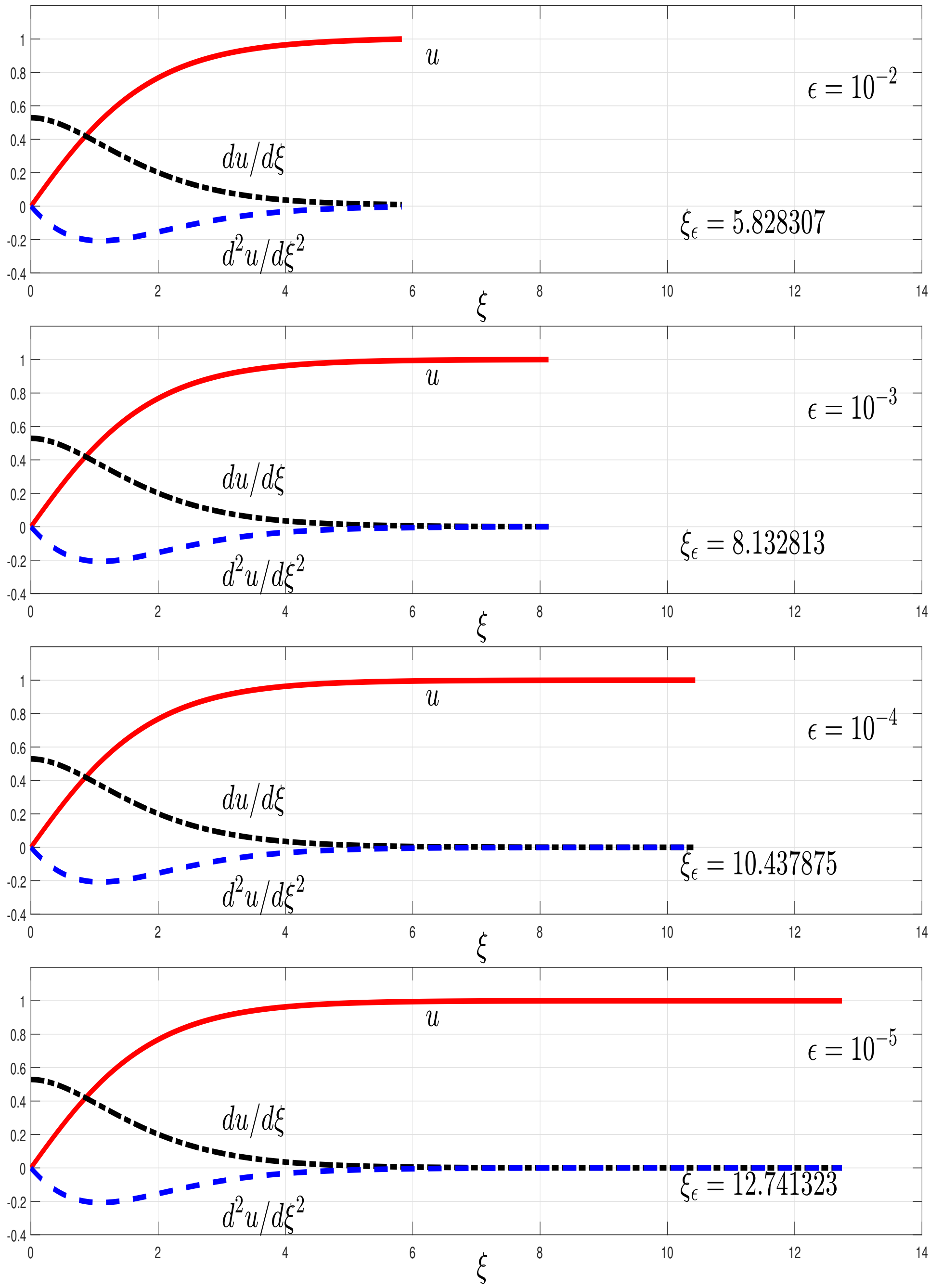

FIGURE 2. Numerical solution of the BVP (7) with slippery conditions (9) by the free boundary approach and initial guess (20).

