Approximate Noether symmetries and approximate conservation laws

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General framework of conservations laws

Lie groups methods to construct conservation laws:

- Variational systems $\longrightarrow$ Noether's theorem [Noether, 1918];
- Extension of Noether's theorem with generalized symmetries [Boyer, 1967];
- Extension of Noether's theorem with differential operators [Vinogradov, 1984];
- Formal Lagrangian and self–adjointness [Ibragimov, 2007];
- Partial Lagrangians [Kara, Mahomed, 2006]
- ...

Not via Lie symmetries...

In general, for DEs not admitting a variational principle, conservation laws may be found through a direct approach [Anco, Bluman, 2002].

Approximate Lie symmetries

Given a differential equation involving a small parameter $\varepsilon \ll 1$,

$$\Delta\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right) = 0,$$

$\mathrm{x} \in X \subseteq \mathbb{R}^n$, $\mathrm{u} \in U \subseteq \mathbb{R}^m$ are the independent variable and dependent variables, $\mathrm{u}^{(r)}$ the derivatives of u w.r.t. x up to the order r, it is not uncommon that this equation possesses few symmetries compared with the unperturbed equation

$$\Delta\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; 0\right) = 0.$$

The applicability of Lie group methods is limited!

Remark

Differential equations containing small terms are commonly and successfully investigated by means of perturbative techniques!

Approximate symmetry theories

Baikov, Gazizov, Ibragimov, Mat. Sb., 1988

Considering a differential equation involving a small parameter

$$\Delta\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\widetilde{\Delta}_{(k)}\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(r)}\right)=O(\varepsilon^{p+1}),$$

the Lie generator is expanded in a perturbation series:

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathrm{x},\mathrm{u};\varepsilon)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathrm{x},\mathrm{u};\varepsilon)\frac{\partial}{\partial u_\alpha}\equiv\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}\widetilde{\xi}_{(k)i}(\mathrm{x},\mathrm{u})\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\widetilde{\eta}_{(k)\alpha}(\mathrm{x},\mathrm{u})\frac{\partial}{\partial u_\alpha}\right).$$

Then, the approximate invariance is defined:

$$\Xi^{(r)}(\Delta)\Big|_{\Delta=O(\varepsilon^{p+1})}=O(\varepsilon^{p+1}).$$

- Pros: quite elegant theory, since all the useful properties of exact Lie symmetries are moved to the approximate world;
- Cons: the expanded generator is not consistent with principles of perturbation analysis since the dependent variables are not expanded!

Approximate symmetry theories

Fushchich and Shtelen, J. Phys. A., 1989

The dependent variables are expanded in a perturbation series as done in usual perturbation analysis:

$$\mathrm{u}(\mathrm{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}_{(k)}(\mathrm{x}) + O(\varepsilon^{p+1});$$

by separating at each order of approximation, a coupled system to be solved in hierarchy is obtained:

$$\widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}_{(0)}, \mathrm{u}_{(0)}^{(r)}, \dots, \mathrm{u}_{(k)}, \mathrm{u}_{(k)}^{(r)}\right) = 0, \qquad k = 0, \dots, p.$$

Approximate symmetries of the original DE defined as the *exact symmetries* of the DE obtained from perturbations!

- Pros: approach with a simple and coherent basis.
- Cons: a lot of algebra (especially for higher-order perturbations) is required; the basic assumption of a fully coupled system is too strong, since the equations at a level should not be influenced by those at higher levels. No possibility to work in a hierarchy!

A consistent approach[1]

Consider DEs containing a small term ε,

$$\Delta\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right) = 0,$$

and take a Lie generator with infinitesimals depending on ε,

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathrm{x}, \mathrm{u}; \varepsilon)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathrm{x}, \mathrm{u}; \varepsilon)\frac{\partial}{\partial u_\alpha}.$$

Expand the dependent variables in power series of ε

$$\mathrm{u}(\mathrm{x}; \varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}_{(k)}(\mathrm{x}) + O(\varepsilon^{p+1}),$$

whereupon DEs write as

$$\Delta \equiv \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)}\left(\mathrm{x}, \mathrm{u}_{(0)}, \mathrm{u}_{(0)}^{(r)}, \ldots, \mathrm{u}_{(k)}, \mathrm{u}_{(k)}^{(r)}\right) = O(\varepsilon^{p+1}).$$

