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Numerical Scaling Invariance Applied to the van der Pol Model

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Abstract In this study we use the van der Pol model to explain a novel numerical application of scaling invariance. The model in point is not invariant to a scaling group of transformations, but by introducing an embedding parameter we are able to recover it from an extended model which is invariant to an extended scaling group. As well known, within a similarity analysis we can define a family of solutions from a computed one, so that the solution of a target problem can be obtained by rescaling the solution of a reference problem. The main idea is to use scaling invariance and numerical analysis to find a reference problem easier to solve, from a numerical viewpoint, than the target problem. This allows us to save human efforts and computational resources every time we have to solve a challenging problem.

We test our approach using three stiff solvers available within the most recent releases of MATLAB. Independently from the solver used, by employing the described scaling invariance we are able to significantly reduce the computational cost of the numerical solution of the van der Pol model.

Keywords Scaling properties · Ordinary differential equations · Initial value problems

AMS Subject Classification 65L05 · 65L07

1 Introduction

The modern applied mathematical community is well aware of the relevance of scaling invariance of mathematical models. As a general principle, the physical laws should not depend upon the units of measure, and, accordingly, the mathematical models should have the properties that they are invariant under rescalings in the independent and dependent variables, cfr. Barenblatt [1]. A recent survey [2] records the most useful applications of

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scaling properties to numerical analysis and computational methods. Moreover, in the last years, Budd et al. [3, 4] introduced scaling invariant methods for initial value problems (IVPs), within the general topic of geometric integration. On the other hand, as pointed out by many authors, see for instance Na [5, p. 137] and the references quoted by him, the main drawback of scaling based methods is that they are not widely applicable, i.e., several models, in the applied sciences, are not invariant to scaling groups.

In this paper we identify an application of scaling invariance properties that can be easily generalized to classes of problems. In particular, we show that even if the model under study is not invariant under a scaling group we can embed it into an extended model which is invariant to an extended scaling group and we can use a specific numerical solution to generate a family of solutions linked by the extended scaling invariance. This property can be used in order to solve one of the less severe IVP for the extended model, so that the solution of a target model can be computed by rescaling. Of course, this approach is worth of consideration every time the target model is, from numerical viewpoint, a challenging one. Moreover, to solve accurately the challenging problem we should be able to compute a reliable numerical reference solution, for instance by using a high order method or better an adaptive approach.

For the sake of brevity, this study is limited to consider only the van der Pol model. However, by using the analysis developed in [6] or in [7], our approach can be easily extended to classes of problems, and this is the topic of a research in progress.

2 van der Pol Model

The van der Pol oscillator is a model developed to describe the behavior of nonlinear vacuum tube circuits in the relatively early days of the development of electronics technology. The circuit scheme, was designed by Balthasar van der Pol in the 1920's [8]. The van der Pol model, in what is now considered to be a standard form, is given by

$$\begin{cases} \dfrac{d^2y}{dt^{*2}} + \epsilon^*(y^2-1)\dfrac{dy}{dt^*} + y = 0, \\ y(0) = y_0, \quad \dfrac{dy}{dt^*}(0) = y_1. \end{cases} \tag{2.1}$$

The IVP (2.1) is often used, when $\epsilon^* \gg 1$, as a test problem for ODEs solvers, see Mazzia and Iavernaro [9]. The governing equation has two periodic solutions, the constant solution, $y(t) = 0$, that is unstable, and the nontrivial periodic solution (roughly corresponding to $y_0 = 2$ and $y_1 = 0$, according to Shampine [10, pp. 398–403]), that is named 'limit cycle' because all the other nontrivial solutions converge to this one as $t \to \infty$. When ϵ^* is 'large' the approach to the limit cycle is quite rapid and the van der Pol equation is more interesting because of the non negligible influence of the nonlinear term. It is well known that the limit cycle can be described in terms of regions where the solution components change slowly and the problem is quite stiff, alternating with parts of very sharp change (quasi-discontinuities) where it is non-stiff, see Shampine [10, pp. 398–403] and the references quoted therein. Thus, the problem switches from stiff to non stiff with a very sharp changing solution that makes the equation quite challenging for ODEs solvers.