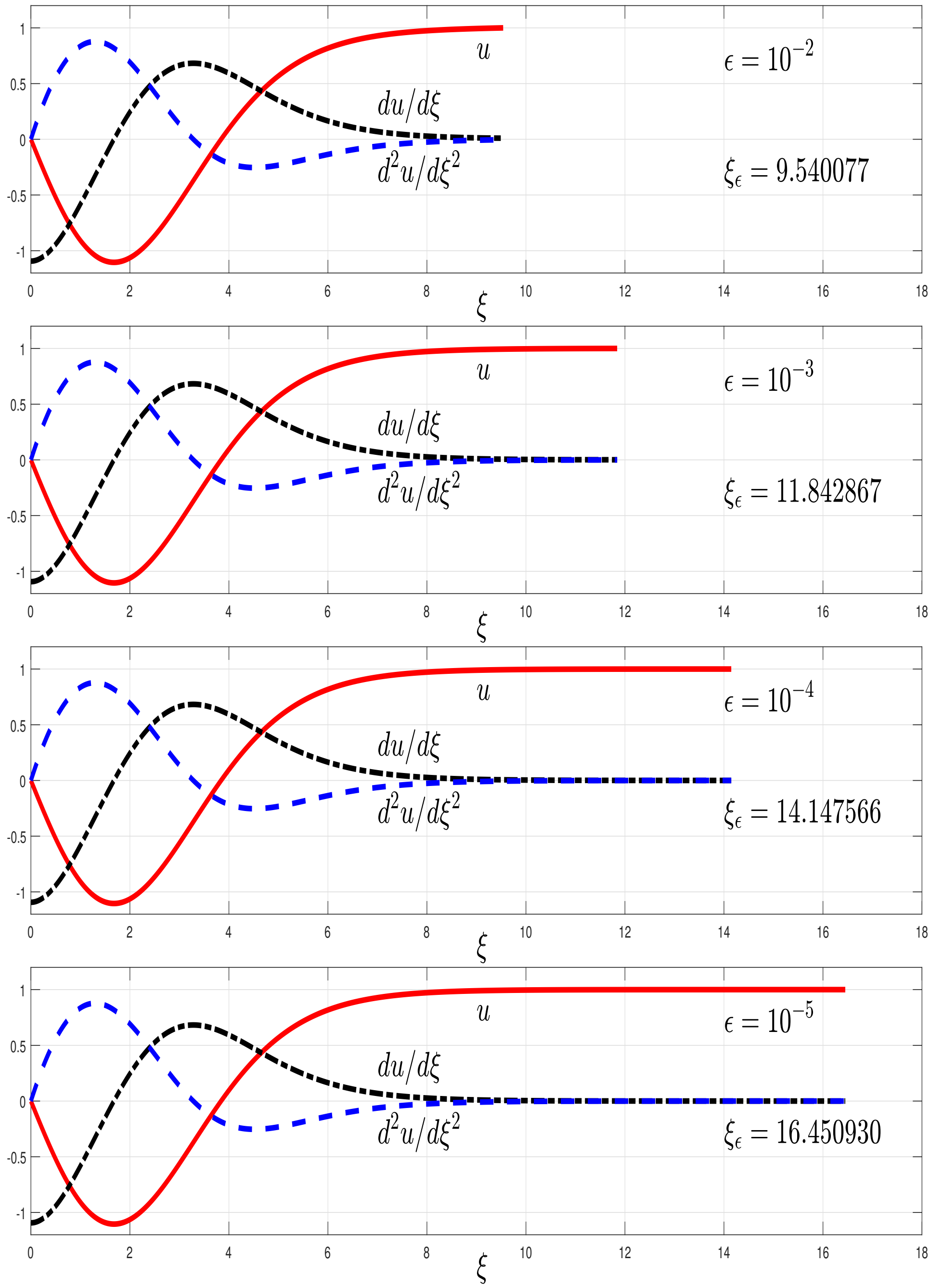

FIGURE 3. Numerical solution of the BVP (7) with slippery conditions (9) by the free boundary approach and initial guess (21).