[1] Di Salvo, Gorgone, Oliveri, Nonlinear Dyn., 2018

Expansions of infinitesimals

$$\xi_i \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\xi}_{(k)i}, \qquad \eta_\alpha \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\eta}_{(k)\alpha},$$

where

$$\begin{aligned}
\widetilde{\xi}_{(0)i} &= \xi_{(0)i} = \xi_i(\mathrm{x}, \mathrm{u}_{(0)}; 0), & \widetilde{\eta}_{(0)\alpha} &= \eta_{(0)\alpha} = \eta_\alpha(\mathrm{x}, \mathrm{u}_{(0)}; 0),\\
\widetilde{\xi}_{(k+1)i} &= \frac{1}{k+1}\mathcal{R}[\widetilde{\xi}_{(k)i}], & \widetilde{\eta}_{(k+1)\alpha} &= \frac{1}{k+1}\mathcal{R}[\widetilde{\eta}_{(k)\alpha}],
\end{aligned}$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives and such that

$$\begin{aligned}
&\mathcal{R}\left[\frac{\partial^{|\tau|} f_{(k)}(\mathrm{x}, \mathrm{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right] = \frac{\partial^{|\tau|} f_{(k+1)}(\mathrm{x}, \mathrm{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}} + \sum_{i=1}^{m} \frac{\partial}{\partial u_{(0)i}} \left(\frac{\partial^{|\tau|} f_{(k)}(\mathrm{x}, \mathrm{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right) u_{(1)i},\\
&\mathcal{R}[u_{(k)j}] = (k+1) u_{(k+1)j},
\end{aligned}$$

where $k \geq 0$, $j = 1, \dots, m$, $|\tau| = \tau_1 + \dots + \tau_m$.

...

We get the approximate Lie generator

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathrm{x}, \mathrm{u}_{(0)}, \dots, \mathrm{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathrm{x}, \mathrm{u}_{(0)}, \dots, \mathrm{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right).$$

Then define prolongations in the usual way (*i.e.*, preserving contact conditions) and impose the approximate invariance condition:

$$\sum_{k=0}^{p} \varepsilon^k \sum_{\ell=0}^{k} \widetilde{\Xi}^{(r)}_{(\ell)} \widetilde{\Delta}_{(k-\ell)} \Big|_{\Delta = O(\varepsilon^{p+1})} = O(\varepsilon^{p+1}).$$

Remark

The approximate Lie point symmetries of a DE are the elements of an approximate Lie algebra.

Computational cost

The consistent approach requires more computations than that required for determining exact Lie symmetries; nevertheless, there is the general and freely available package ReLie[a] able to do automatically all the needed work.

[a] Oliveri, Symmetry, 2021

Approximate conservation laws

Given a system of DEs,

$$\Delta\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(r)};\varepsilon\right)\equiv\sum_{k=0}^{p}\varepsilon^k\widetilde{\Delta}_{(k)}\left(\mathrm{x},\mathrm{u}_{(0)},\mathrm{u}_{(0)}^{(r)},\dots,\mathrm{u}_{(k)},\mathrm{u}_{(k)}^{(r)}\right)=O(\varepsilon^{p+1}),$$

an approximate conservation law of order r, compatible with the system, is a divergence expression

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}D_i\left(\widetilde{\Phi}^i_{(k)}\left(\mathrm{x},\mathrm{u}_{(0)},\mathrm{u}_{(0)}^{(r-1)},\dots,\mathrm{u}_{(k)},\mathrm{u}_{(k)}^{(r-1)}\right)\right)\right)=O(\varepsilon^{p+1}),$$

holding for all solutions of the system, where

$$\sum_{k=0}^{p}\varepsilon^k\widetilde{\Phi}^i_{(k)}\left(\mathrm{x},\mathrm{u}_{(0)},\mathrm{u}_{(0)}^{(r-1)},\dots,\mathrm{u}_{(k)},\mathrm{u}_{(k)}^{(r-1)}\right),\qquad i=1,\dots,n$$

are the expansions at order p of the fluxes $\Phi^i\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(r-1)};\varepsilon\right)$ of the conservation law, and

$$D_i=\frac{D}{Dx_i}=\frac{\partial}{\partial x_i}+\sum_{k=0}^{p}\sum_{\alpha=1}^{m}\left(u_{(k)\alpha,i}\frac{\partial}{\partial u_{(k)\alpha}}+\sum_{j=1}^{n}u_{(k)\alpha,ij}\frac{\partial}{\partial u_{(k)\alpha,j}}+\dots\right)$$

is the approximate Lie derivative, with $u_{(k)\alpha,i}=\dfrac{\partial u_{(k)\alpha}}{\partial x_i}$, $u_{(k)\alpha,ij}=\dfrac{\partial^2 u_{(k)\alpha}}{\partial x_i\partial x_j}$, ...

