# Reduction of Quasilinear First Order PDE's to Partially or Fully Decoupled Systems

MATTEO GORGONE

joint work with F. Oliveri & M. P. Speciale

Department MIFT, University of Messina

Topics in Nonlinear Analysis and Applications
Università di Milano "Bicocca"
March 15 – 16, 2017

## Problem

*When can a system like*

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

*where* $\mathbf{u} \equiv (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ *and* $A$ *is an* $n \times n$ *matrix with real entries depending on* $\mathbf{u}$*, be locally decoupled in some coordinates* $v_1(\mathbf{u}), \ldots, v_n(\mathbf{u})$ *into* $k$ *non–interacting subsystems of some orders* $n_1, \ldots, n_k$ *with* $n_1 + \cdots + n_k = n$*?*

# Necessary and Sufficient Conditions

## Theorem [Nijenhuis]

The necessary and sufficient condition for the complete decoupling of the system

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}$$

into $n$ non–interacting one–dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor[a]

$$N_{jik} = A_{\alpha i}\frac{\partial A_{jk}}{\partial u_\alpha} - A_{\alpha k}\frac{\partial A_{ji}}{\partial u_\alpha} + A_{j\alpha}\frac{\partial A_{\alpha i}}{\partial u_k} - A_{j\alpha}\frac{\partial A_{\alpha k}}{\partial u_i},$$

provided that all eigenvalues of $A$ are real and distinct.

[a] A. Nijenhuis. Proc. Kon. Ned. Akad. Amsterdam, **54**, 200–212, 1951.

## Necessary and Sufficient Conditions

Bogoyavlenskij[a][b] provided necessary and sufficient conditions to reduce quasilinear first order systems into block–diagonal form. Tunitsky[c] established necessary and sufficient conditions for transforming quasilinear first order systems into triangular blocks.

[a] O. I. Bogoyavlenskij. J. Math. Phys., **47**, 063502, 2006.
[b] O. I. Bogoyavlenskij. Commun. Math. Phys., **269**, 545–556, 2007.
[c] D. V. Tunitsky. Sbornik: Mathematics, **204**, 438–462, 2013.

## Remark

Both the results are based on Nijenhuis and Haantjes[a] tensors.

[a] J. Haantjes. Ned. Akad. Wetensch. Proc. Ser. A, **58**, 158–162, 1958.

## Notation

Let $\mathbf{U} \equiv (U_1, U_2, \dots, U_n)^T \in \mathbb{R}^n$. Let us relabel and group the components of $\mathbf{U}$ as follows:

$$\left\{\{U^{(1,1)}, \dots, U^{(1,n_1)}\}, \dots, \{U^{(k,1)}, \dots, U^{(k,n_k)}\}\right\},$$

with $k > 1$, and $n_1 + \dots + n_k = n$.
We set

$$\mathcal{U}_i = \bigcup_{r=1}^{i} \left\{U^{(r,1)}, \dots U^{(r,n_r)}\right\},$$

$$\overline{\mathcal{U}}_i = \bigcup_{r=i+1}^{k} \left\{U^{(r,1)}, \dots, U^{(r,n_r)}\right\};$$

the cardinality of the set $\mathcal{U}_i$ is $m_i$, whereas the cardinality of the set $\overline{\mathcal{U}}_i$ is $n - m_i$, where $m_i = n_1 + \dots + n_i$.

## Definition [Partially Decoupled Systems]

The first order quasilinear system

$$\frac{\partial \mathbf{U}}{\partial t} + T(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}, \qquad \mathbf{U} \in \mathbb{R}^n, \quad T \quad n \times n \text{ real matrix}, \qquad (1)$$

is partially decoupled in $2 \leq k \leq n$ subsystems of some orders $n_1, \ldots, n_k$ $(n_1 + \ldots + n_k = n)$ if, relabelling and suitably collecting the components of $\mathbf{U}$ in $k$ subgroups, we recognize $k$ subsystems such that the $i$-th subsystem $(i = 1, \ldots, k)$ involves at most the $m_i$ field variables of $\mathcal{U}_i$.

## Definition [Fully Decoupled Systems]

The system (1) is fully decoupled in $2 \leq k \leq n$ subsystems of some orders $n_1, \ldots, n_k$ $(n_1 + \ldots + n_k = n)$ if we recognize $k$ subsystems such that the $i$-th subsystem $(i = 1, \ldots, k)$ involves exactly the $n_i$ field variables $\{U^{(i,1)}, \ldots, U^{(i,n_i)}\}$.