TABLE 3. Free boundary formulation for the BVP (7)-(9) and initial guess (21). Equation (11) gives $\frac{du}{d\xi}(0) = -1.130662$.

ε	iter	ξ_ε	$\frac{du}{d\xi}(0)$
1E−2	9	9.540077	−1.093016
1E−3	10	11.842867	−1.093018
1E−4	11	14.147566	−1.093019
1E−5	13	16.450930	−1.093021

and by using the results as initial guesses for $\varepsilon = 1E-3$, proceeding in this way for the next values of ε, the number of iterations of the relaxation method is reduced to 6,6,6,6 for the results reported in Tables 1-2 and to 9,6,6,6 for those of Table 3.

3.2. Finite difference method on a quasi-uniform grid. Let us consider the smooth strict monotone quasi-uniform map $\xi = \xi(\eta)$, the so-called grid generating function,

$$\xi = -p \cdot \ln(1-\eta) \ , \tag{22}$$

where $\eta \in [0,1]$, $\xi \in [0,\infty]$, and $p > 0$ is a control parameter. We notice that more than half of the intervals are in the domain with length approximately equal to p and $\xi_{J-1} = p \ln J$. Moreover, the mesh in ξ is nonuniform with the most rapid variation occurring with $p << \xi$ and the map gives slightly better resolution near $\xi = 0$. The problem under consideration can be discretized by introducing a uniform grid η_j of $J+1$ nodes on $[0,1]$ with $\eta_0 = 0$ and $\eta_{j+1} = \eta_j + h$ with $h = 1/J$, so that ξ_j defines a quasi-uniform grid on $[0,\infty]$. The last interval in (22), namely $[\xi_{J-1}, \xi_J]$, is infinite but the point $\xi_{J-1/2}$ is finite, because the non integer nodes are defined by

$$\xi_{j+\alpha} = \xi\left(\eta = \frac{j+\alpha}{J}\right) \ , \tag{23}$$

with $j \in \{0,1,\ldots,J-1\}$ and $0 < \alpha < 1$. The map allow us to describe the infinite domain by a finite number of intervals. The last node of such grid is placed on infinity so that right boundary condition is taken into account correctly.

For the sake of simplicity we consider here the simple scalar case. The finite difference formulae can be applied component-wise to a system of differential equations. We can define the values of $u(\xi)$ on the middle-points of the grid

$$u_{j+1/2} \approx \frac{\xi_{j+1} - \xi_{j+1/2}}{\xi_{j+1} - \xi_j} u_j + \frac{\xi_{j+1/2} - \xi_j}{\xi_{j+1} - \xi_j} u_{j+1} \ . \tag{24}$$

As far as the first derivative is concerned we can apply the following approximation

$$\left.\frac{du}{d\xi}\right|_{j+1/2} \approx \frac{u_{j+1} - u_j}{2\left(\xi_{j+3/4} - \xi_{j+1/4}\right)} \ . \tag{25}$$

These formulae use the value $u_J = u_\infty$, but not $\xi_J = \infty$. Both finite difference approximations (24) and (25) have order of accuracy $O(J^{-2})$.

A finite difference scheme on a quasi-uniform mesh for the class of BVPs (12) can be defined by using the approximations given by (24) and (25). We denote by the 3−dimensional vector $\mathbf{U}_j$ the numerical approximation to the solution $\mathbf{u}(\xi_j)$ of (12) at the points of the mesh, that is for $j = 0, 1, \ldots, J$. We can define a second order finite difference scheme for (12) as

$$\begin{aligned} &\mathbf{U}_{j+1} - \mathbf{U}_j - a_{j+1/2}\mathbf{f}\left(\xi_{j+1/2}, b_{j+1/2}\mathbf{U}_{j+1} + c_{j+1/2}\mathbf{U}_j\right) = \mathbf{0} \,, \\ &\mathbf{g}\left(\mathbf{U}_0, \mathbf{U}_J\right) = \mathbf{0} \,, \end{aligned} \tag{26}$$

for $j = 0, 1, \ldots, J-1$, where

$$\begin{aligned} a_{j+1/2} &= 2\left(\xi_{j+3/4} - \xi_{j+1/4}\right) \,, \\ b_{j+1/2} &= \frac{\xi_{j+1/2} - \xi_j}{\xi_{j+1} - \xi_j} \,, \\ c_{j+1/2} &= \frac{\xi_{j+1} - \xi_{j+1/2}}{\xi_{j+1} - \xi_j} \,. \end{aligned} \tag{27}$$