It is a simple matter to verify that the van der Pol model is not invariant to scaling groups! In fact, a classical scaling group, involving the independent and dependent variables only, can be written as follows:

$$t^* = \lambda^\alpha t', \qquad y = \lambda^\beta y',$$

where λ is the group parameter, α and β are group exponents to be determined by the required invariance and t' and y' are rescaled variables. To be invariant (2.1) has to be transformed under (2.2) into a model independent on λ. For the invariance of the governing differential equation and of the initial conditions, α and β have to verify the algebraic relations:

$$\beta - 2\alpha = \beta - \alpha = 3\beta - \alpha = \beta$$

and

$$\beta = 0, \qquad \alpha = \beta,$$

respectively. The above relations can be satisfied only for $\alpha = \beta = 0$, and therefore the van der Pol model is not invariant to scaling groups.

As a consequence, the adaptive approach developed by Budd et al. [3], within the scaling invariance theory, cannot be applied to the van der Pol model. However, let us introduce an embedding parameter h and consider the extended model

$$\begin{cases} \dfrac{d^2y}{dt^2} + \epsilon(y^2 - 1)\dfrac{dy}{dt} + hy = 0, \\ y(0) = y_0, \qquad \dfrac{dy}{dt}(0) = h^{1/2}y_1, \end{cases} \tag{2.2}$$

which is invariant with respect to the extended scaling group

$$t = \lambda^{-1}t^*, \qquad \epsilon = \lambda\epsilon^*, \qquad h = \lambda^2 h^*. \tag{2.3}$$

In fact, if we insert the transformation (2.3) into (2.2), then in the transformed model equal powers of λ can be simplified and the model results to be invariant.

We remark that the original IVP (2.1) is recoved from (2.2) by setting $t^* = t$, $\epsilon^* = \epsilon$, and $h = 1$. Therefore, the solution of (2.1) can be computed by rescaling the solution of (2.2) on condition that under (2.3) h will be transformed into $h^* = 1$. In order to approximate numerically the IVP (2.1) with a large value of ϵ^*, we can solve numerically the IVP (2.2) using a small value of ϵ and the value of h corresponding to $h^* = 1$. This idea is explained in more details in the next section.

3 Numerical Results

We applied our scaling approach, described above, to the van der Pol model for several values of ϵ^* and different initial conditions. Indeed, to verify the usefulness of this study we started with the modest case where $\epsilon^* = 10$ and $\epsilon = 1$. This simple case is not reported here for the sake of brevity. We report below the numerical results related to the usual stiff case and consider as a target the IVP (2.1) with $\epsilon^* = 10^3$ and integrate numerically the IVP (2.2) with $\epsilon = 10^{-3}$. Larger values of ϵ^* can be treated in a similar way. By using (2.3) we can find the value of the group parameter which transforms the extended model solution to the van der Pol solution, and the value of h to be used in (2.2)

$$\lambda = 10^{-6}, \qquad h = 10^{-12},$$

where λ is defined by $\epsilon^* = 10^3$ and $\epsilon = 10^{-3}$, whereupon h can be computed from $h = \lambda^2$ because our target is $h^* = 1$.

By setting $\epsilon = 10^{-3}$ as a reference value, we chose to modify the problem under study as follows:

1. we stretch the time scale by a factor 10^6;
2. we reduce the scale of the first and second derivative of $y(t)$ by a factor equal to 10^{-6} and 10^{-12}, respectively, because these derivatives transform under (2.3) as follows

$$\frac{dy}{dt^*}(t^*) = \lambda^{-1}\frac{dy}{dt}(t), \qquad \frac{d^2y}{dt^{*2}}(t^*) = \lambda^{-2}\frac{d^2y}{dt^2}(t);$$

3. we multiply both sides of the van der Pol equation by a factor 10^{-12}.