## Lemma

*Let $T$ be an $n \times n$ lower triangular block matrix with real eigenvalues and a complete set of eigenvectors,*

$$T = \begin{bmatrix} T_1^1 & 0_2^1 & 0_3^1 & \dots & 0_{k-1}^1 & 0_k^1 \\ T_1^2 & T_2^2 & 0_3^2 & \dots & 0_{k-1}^2 & 0_k^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T_1^{k-1} & T_2^{k-1} & T_3^{k-1} & \dots & T_{k-1}^{k-1} & 0_k^{k-1} \\ T_1^k & T_2^k & T_3^k & \dots & T_{k-1}^k & T_k^k \end{bmatrix}.$$

*The entries of matrices $T_j^i$ $(i = 1, \dots, k;\ j = 1, \dots, i)$ depend at most on the $m_i$ variables of the set $\mathcal{U}_i$ if and only if:*

$\dots$

## . . . Lemma

1. the set of the eigenvalues of $T$ and the corresponding left and right eigenvectors can be divided into $k$ subsets

$$\left\{\{\Lambda^{(1,1)},\dots,\Lambda^{(1,n_1)}\},\dots,\{\Lambda^{(k,1)},\dots,\Lambda^{(k,n_k)}\}\right\},$$
$$\left\{\{\mathbf{L}^{(1,1)},\dots,\mathbf{L}^{(1,n_1)}\},\dots,\{\mathbf{L}^{(k,1)},\dots,\mathbf{L}^{(k,n_k)}\}\right\},$$
$$\left\{\{\mathbf{R}^{(1,1)},\dots,\mathbf{R}^{(1,n_1)}\},\dots,\{\mathbf{R}^{(k,1)},\dots,\mathbf{R}^{(k,n_k)}\}\right\};$$

2. the following structure conditions hold true:

$$\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,$$
$$\mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0,$$
$$i=1,\dots,k-1,\quad \ell=1,\dots,i,$$
$$\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta\ \text{ if }\ i=\ell,$$
$$j=i+1,\dots,k,\quad \gamma=1,\dots,n_j.$$

# Proof: Necessary Condition

Let us assume that $T^i_j$ $(i = 1, \ldots, k,\ j = 1, \ldots, i)$ depend at most on the elements of the set $\mathcal{U}_i$; it follows that also $\Lambda^{(i,\alpha)}$ depend at most on the elements of the set $\mathcal{U}_i$.

Let us group the components of a right (left, resp.) eigenvector $\mathbf{R}^{(r,\alpha)}$ ($\mathbf{L}^{(r,\alpha)}$, resp.) as follows:

$$\mathbf{R}^{(r,\alpha)} = \begin{pmatrix} \mathbf{R}^{(r,\alpha)}_1 \\ \mathbf{R}^{(r,\alpha)}_2 \\ \ldots \\ \mathbf{R}^{(r,\alpha)}_k \end{pmatrix}, \qquad \mathbf{L}^{(r,\alpha)} = \left(\mathbf{L}^{(r,\alpha)}_1, \mathbf{L}^{(r,\alpha)}_2, \ldots, \mathbf{L}^{(r,\alpha)}_k\right).$$

# ...Proof: Necessary Condition

Taking into account the relations for the left eigenvectors,

$$
\begin{aligned}
\mathbf{L}_1^{(r,\alpha)} T_1^1 + \mathbf{L}_2^{(r,\alpha)} T_1^2 + \ldots + \mathbf{L}_{k-1}^{(r,\alpha)} T_1^{k-1} + \mathbf{L}_k^{(r,\alpha)} T_1^k &= \Lambda^{(r,\alpha)} \mathbf{L}_1^{(r,\alpha)},\\
\mathbf{L}_2^{(r,\alpha)} T_2^2 + \ldots + \mathbf{L}_{k-1}^{(r,\alpha)} T_2^{k-1} + \mathbf{L}_k^{(r,\alpha)} T_2^k &= \Lambda^{(r,\alpha)} \mathbf{L}_2^{(r,\alpha)},\\
&\cdots\cdots\\
\mathbf{L}_{k-1}^{(r,\alpha)} T_{k-1}^{k-1} + \mathbf{L}_k^{(r,\alpha)} T_{k-1}^k &= \Lambda^{(r,\alpha)} \mathbf{L}_{k-1}^{(r,\alpha)},\\
\mathbf{L}_k^{(r,\alpha)} T_k^k &= \Lambda^{(r,\alpha)} \mathbf{L}_k^{(r,\alpha)}.
\end{aligned}
$$

If $r < k$, $\mathbf{L}^{(r,\alpha)} = \left(\mathbf{L}_1^{(r,\alpha)}, \ldots, \mathbf{L}_r^{(r,\alpha)}, \mathbf{0}_{r+1}, \ldots, \mathbf{0}_k\right)$ and $\mathbf{L}_s^{(r,\alpha)}$ $(s = 1, \ldots, r)$ depend at most on the elements of $\mathcal{U}_r$.