It is evident that (26) is a nonlinear system with respect to the unknown $3(J+1)$-dimensional vector $\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_J)^T$. We notice that $b_{j+1/2} \approx c_{j+1/2} \approx 1/2$ for all $j = 0, 1, \ldots, J-2$, but when $j = J-1$, then $b_{J-1/2} = 0$ and $c_{J-1/2} = 1$. On the contrary, we choose to set $b_{J-1/2} = b_{J-3/2}$ and $c_{J-1/2} = c_{J-3/2}$ in order to avoid a suddenly jump for the coefficients of (26). As reported by Fazio and Jannelli (2014), this choice produces a much smaller error in the numerical solution of the system at ξ_J.

For the solution of (26) we can apply the classical Newton's method along with the simple termination criterion

$$\frac{1}{3(J+1)} \sum_{\ell=1}^{3} \sum_{j=0}^{J} |\Delta U_{j\ell}| \leq \mathrm{TOL} \,, \tag{28}$$

where $\Delta U_{j\ell}$, $j = 0, 1, \ldots, J$ and $\ell = 1, 2, 3$, is the difference between two successive iterate components and TOL is a fixed tolerance. The computed numerical results are obtained by setting $\mathrm{TOL} = 1\mathrm{E}-6$. Figures 4 and 5 show the numerical solution of ocean model (7)-(8) and (7)-(9). From these figures we notice how the grid is denser close to the origin in comparison with the side of the far boundary at infinity. As far as the BVP (7) with no-slip boundary conditions (8) is concerned we found a missing value of $\frac{d^2u}{d\xi^2}(0) = 0.826180$.

On the other hand, for the BVP (7) with slippery boundary conditions (9) we get a missing value of $\frac{du}{d\xi}(0) = 0.528927$ when using the initial iterate given by (29) and $\frac{du}{d\xi}(0) = -1.093088$ with the initial iterate provided by (30). The initial iterate for the solutions of the BVP (7) with no-slip conditions (8) and with slippery conditions (9), shown in figure 4 and in the top frame of figure 5, is the following

$$u_1(\xi) = 1 \,, \quad u_2(\xi) = u_3(\xi) = 0.1 \,, \tag{29}$$

the initial iterare for the solution of the BVP (7) with slippery conditions (9), shown in the bottom frame of figure 5, is the following

$$u_1(\xi) = -\xi \,, \quad u_2(\xi) = -0.5 \,, \quad u_3(\xi) = 0.1 \,. \tag{30}$$