Approximate variational problems

Unperturbed variational problems

The determination of CLaws is ruled by Noether's theorem, establishing a correspondence between symmetries of the action integral and conservation laws through an explicit formula involving the infinitesimals and the Lagrangian itself. The same can be done in the approximate framework!

Perturbed first order Lagrangian function and Lagrangian action

$$\mathcal{L}\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(1)};\varepsilon\right) \equiv \mathcal{L}_0\left(\mathrm{x},\mathrm{u}_{(0)},\mathrm{u}^{(1)}_{(0)}\right) + \sum_{k=1}^{p}\varepsilon^k\mathcal{L}_k\left(\mathrm{x},\mathrm{u}_{(0)},\dots,\mathrm{u}_{(k)},\mathrm{u}^{(1)}_{(0)},\dots,\mathrm{u}^{(1)}_{(k)}\right) + O(\varepsilon^{p+1})$$

$$\mathcal{J}\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(1)};\varepsilon\right) = \int_\Omega \mathcal{L}\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(1)};\varepsilon\right)d\mathrm{x} \equiv \int_\Omega\left(\sum_{k=0}^{p}\varepsilon^k\mathcal{L}_k\left(\mathrm{x},\mathrm{u}_{(0)},\dots,\mathrm{u}_{(k)},\mathrm{u}^{(1)}_{(0)},\dots,\mathrm{u}^{(1)}_{(k)}\right)\right)d\mathrm{x} + O(\varepsilon^{p+1})$$

Approximate Euler–Lagrange equations

By requiring $\delta\mathcal{J} = O(\varepsilon^{p+1})$ under variations of order $O(\varepsilon^{p+1})$ at the boundary of Ω, we obtain

$$\sum_{k=0}^{p}\varepsilon^k\left(\frac{\partial\mathcal{L}_k}{\partial u_{(0)\alpha}} - \sum_{i=1}^{n}D_i\left(\frac{\partial\mathcal{L}_k}{\partial u_{(0)\alpha,i}}\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha = 1,\dots,m.$$

Approximate Noether theorem[2]

Let us consider a variational system of DEs arising from a first order perturbed Lagrangian function.
The generator

$$\Xi = \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathrm{x}, \mathrm{u}_{(0)}, \dots, \mathrm{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathrm{x}, \mathrm{u}_{(0)}, \dots, \mathrm{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right).$$

of an approximate Lie symmetry leaves the Lagrangian action approximately invariant if

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{j=0}^{k} \left(\widetilde{\Xi}^{(1)}_{(j)} \mathcal{L}_{k-j} + \mathcal{L}_{k-j} \sum_{i=1}^{n} D_i \widetilde{\xi}_{(j)i} \right) - \sum_{i=1}^{n} D_i \phi^i_{(k)} \right) = O(\varepsilon^{p+1}),$$

with $\phi^i_{(k)}\left(\mathrm{x}, \mathrm{u}_{(0)}, \dots, \mathrm{u}_{(k)}\right)$ $(i = 1, \dots, n)$ functions to be suitably determined.
Then, we obtain the approximate conservation law

$$\sum_{k=0}^{p} \varepsilon^k \sum_{i=1}^{n} D_i \widetilde{\Phi}^i_{(k)} = O(\varepsilon^{p+1}),$$

where

$$\widetilde{\Phi}^i_{(k)} = \sum_{\ell=0}^{k} \left(\sum_{\alpha=1}^{m} \left(\left(\widetilde{\eta}_{(\ell)\alpha} - \sum_{j=1}^{n} \widetilde{\xi}_{(\ell)j} u_{(\ell)\alpha,j} \right) \sum_{q=0}^{k-\ell} \frac{\partial \mathcal{L}_{k-\ell}}{\partial u_{(q)\alpha,i}} \right) + \widetilde{\xi}_{(\ell)i} \mathcal{L}_{k-\ell} \right) - \phi^i_{(k)}.$$