# . . . Proof: Necessary Condition

Analogously, by considering the relations for the right eigenvectors,

$$
\begin{aligned}
&T_1^1\mathbf{R}_1^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_1^{(r,\alpha)},\\
&T_1^2\mathbf{R}_1^{(r,\alpha)} + T_2^2\mathbf{R}_2^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_2^{(r,\alpha)},\\
&\dots\dots\\
&T_1^{k-1}\mathbf{R}_1^{(r,\alpha)} + T_2^{k-1}\mathbf{R}_2^{(r,\alpha)} + \dots + T_{k-1}^{k-1}\mathbf{R}_{k-1}^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_{k-1}^{(r,\alpha)},\\
&T_1^{k}\mathbf{R}_1^{(r,\alpha)} + T_2^{k}\mathbf{R}_2^{(r,\alpha)} + \dots + T_{k-1}^{k}\mathbf{R}_{k-1}^{(r,\alpha)} + T_k^k\mathbf{R}_k^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_{k}^{(r,\alpha)}.
\end{aligned}
$$

If $r > 1$, $\mathbf{R}^{(r,\alpha)} = \left(\mathbf{0}_1, \dots, \mathbf{0}_{r-1}, \mathbf{R}_r^{(r,\alpha)}, \dots, \mathbf{R}_k^{(r,\alpha)}\right)^T$ and $\mathbf{R}_s^{(r,\alpha)}$ $(s = r, \dots, k)$ depend at most on the elements of $\mathcal{U}_s$.

# . . . Proof: Necessary Condition

As a consequence, the structure conditions are trivially satisfied. In fact:

1.
   - only the first $m_i$ components of the vector $\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}$ may be non–vanishing;
   - the first $m_{j-1}$ components of $\mathbf{R}^{(j,\gamma)}$ are vanishing;

$$\Longrightarrow \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)} = 0, \qquad j > i;$$

2.
   - the first $m_{\ell-1}$ components of $\mathbf{R}^{(\ell,\beta)}$ are vanishing;
   - the components of $\mathbf{R}_s^{(\ell,\beta)}$ ($s = \ell, \dots, k$) depend at most on $\mathcal{U}_s$, the first $m_{j-1}$ of $\mathbf{R}^{(j,\gamma)}$ are vanishing;
     $\Longrightarrow$ the first $m_j$ components of the vector $(\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}$ are vanishing;

$$\Longrightarrow \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right) = 0, \qquad j > i \geq \ell.$$

# ...Proof: Sufficient Condition

Let us assume the structure conditions hold.
Let be $\Lambda = \Lambda^{(r,\alpha)}$ one of the eigenvalues of the matrix $T_r^r$ $(1 \leq r < k)$, and

$$\nabla_{\mathbf{U}} \equiv (\nabla_1, \ldots, \nabla_k),$$

where

$$\nabla_i \equiv \left(\frac{\partial}{\partial U^{(i,1)}}, \ldots, \frac{\partial}{\partial U^{(i,n_i)}}\right), \qquad i = 1, \ldots, k.$$

For $j > r$, it is

$$\begin{aligned}(\nabla_{r+1}\Lambda) \cdot \mathbf{R}_{r+1}^{(r+1,\gamma)} + (\nabla_{r+2}\Lambda) \cdot \mathbf{R}_{r+2}^{(r+1,\gamma)} + \ldots + (\nabla_k\Lambda) \cdot \mathbf{R}_k^{(r+1,\gamma)} &= 0,\\ (\nabla_{r+2}\Lambda) \cdot \mathbf{R}_{r+2}^{(r+2,\gamma)} + \ldots + (\nabla_k\Lambda) \cdot \mathbf{R}_k^{(r+2,\gamma)} &= 0,\\ &\ldots\\ (\nabla_k\Lambda) \cdot \mathbf{R}_k^{(k,\gamma)} &= 0,\end{aligned}$$

and it follows that

$$\frac{\partial\Lambda}{\partial U^{(r+1,1)}} = \ldots = \frac{\partial\Lambda}{\partial U^{(r+1,n_{r+1})}} = \ldots = \frac{\partial\Lambda}{\partial U^{(k,1)}} = \ldots = \frac{\partial\Lambda}{\partial U^{(k,n_k)}} = 0.$$

# ...Proof: Sufficient Condition

$$\sum_{r=\ell}^{i}\left(\mathbf{L}_r^{(i,\alpha)}\cdot\left((\nabla_j\mathbf{R}_r^{(\ell,\beta)})\mathbf{R}_j^{(j,\gamma)}+\ldots+(\nabla_k\mathbf{R}_r^{(\ell,\beta)})\mathbf{R}_k^{(j,\gamma)}\right)\right)=0,$$

for $i=1,\ldots,k-1,\ \ell\leq i,\ j=i+1,\ldots,k$, and $\alpha\neq\beta$ for $\ell=i$.