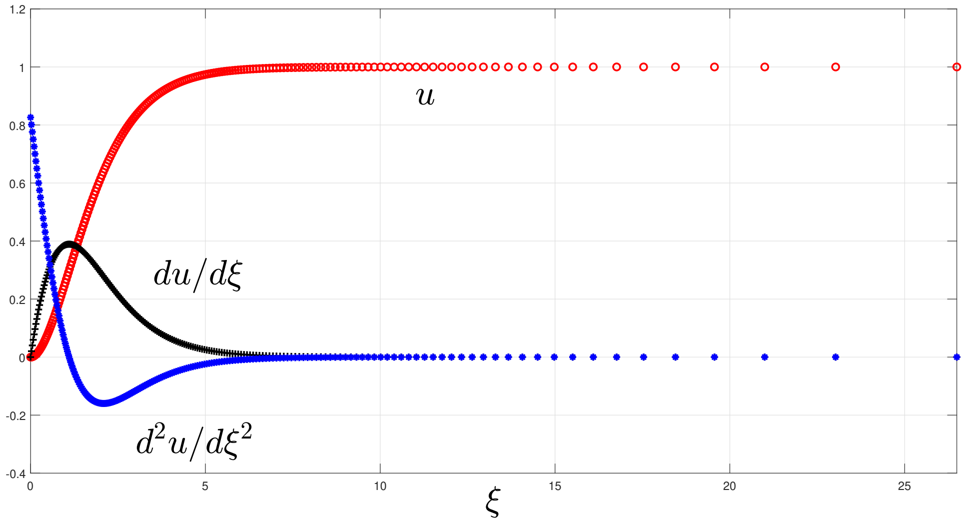

FIGURE 4. Numerical solution for the BVP (7)-(8) obtained with the map (22) and $p = 5$ for $J = 200$.

4. Final remarks and conclusions

In this paper we described two different approaches for the numerical solution of a simple wind driven circulation model arising in physical oceanography. Our final aim is the comparison of numerical results. This is provided in Tables 4, 5 and 6, where we used the abbreviations FBF (free boundary formulation) and QUG (quasi-uniform grid). For the sake of simplicity we limited ourselves to compare the computed values of the missing initial condition. We applied finite difference methods to both the free boundary approach and the quasi-uniform grid treatment of the original BVP. For the sake of comparison, all numerical methods used in this study are second order methods. To test the proposed numerical methods, the simulations were performed by an Intel Core i5 processors under Windows 10 using the programming language Fortran 77.

In the free boundary formulation, the problem involves a system of $N = 4$ coupled first-order ordinary differential equations (16) and is replaced with a nonlinear system of equations (18) with respect to the unknown $4(J+1)-$dimensional vector. On the other side, by using the quasi-uniform mesh, the problem involves a system of $N = 3$ coupled first-order ordinary differential equations and is replaced with a nonlinear system of equations (26) with respect to the unknown $3(J+1)-$dimensional vector. Moreover, as root finding solver, we applied the classical Newton's method along with a termination criterion. In both cases, the solution consists of values for N dependent functions given at each of the $J+1$ mesh points, or $N(J+1)$ variables in all, and is found by starting with an initial guess and improving it, iteratively. The iterations improve the solution. Note that the Newton's method for the free boundary formulation required a higher iterations number than the iteration number of QUG approach. Moreover, in order to improve the accuracy of the solution obtained by the free boundary formulation, a larger grid point number respect to