We remark that in order to get an accurate solution of the target IVP, it is necessary to solve the extended IVP as accurately as possible, and therefore an adaptive numerical strategy is suggested. The numerical results reported below were obtained by the `ode23s` solver, which is available within the most recent releases of MATLAB. Figure 1 shows the numerical solution of the extended model (2.2) with $\epsilon = 10^{-3}$, $h = 10^{-12}$ and the initial conditions $y_0 = 2$, and $y_1 = 0$. In Fig. 2 we plot the solution of the IVP (2.1) with $\epsilon^* = 10^3$, $y_0 = 2$, and $y_1 = 0$ computed by rescaling. By comparing the two Figs. 1 and 2 we can appreciate how the time variable and the first derivative of $y(t)$ are rescaled, so that in comparison with the behavior of the solution of the extended problem the greater variability of the first derivative in the target problem takes place in a shorter time interval.

Figure 3 shows the two solutions in the phase plane. Due to the scales used, the computed reference solution seems to be a straight segment, joining the two points $(-2, 0)$ and $(2, 0)$, in the phase plane. Figure 3 helps us to explain way the computational complexities of the two models, shown in Table 1, are so different.

Table 1 provides the performance of our approach compared to the direct solution of the van der Pol model (2.1) with $\epsilon^* = 10^3$. In this statistics we compare the performance of three stiff solvers available within the so called MATLAB ODE suite developed by Samphine and Reichelt [11]. These solvers are: ode23s (a Rosenbrock type method of order two), ode23tb (an implicit Runge-Kutta method with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two), and ode15s (a variable order method based on the numerical differentiation formulas). As far as the adaptivity parameters are concerned we used the default values AbsTol $= 10^{-6}$ (a scalar absolute error tolerance) and RelTol $= 10^{-3}$ (a relative error tolerance) because that would be the choice of a typical user working in the applied sciences.

At each step, the MATLAB solvers estimate the local error $e(i)$ in the i-th component of the solution, call it $u(i)$. This error must be less than or equal to the acceptable error, which is a function of the specified relative tolerance, RelTol, and the specified component-wise absolute tolerance, AbsTol(i):

$$|e(i)| \leq \max(\text{RelTol} \cdot |u(i)|, \text{AbsTol}(i)).$$

RelTol is a measure of the error relative to the size of each solution component. Roughly speaking, it controls the number of correct digits in all solution components, except those smaller than thresholds AbsTol(i). Its default value, 10^{-3}, corresponds to 0.1% accuracy. AbsTol(i) is a threshold below which the value of the i-th solution component is unimportant. The absolute error tolerances determine the accuracy when the solution approaches zero. When AbsTol is set to a scalar value, as in our case, all solution components have to verify the same criterion.

It is evident, from the values listed in Table 1, that our scaling approach is competitive for all the considered solvers. Independently from the solver used, the computational cost of the numerical solution of the van der Pol model has been significantly reduced. Moreover, it