From the relations

$$T_r^r\mathbf{R}_r^{(r,\alpha)}=\Lambda^{(r,\alpha)}\mathbf{R}_r^{(r,\alpha)},\qquad r=1,\ldots,k-1,$$

for $j>r$, we obtain

$$\left(\sum_{s=j}^{k}(\nabla_s T_r^r)\mathbf{R}_s^{(j,\gamma)}\right)\mathbf{R}_r^{(r,\alpha)}+(T_r^r-\Lambda^{(r,\alpha)}\mathbb{I}_r)\left(\sum_{s=j}^{k}(\nabla_s\mathbf{R}_r^{(r,\alpha)})\mathbf{R}_s^{(j,\gamma)}\right)=\mathbf{0},$$

$\mathbb{I}_r$ being the $r\times r$ identity matrix, whereupon

$$\mathbf{L}_r^{(r,\beta)}\cdot\left(\sum_{s=j}^{k}(\nabla_s T_r^r)\mathbf{R}_s^{(j,\gamma)}\right)\mathbf{R}_r^{(r,\alpha)}+(\Lambda^{(r,\beta)}-\Lambda^{(r,\alpha)})\mathbf{L}_r^{(r,\beta)}\cdot\left(\sum_{s=j}^{k}(\nabla_s\mathbf{R}_r^{(r,\alpha)})\mathbf{R}_s^{(j,\gamma)}\right)=0.$$

For $j=k,k-1,\ldots,r+1$, it follows that $T_r^r$ $(r=1,\ldots,k-1)$ and $\mathbf{R}_r^{(r,\alpha)}$, depend at most on the elements of $\mathcal{U}_r$.

# ...Proof: Sufficient Condition

By considering the relations

$$\sum_{r=\ell}^{i}\left(\mathbf{L}_r^{(i,\alpha)}\cdot\left((\nabla_j\mathbf{R}_r^{(\ell,\beta)})\mathbf{R}_j^{(j,\gamma)}+\ldots+(\nabla_k\mathbf{R}_r^{(\ell,\beta)})\mathbf{R}_k^{(j,\gamma)}\right)\right)=0,$$

for $\ell=i-1,i-2,\ldots,1$ and $j=k,k-1,\ldots,i+1$, it is immediately deduced that $\mathbf{R}_s^{(r,\alpha)}$ for $s\geq r$ depend at most on the elements of $\mathcal{U}_s$.
Finally, from the relations

$$\sum_{s=r}^{i} T_s^i\mathbf{R}_s^{(r,\alpha)}=\Lambda^{(r,\alpha)}\mathbf{R}_i^{(r,\alpha)},\qquad r=1,\ldots,i,\quad i<k,$$

for $j>i$, we obtain

$$\sum_{s=r}^{i}\left(\left(\sum_{t=j}^{k}(\nabla_t T_s^i)\mathbf{R}_t^{(j,\gamma)}\right)\mathbf{R}_s^{(r,\alpha)}\right)=\mathbf{0}.$$

For $r=i-1,i-2,\ldots,1$ and $j=k,k-1,\ldots,i+1$, it follows that $T_s^i$ $(s=1,\ldots,i)$ depend at most on the $m_i$ variables of $\mathcal{U}_i$.

Q.E.D.

## Remark

The relations

$$\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,\quad \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0,$$
$$i=1,\dots,k-1,\quad \ell=1,\dots,i,$$
$$\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta\ \text{ if }\ i=\ell,$$
$$j=i+1,\dots,k,\quad \gamma=1,\dots,n_j,$$

provide

$$\sum_{i=1}^{k} n_i(n-m_i)+\sum_{i=1}^{k} n_i(m_i-1)(n-m_i)=\sum_{i=1}^{k} n_i m_i(n-m_i)$$

constraints, and this is exactly the number of conditions required to ensure that the entries of matrices $T_j^i$ $(i=1,\dots,k;\ j=1,\dots,i)$ are independent of the elements of the set $\overline{\mathcal{U}}_i$. In fact, the number of entries of the matrices $T_j^i$ are $n_i m_i$, and the cardinality of the set $\overline{\mathcal{U}}_i$ is $n-m_i$.

## Remark

If matrix $T$ has the lower triangular block structure then, since the first $m_{j-1}$ components of $\mathbf{R}^{(j,\gamma)}$ are vanishing, and $j > i \geq \ell$, the first $m_i$ components of the vector $(\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}$ can not be different from zero; therefore, it is identically

$$\begin{aligned}
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right) = 0,\\
&i = 1, \dots, k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \text{ if } i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned}$$

Consequently, the structure conditions may be written as well as

$$\begin{aligned}
&\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)} = 0,\\
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right) = 0.
\end{aligned}$$

## Theorem [Partial Decoupling]

The first order quasilinear system

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \qquad \mathbf{u} \in \mathbb{R}^n, \quad A \quad n \times n \text{ real matrix},$$

assumed to be hyperbolic in the $t$–direction, can be transformed by a smooth (locally) invertible transformation

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \text{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u}),$$

into a system like

$$\frac{\partial \mathbf{U}}{\partial t} + T(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0},$$

where $T = (\nabla_{\mathbf{u}}\mathbf{H})\, A\, (\nabla_{\mathbf{u}}\mathbf{H})^{-1}$ is a lower triangular block matrix, with $T_j^i$ $(i = 1, \dots, k;\ j = 1, \dots, i)$ $n_i \times n_j$ matrices such that their entries are smooth functions depending at most on the elements of the set $\mathcal{U}_i$, if and only if:

...