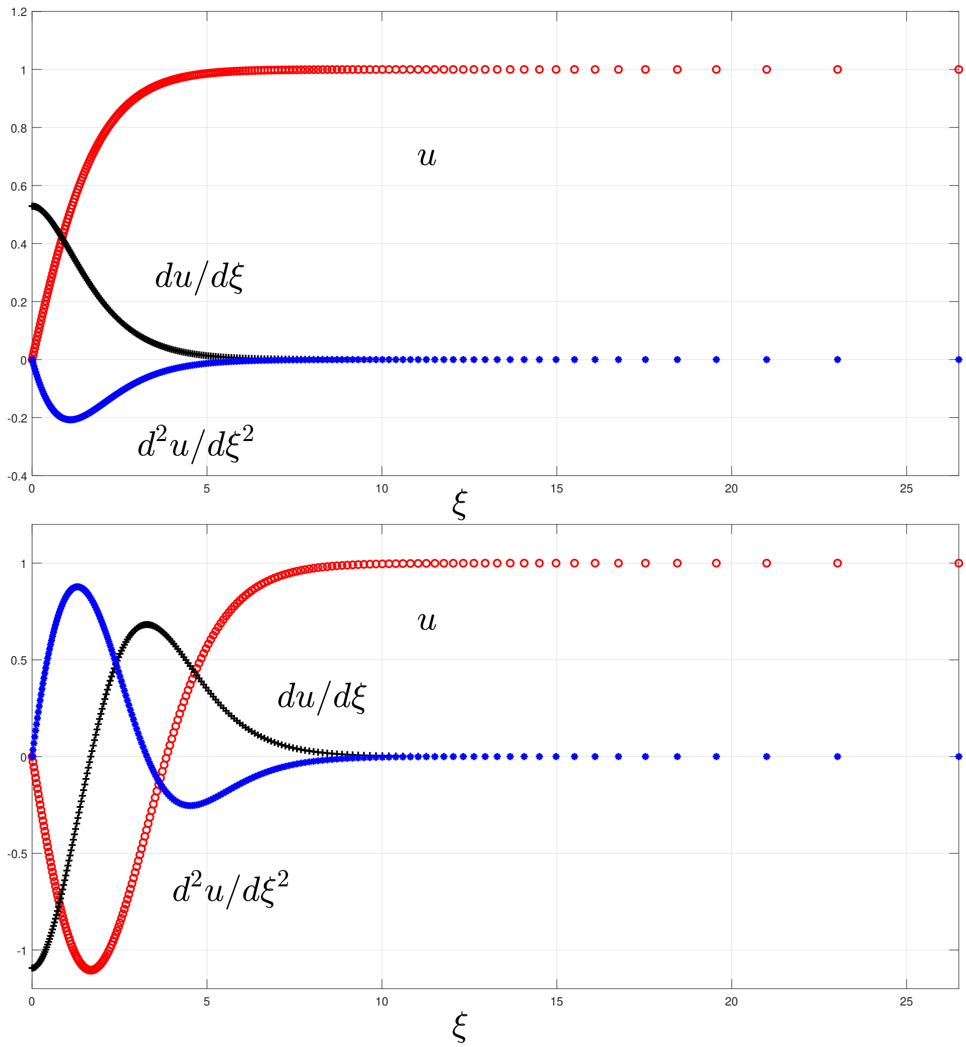

FIGURE 5. The two numerical solutions for the BVP (7)-(9) obtained with the map (22) and $p = 5$ for $J = 200$. Top frame: initial iterate (29), bottom frame: initial iterate (30).

QUG approach was necessary. It is evident that, the free boundary formulation requires a higher computational cost than the QUG approach.

The reported numerical results allow to point out that the non-standard finite difference method with a quasi-uniform grid is the least demanding of the two approaches and that the free boundary approach provides a more reliable formulation than the classical truncated boundary one. Let us remark here that a further advantage in using finite difference schemes on uniform, or quasi-uniform, grids in calculations is the possibility to apply Richardson's

TABLE 4. Comparison of numerical results for the ocean model (7) with no-slip boundary conditions (8). The positive root of equation (10) gives $\frac{d^2u}{d\xi^2}(0) = 0.828336$.

Method	grid-points	iter	boundary	$\frac{d^2u}{d\xi^2}(0)$
FBF	2000	10	$\xi_\varepsilon = 13.402219$	0.826142
FBF	4000	10	$\xi_\varepsilon = 13.402251$	0.826140
QUG	200	5	$\xi_J = \infty$	0.826180
QUG	400	5	$\xi_J = \infty$	0.826150

TABLE 5. Comparison of numerical results for the ocean model (7) with slippery boundary conditions (9). Equation (11) gives $\frac{du}{d\xi}(0) = 0.530662$.

Method	grid-points	iter	boundary	$\frac{du}{d\xi}(0)$
FBF	2000	10	$\xi_\varepsilon = 12.741323$	0.528921
FBF	4000	10	$\xi_\varepsilon = 12.741353$	0.528921
QUG	200	4	$\xi_J = \infty$	0.528927
QUG	400	4	$\xi_J = \infty$	0.528922

TABLE 6. Comparison of numerical results for the ocean model (7) with slippery boundary conditions (9). Equation (11) gives $\frac{du}{d\xi}(0) = -1.130662$.

Method	grid-points	iter	boundary	$\frac{du}{d\xi}(0)$
FBF	2000	13	$\xi_\varepsilon = 16.450930$	-1.093021
FBF	4000	13	$\xi_\varepsilon = 16.450951$	-1.093016
QUG	200	10	$\xi_J = \infty$	-1.093088
QUG	400	9	$\xi_J = \infty$	-1.093033

extrapolation (Richardson and Gaunt 1927), in order to improve the numerical accuracy, see for instance Fazio and Jannelli (2014, 2017) and Fazio *et al.* (2018) for details.

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