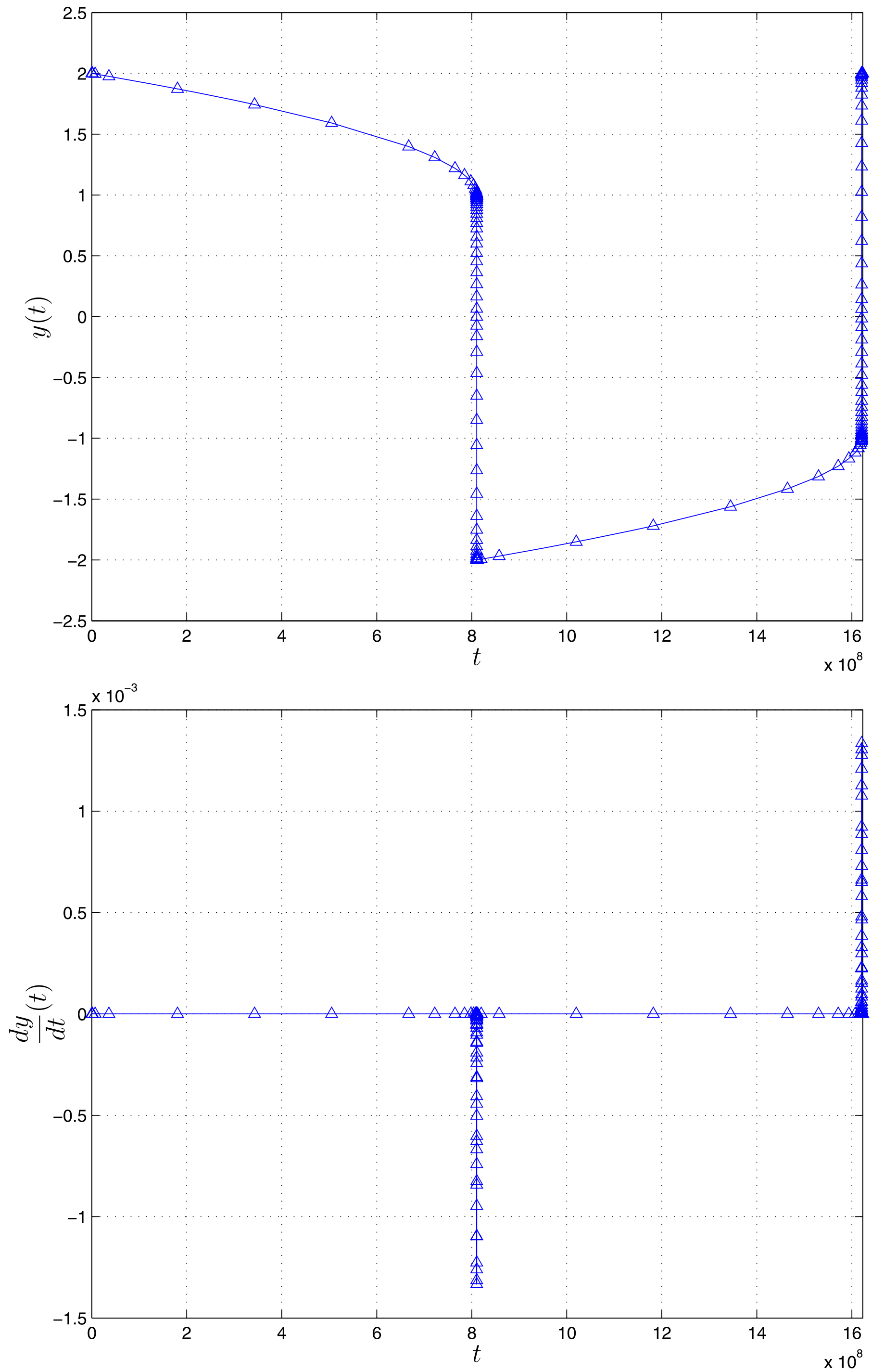

Fig. 1 One period of the solution of the IVP (2.2) with $\epsilon = 10^{-3}$, $h = 10^{-12}$, $y_0 = 2$, and $y_1 = 0$