## . . . Theorem [Partial Decoupling]

1. the set of the eigenvalues of matrix $A$, and the associated left and right eigenvectors can be divided into $k$ subsets

$$\left\{\{\lambda^{(1,1)},\dots,\lambda^{(1,n_1)}\},\dots,\{\lambda^{(k,1)},\dots,\lambda^{(k,n_k)}\}\right\},$$
$$\left\{\{\mathbf{l}^{(1,1)},\dots,\mathbf{l}^{(1,n_1)}\},\dots,\{\mathbf{l}^{(k,1)},\dots,\mathbf{l}^{(k,n_k)}\}\right\},$$
$$\left\{\{\mathbf{r}^{(1,1)},\dots,\mathbf{r}^{(1,n_1)}\},\dots,\{\mathbf{r}^{(k,1)},\dots,\mathbf{r}^{(k,n_k)}\}\right\};$$

2. the following structure conditions hold true:

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,\qquad \mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right)=0,$$
$$i=1,\dots,k-1,\quad \ell=1,\dots,i,\quad \alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,$$
$$\alpha\neq\beta\ \text{ if }\ i=\ell,\quad j=i+1,\dots,k,\quad \gamma=1,\dots,n_j.$$

Moreover, the decoupling variables $U^{(i,\alpha)}=H^{(i,\alpha)}(\mathbf{u})$ ($i=1,\dots,k-1$; $\alpha=1,\dots,n_i$) are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,\qquad j=i+1,\dots,k\qquad \gamma=1,\dots,n_j.$$

# Proof

Let us assume that the structure conditions are satisfied.
By introducing $\mathbf{U} = \mathbf{H}(\mathbf{u})$ such that

$$\left(\nabla_{\mathbf{u}} H^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0, \qquad \begin{array}{ll} i = 1, \dots, k-1, & \alpha = 1, \dots, n_i, \\ j = i+1, \dots, k, & \gamma = 1, \dots, n_j, \end{array}$$

we obtain that $T$ is a lower triangular block matrix.
Since

$$\begin{array}{ll} \lambda^{(i,\alpha)} = \Lambda^{(i,\alpha)}, & \nabla_{\mathbf{u}}(\cdot) = \nabla_{\mathbf{U}}(\cdot)(\nabla_{\mathbf{u}}\mathbf{H}), \\ \mathbf{l}^{(i,\alpha)} = \mathbf{L}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H}), & \mathbf{r}^{(i,\alpha)} = (\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(i,\alpha)}, \end{array}$$

we have

$$0 = \left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{U}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)} =$$

$$= \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)}.$$

# …Proof

$$
\begin{aligned}
0 &= \mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right)=\\
&= \mathbf{L}^{(i,\alpha)}\left(\nabla_{\mathbf{u}}\mathbf{H}\right)\left(\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right.\\
&\qquad\left.-\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)=\\
&= \mathbf{L}^{(i,\alpha)}\left(\nabla_{\mathbf{u}}\mathbf{H}\right)\left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right.\\
&\qquad\left.-\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)=\\
&= \mathbf{L}^{(i,\alpha)}\left(\nabla_{\mathbf{u}}\mathbf{H}\right)\left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)\mathbf{R}^{(j,\gamma)}-\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)\mathbf{R}^{(\ell,\beta)}\right)=\\
&= \mathbf{L}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H})\left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\right)\left(\mathbf{R}^{(\ell,\beta)}\mathbf{R}^{(j,\gamma)}-\mathbf{R}^{(j,\gamma)}\mathbf{R}^{(\ell,\beta)}\right)\right.\\
&\qquad\left.+(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}-(\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right)\right)=\\
&= \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)-\mathbf{L}^{(i,\alpha)}\left((\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right)=\\
&= \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right).
\end{aligned}
$$

# . . . Proof

Therefore:

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0 \quad\Leftrightarrow\quad \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\mathbf{R}^{(j,\gamma)}=0,$$

and

$$\mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right)=0 \quad\Leftrightarrow$$
$$\Leftrightarrow\quad \mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right)=0.$$

Q.E.D.

## Theorem [Full Decoupling]

For a hyperbolic system of first order homogeneous and autonomous quasilinear PDEs to be locally reducible into $k$ non–interacting subsystems of some orders $n_1, \dots, n_k$, with $n_1 + \dots + n_k = n$, it is necessary and sufficient that:

1. its characteristic velocities, and the corresponding left and right eigenvectors can be divided into $k$ subsets;
2. the following structure conditions hold true:

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,\quad \mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(i,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(i,\beta)}\right)=0,$$
$$\forall\, i,j=1,\dots,k,\quad i\neq j,\quad \alpha,\beta=1,\dots,n_i,\quad \alpha\neq\beta,\quad \gamma=1,\dots,n_j.$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$, are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,$$

where $i,j=1,\dots,k,\quad i\neq j,\quad \alpha=1,\dots,n_i,\quad \gamma=1,\dots,n_j$.
The coefficient matrix for a fully decoupled system results in block–diagonal form (diagonal if $k=n$).