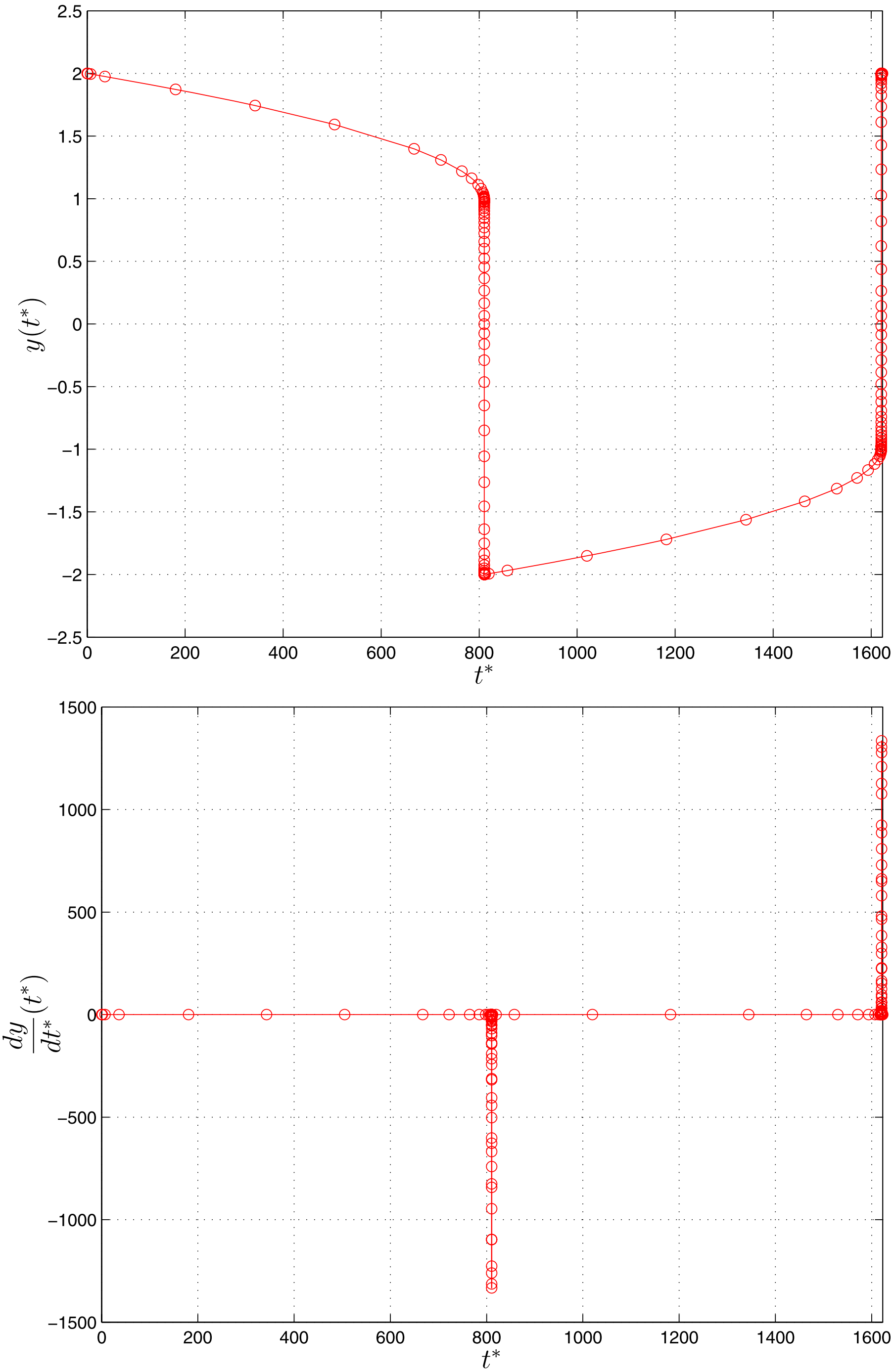

Fig. 2 One period of the van der Pol solution, of the IVP (2.1) with $\epsilon^* = 10^3$, $y_0 = 2$, and $y_1 = 0$ computed by rescaling