# Physical Meaning

The structure conditions

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\mathbf{r}^{(j,\gamma)}=0,$$

$$\mathbf{l}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(i,\beta)})\mathbf{r}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(i,\beta)}\right)=0,$$

$$\forall\, i,j=1,\dots,k,\quad i\neq j,\quad \alpha,\beta=1,\dots,n_i,\quad \alpha\neq\beta,\quad \gamma=1,\dots,n_j,$$

guaranteeing the decoupling of a hyperbolic first order quasilinear system in $k$ non–interacting subsystems, have the following (obvious) meaning:

1. the change in the characteristic speeds of a subsystem across a wave of a different subsystem must be vanishing;
2. waves of different subsystems do not interact.

# General (not necessarily hyperbolic) Homogeneous and Autonomous Systems

## Definition

Let $A$ be an $n \times n$ real matrix whose entries are smooth functions depending on $\mathbf{u} \in \mathbb{R}^n$. If the matrix $A$ has not a complete set of eigenvectors and/or has complex–valued eigenvalues, let us associate:

- to each real eigenvalue its (left and right) eigenvectors and, if needed, its generalized (left and right) eigenvectors in such a way we have as many linearly independent vectors as the multiplicity of the eigenvalue;
- to each couple of conjugate complex eigenvalues the real part and the imaginary part of its (left and right) eigenvectors (or generalized eigenvectors, if needed) in such a way we have as many couples of linearly independent vectors as the multiplicity of the conjugate complex eigenvalues.

# Nonhomogeneous and/or Nonautonomous Systems

Three different situations may occur:

1. System can be mapped to an equivalent autonomous and homogeneous form. It is required that it admits as subalgebra of its algebra of Lie point symmetries a three–dimensional solvable Lie algebra spanned by the vector fields

$$\Xi_i = \tau_i(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x} + \sum_{j=1}^{n} \eta_i^j(t,x,\mathbf{u})\frac{\partial}{\partial u_j}, \qquad i = 1,\dots,3;$$

2. System can be mapped to an equivalent autonomous and nonhomogeneous form. It is required that it admits as subalgebra of its algebra of Lie point symmetries a two–dimensional abelian Lie algebra spanned by the vector fields

$$\Xi_i = \tau_i(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x} + \sum_{j=1}^{n} \eta_i^j(t,x,\mathbf{u})\frac{\partial}{\partial u_j}, \qquad i = 1,2;$$

3. System can not be transformed to autonomous form.

## Theorem [Partial Decoupling for Quasilinear Systems]

The first order quasilinear system

$$\frac{\partial \mathbf{u}}{\partial t} + A(t, x, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}), \quad \mathbf{u}, \mathbf{g} \in \mathbb{R}^n, \quad A \quad n \times n \text{ real matrix}$$

can be transformed by a smooth (locally) invertible transformation

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \text{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u}),$$

into a system like

$$\frac{\partial \mathbf{U}}{\partial t} + T(t, x, \mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{G}(t, x, \mathbf{U}),$$

where $T = (\nabla_{\mathbf{u}}\mathbf{H})\,A\,(\nabla_{\mathbf{u}}\mathbf{H})^{-1}$, $\mathbf{G} = (\nabla_{\mathbf{u}}\mathbf{H})\mathbf{g}$, such that $T_j^i$ and $G^{(i,\alpha)}$ $(i = 1, \dots, k;\ j = 1, \dots, i;\ \alpha = 1, \dots, n_i)$ depend at most on $t$, $x$ and the elements of the set $\mathcal{U}_i$, respectively, if and only if:

...

## ...Theorem [Partial Decoupling for Quasilinear Systems]

1. the set of the eigenvalues of matrix $A$, and the associated left and right *autovectors* can be divided into $k$ subsets each containing $n_i$ $(i = 1, \dots, k)$ elements;
2. the following structure conditions hold true:

$$\left(\nabla_{\mathbf{u}} \lambda^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0, \qquad \left(\nabla_{\mathbf{u}} (\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0,$$

$$\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \left( (\nabla_{\mathbf{u}} \widehat{\mathbf{r}}^{(\ell,\beta)}) \widehat{\mathbf{r}}^{(j,\gamma)} - (\nabla_{\mathbf{u}} \widehat{\mathbf{r}}^{(j,\gamma)}) \widehat{\mathbf{r}}^{(\ell,\beta)} \right) = 0,$$

$$i = 1, \dots, k-1, \quad \ell = 1, \dots, i, \quad j = i+1, \dots, k,$$

$$\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \text{ if } i = \ell, \quad \gamma = 1, \dots, n_j.$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$ $(i = 1, \dots, k-1;$ $\alpha = 1, \dots, n_i)$ are found from

$$\left(\nabla_{\mathbf{u}} H^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0, \qquad j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.$$

# Proof

It is

$$
\begin{aligned}
0 &= \left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = \\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) (\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\widehat{\mathbf{R}}^{(j,\gamma)} = \\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)} = \\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)} \cdot \mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)}.
\end{aligned}
$$

Therefore,

$$
\left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0 \quad \Leftrightarrow \quad \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)} \cdot \mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)} = 0.
$$

Conditions

$$
\left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)} \cdot \mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)} = 0
$$

are necessary and sufficient in order the components $\{\mathbf{G}^{(r,1)}, \dots, \mathbf{G}^{(r,n_r)}\}$ to be dependent at most on $t$, $x$ and the elements of the set $\mathcal{U}_r$.