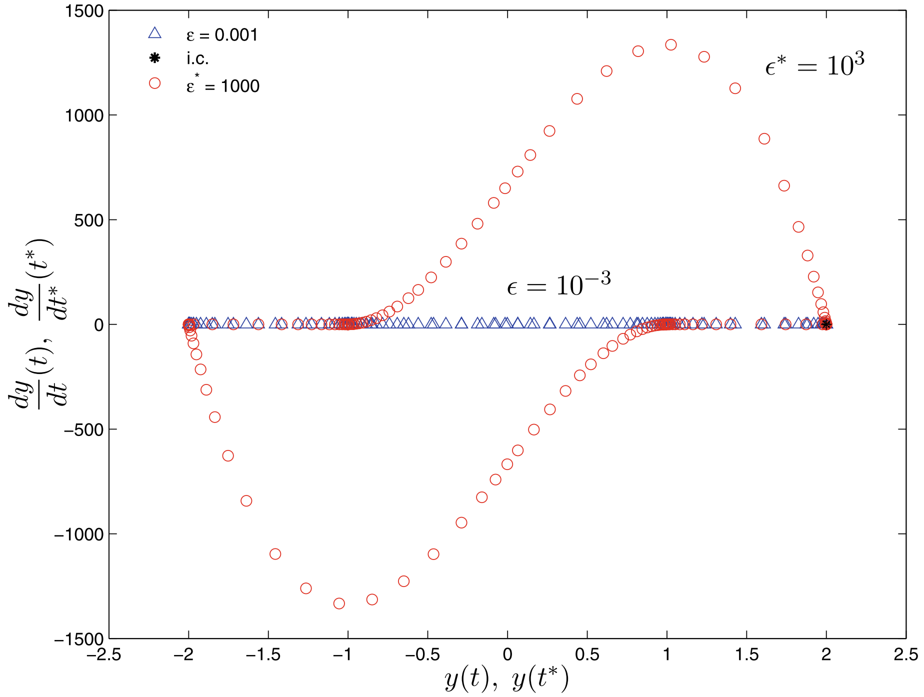

Fig. 3 Limit cycles in the phase plane: the solution of the IVP (2.2) with $\epsilon = 10^{-3}$, $h = 10^{-12}$, $y_0 = 2$ and $y_1 = 0$ is marked by *triangles*; the rescaled van der Pol solution, $\epsilon^* = 10^3$, and $h^* = 1$ is marked by *circles*; an *asterisk* marks the common initial conditions

Table 1 Statistics for the van der Pol model. The data on the columns marked by a S.A. means that we have used the proposed scaling approach

Statistics	ode23s		ode23tb		ode15s	
		S.A.		S.A.		S.A.
Successful steps	141	490	172	386	200	402
Failed attempts	53	10	72	62	73	149
Function evaluation	814	2472	953	1954	583	1255
Partial derivatives	141	490	28	31	29	30
LU decomposition	194	500	140	180	107	195
Solution of linear systems	582	1500	1066	2266	495	1164

seems that the best results can be achieved by setting $\epsilon \approx 10^{-4}$. In this case, for instance, our scaling approach required only 126, 163, and 176 successful steps for the ode23s, ode23tb, and ode15s, respectively.

A preliminary numerical study shows that the data reported in Table 1 are not related to a different stiffness of the two models but rather to the dependency of the local truncation errors of the methods applied to (2.1) on ϵ^*, and to (2.2) on ϵ and h. In fact, if we rewrite the governing differential equations in (2.1) and (2.2) as first order systems, then we can define the Jacobian of the right hand side of these systems, cf. for the van der Pol equation Shampine [10, pp. 398–399], and the Jacobian eigenvalues of the two models will be transformed by (2.3) in the following way

$$\mu_{1/2} = \lambda \mu^*_{1/2},$$

where we have denoted by $\mu_{1/2}$ and $\mu^*_{1/2}$ the Jacobian eigenvalues of the extended model and the van der Pol model, respectively. The stiffness ratio R, for these eigenvalues, is defined by

$$R = \frac{\max\{|Re(\mu_1)|, |Re(\mu_2)|\}}{\min\{|Re(\mu_1)|, |Re(\mu_2)|\}},$$

where $Re(\cdot)$ stands for the real part of its argument. It is a simple matter to verify that R is an invariant with respect to (2.3). We might conclude that the problem stiffness does not depend on rescaling the extended model. However, it is well known that often the stiffness ratio is not a good measure of stiffness even for linear systems, since if the minimum eigenvalue is zero, the problem has infinite stiffness ratio, but may not be stiff at all if the other eigenvalues are of moderate size, see Lambert [12, pp. 216–224 and pp. 261–264]. Moreover, if we consider the stiffness definition $S \gg 1$, where

$$S = t_{\max} \max\{|Re(\mu_1)|, |Re(\mu_2)|\},$$

used by Shampine [10, pp. 382–383], then we can easily verify that S is invariant under (2.3).

Some topics are left for a future research, namely: the extension of our approach to classes of problems, a study concerning the order of accuracy of the rescaled solution, as well as the invariance properties of different step selection strategies.

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