## Theorem [Full Decoupling for Quasilinear Systems]

For a first order nonhomogeneous and/or nonautonomous quasilinear system to be locally reducible into $k$ non–interacting subsystems of some orders $n_1, \ldots, n_k$, with $n_1 + \ldots + n_k = n$, it is necessary and sufficient that:

1. the eigenvalues of the coefficient matrix, and the corresponding left and right autovectors can be divided into $k$ subsets;
2. the following structure conditions hold true:

$$
\begin{gathered}
\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)} = 0, \qquad \left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)}\cdot\mathbf{g})\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)} = 0,\\
\widehat{\mathbf{l}}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(i,\beta)})\widehat{\mathbf{r}}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(j,\gamma)})\widehat{\mathbf{r}}^{(i,\beta)}\right) = 0,\\
\forall\, i,j = 1,\ldots,k, \quad i \neq j, \quad \alpha,\beta = 1,\ldots,n_i, \quad \alpha\neq\beta, \quad \gamma = 1,\ldots,n_j.
\end{gathered}
$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$, are found from

$$
\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)} = 0,
$$

where

$$
i,j = 1,\ldots,k, \quad i\neq j, \quad \alpha = 1,\ldots,n_i, \quad \gamma = 1,\ldots,n_j.
$$

# One–dimensional isentropic gas dynamics equations

Governing equation:

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

with

$$\mathbf{u} = \left[\begin{array}{c} \rho \\ v \\ s \end{array}\right], \qquad A = \left[\begin{array}{ccc} v & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial s} \\ 0 & 0 & v \end{array}\right],$$

$\rho(t,x)$ mass density, $v(t,x)$ velocity, $s(t,x)$ entropy, and $p(\rho,s)$ pressure.
Eigenvalues of matrix $A$:

$$\lambda_{1,2} = v \pm \sqrt{\frac{\partial p}{\partial \rho}}, \qquad \lambda_3 = v,$$

and associated left (right resp.) eigenvectors

$$\mathbf{l}_{1,2} = \left(\sqrt{\frac{\partial p}{\partial \rho}}, \pm\rho, \frac{\rho}{s}\sqrt{\frac{\partial p}{\partial \rho}}\right), \qquad \mathbf{l}_3 = (0,0,1),$$

# . . . One–dimensional isentropic gas dynamics equations

$$\mathbf{r}_{1,2} = \begin{pmatrix} \rho \\ \pm\sqrt{\frac{\partial p}{\partial \rho}} \\ 0 \end{pmatrix}, \qquad \mathbf{r}_3 = \begin{pmatrix} \rho \\ 0 \\ -s \end{pmatrix}.$$

The relations

$$(\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_2 = 0, \qquad (\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_3 = 0,$$
$$(\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_1 = 0, \qquad (\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_3 = 0,$$

are satisfied with the constitutive law

$$p(\rho, s) = p_0\rho^3 s^2 + f(s),$$

with $p_0$ constant and $f(s)$ function of its argument.

# . . . One–dimensional isentropic gas dynamics equations

Using

$$(\nabla_{\mathbf{u}} H_1)\cdot \mathbf{r}_2 = 0, \qquad (\nabla_{\mathbf{u}} H_1)\cdot \mathbf{r}_3 = 0,$$
$$(\nabla_{\mathbf{u}} H_2)\cdot \mathbf{r}_1 = 0, \qquad (\nabla_{\mathbf{u}} H_2)\cdot \mathbf{r}_3 = 0,$$

we choose the new field variables as

$$\begin{aligned} U_1 &= H_1(\rho, v, s) = v + \sqrt{3p_0}\rho s, \\ U_2 &= H_2(\rho, v, s) = v - \sqrt{3p_0}\rho s, \\ U_3 &= H_3(\rho, v, s) = s, \end{aligned}$$

and obtain the following partially decoupled system

$$\begin{cases} \dfrac{\partial U_1}{\partial t} + U_1 \dfrac{\partial U_1}{\partial x} = 0, \\ \dfrac{\partial U_2}{\partial t} + U_2 \dfrac{\partial U_2}{\partial x} = 0, \\ \dfrac{\partial U_3}{\partial t} + \dfrac{1}{2}(U_1 + U_2) \dfrac{\partial U_3}{\partial x} = 0. \end{cases}$$

# Model of a travelling threadline

Governing equation[1]:

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ V^x \\ v \\ \varepsilon \end{pmatrix}, \qquad A = \begin{bmatrix} V^x & \rho & 0 & 0 \\ \frac{-T'}{\rho(1+\varepsilon^2)} & V^x & 0 & \frac{\varepsilon}{1+\varepsilon^2)^2}\left(T' + \frac{T}{m}\right) \\ 0 & 0 & 2V^x & (V^x)^2 - \frac{T}{m(1+\varepsilon^2)} \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$\rho$ mass density, $V^x$ and $v$ the components of velocity, $\varepsilon$ transverse displacement and $T(m)$ tension, $\rho = m\sqrt{1+\varepsilon^2}$, $T'(m) < 0$.

[1] W. F. Ames, S. Y. Lee, J. N. Zaiser. Int. J. Non-Linear Mech., **3**, 449–469, 1968.

# . . . Model of a travelling threadline

Eigenvalues of matrix $A$

$$\lambda_{1,2} = V^x \pm \left(\frac{-T'}{1+\varepsilon^2}\right)^{1/2}, \qquad \lambda_{3,4} = V^x \pm \left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2},$$

with associated left and right eigenvectors

$$\mathbf{l}_{1,2} = \left(\pm\frac{\sqrt{-(1+\varepsilon^2)T'}}{\rho\varepsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}\right)}, \frac{1+\varepsilon^2}{\varepsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}\right)}, \frac{1}{V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}}, 1\right),$$

$$\mathbf{l}_{3,4} = \left(0, 0, \rho, \rho V^x \pm \sqrt{\frac{\rho T}{(1+\varepsilon^2)^{1/2}}}\right),$$

$$\mathbf{r}_{1,2} = \begin{pmatrix} \rho \\ \pm\left(\frac{-T'}{1+\varepsilon^2}\right)^{1/2} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_{3,4} = \begin{pmatrix} \frac{\rho\varepsilon}{1+\varepsilon^2} \\ \pm\left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2}\frac{\varepsilon}{1+\varepsilon^2} \\ -\left(V^x \pm \frac{T}{m(1+\varepsilon^2)}\right)^{1/2} \\ 1 \end{pmatrix}.$$

# . . . Model of a travelling threadline

The structure conditions

$$
\begin{aligned}
&(\nabla_{\mathbf{u}}\lambda_i)\cdot\mathbf{r}_j = 0,\\
&\mathbf{l}_i\cdot((\nabla_{\mathbf{u}}\mathbf{r}_\ell)\mathbf{r}_j - (\nabla_{\mathbf{u}}\mathbf{r}_j)\mathbf{r}_\ell) = 0, \qquad i,\ell = 1,2, \quad i\neq\ell, \quad j = 3,4,
\end{aligned}
$$

are satisfied with the following constitutive law

$$T(m) = \frac{k}{m}, \qquad k \ \text{ constant.}$$

By introducing $\mathbf{U} = \mathbf{H}(\mathbf{u})$ such that

$$(\nabla_{\mathbf{u}} H_i)\cdot\mathbf{r}_j = 0, \qquad\qquad i, = 1,2, \quad j = 3,4,$$

*i.e.*,

$$
\begin{aligned}
&\left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \varepsilon} = 0, \quad \left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \varepsilon} = 0,\\
&\left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \varepsilon} = 0, \quad \left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \varepsilon} = 0,
\end{aligned}
$$

# . . . Model of a travelling threadline

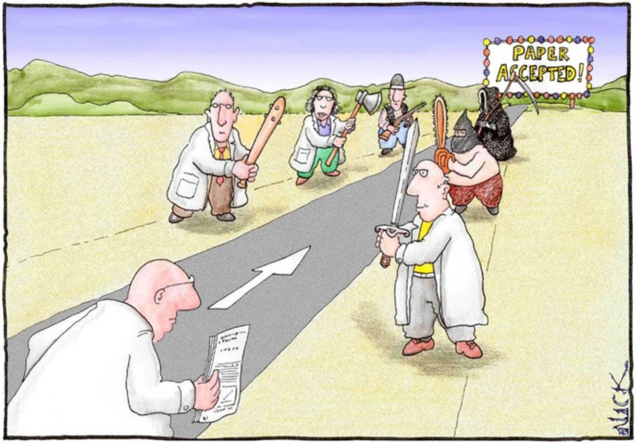
we get

$$H_1 = H_1\left(\rho, V^x\right), \qquad H_2 = H_2\left(\rho, V^x\right).$$

By choosing the identity transformation, we obtain this partially decoupled system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + V^x \dfrac{\partial \rho}{\partial x} + \rho \dfrac{\partial V^x}{\partial x} = 0, \\ \dfrac{\partial V^x}{\partial t} + \dfrac{k}{\rho^3}\dfrac{\partial \rho}{\partial x} + V^x \dfrac{\partial V^x}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + 2V^x \dfrac{\partial v}{\partial x} + \left(\left(V^x\right)^2 - \dfrac{k}{\rho^2}\right)\dfrac{\partial \varepsilon}{\partial x} = 0, \\ \dfrac{\partial \varepsilon}{\partial t} - \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

M. Gorgone, F. Oliveri, M. P. Speciale. On the decoupling problem of general quasilinear first order systems in two independent variables. J. Math. Anal. Appl., **446**, 276–298, 2017.




Most scientists regarded the new streamlined peer-review process as "quite an improvement."

# TH.

# ANKS