[2] Gorgone, Oliveri, Mathematics, 2021

The planar three–body problem

Motion equations:

$$\ddot{r}_1 + Gm_2\frac{\mathrm{r}_{12}}{|\mathrm{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathrm{r}_{13}}{|\mathrm{r}_{13}|^3} = 0,$$
$$\ddot{r}_2 - Gm_1\frac{\mathrm{r}_{12}}{|\mathrm{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathrm{r}_{23}}{|\mathrm{r}_{23}|^3} = 0,$$
$$\ddot{r}_3 - Gm_1\frac{\mathrm{r}_{13}}{|\mathrm{r}_{13}|^3} - Gm_2\frac{\mathrm{r}_{23}}{|\mathrm{r}_{23}|^3} = 0,$$

with $\mathrm{r}_i \equiv (x_i(t), y_i(t), 0)$ $(i = 1, 2, 3)$ position vectors of the three masses m_α in a fixed frame reference, and $\mathrm{r}_{ij} = \mathrm{r}_i - \mathrm{r}_\mathrm{j}$ $(1 \le i < j \le 3)$. The system arises from the Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left(m_1\dot{\mathrm{r}}_1^2 + m_2\dot{\mathrm{r}}_2^2\right) + \frac{Gm_1m_2}{|\mathrm{r}_{12}|} + \varepsilon\left(m_3\dot{\mathrm{r}}_3^2 + \frac{Gm_1m_3}{|\mathrm{r}_{13}|} + \frac{Gm_2m_3}{|\mathrm{r}_{23}|}\right).$$

By expanding the dependent variables at first order in ε, i.e.,

$$\mathrm{r}_i = \mathrm{r}_{(0)i} + \varepsilon\mathrm{r}_{(1)i} + O(\varepsilon^2) \equiv \left(x_{(0)i}(t) + \varepsilon x_{(1)i}(t) + O(\varepsilon^2), y_{(0)i}(t) + \varepsilon y_{(1)i}(t) + O(\varepsilon^2), 0\right),$$

along with

$$\mathrm{r}_{ij} = \mathrm{r}_{(0)ij} + \varepsilon\mathrm{r}_{(1)ij} + O(\varepsilon^2) = \mathrm{r}_{(0)i} - \mathrm{r}_{(0)j} + \varepsilon\left(\mathrm{r}_{(1)i} - \mathrm{r}_{(1)j}\right) + O(\varepsilon^2),$$

we are able to determine the approximate variational Lie symmetries, together with the corresponding approximate conserved quantities.

The planar three–body problem – Results

- From

$$\Xi_1 = \frac{\partial}{\partial t}, \qquad \phi_{(0)} = \phi_{(1)} = 0,$$

we have the approximate conservation of total energy:

$$\Phi_1 = \frac{1}{2}\left(m_1\dot{\mathrm{r}}^2_{(0)1} + m_2\dot{\mathrm{r}}^2_{(0)2}\right) - \frac{Gm_1m_2}{|\mathrm{r}_{(0)12}|}$$
$$+\varepsilon\left(\frac{1}{2}m_3\dot{\mathrm{r}}^2_{(0)3} + m_1\dot{\mathrm{r}}_{(0)1}\cdot\dot{\mathrm{r}}_{(1)1} + m_2\dot{\mathrm{r}}_{(0)2}\cdot\dot{\mathrm{r}}_{(1)2} - \frac{Gm_1m_3}{|\mathrm{r}_{(0)13}|} - \frac{Gm_2m_3}{|\mathrm{r}_{(0)23}|} + \frac{Gm_1m_2}{|\mathrm{r}_{(0)23}|^3}\mathrm{r}_{(0)12}\cdot\mathrm{r}_{(1)12}\right);$$

- From

$$\Xi_{2a} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}, \qquad \Xi_{2b} = \sum_{i=1}^{3}\frac{\partial}{\partial y_i}, \qquad \phi_{(0)} = \phi_{(1)} = 0,$$

we have the approximate conservation of total linear momentum:

$$\Phi_2 = m_1\dot{\mathrm{r}}_{(0)1} + m_2\dot{\mathrm{r}}_{(0)2} + \varepsilon\left(m_1\dot{\mathrm{r}}_{(1)1} + m_2\dot{\mathrm{r}}_{(1)2} + m_3\dot{\mathrm{r}}_{(0)3}\right);$$

The planar three–body problem – Results

- From

$$\Xi_{3a} = t\sum_{i=1}^{3}\frac{\partial}{\partial x_i}, \qquad \phi_{(0)} = -\sum_{i=1}^{2} m_i x_{(0)i}, \quad \phi_{(1)} = -\sum_{i=1}^{2} m_i x_{(1)i} - m_3 x_{(0)3},$$
$$\Xi_{3b} = t\sum_{i=1}^{3}\frac{\partial}{\partial y_i}, \qquad \phi_{(0)} = -\sum_{i=1}^{2} m_i y_{(0)i}, \quad \phi_{(1)} = -\sum_{i=1}^{2} m_i y_{(1)i} - m_3 y_{(0)3},$$

  we have

$$\begin{aligned}\Phi_3 &= m_1(t\dot{\mathrm{r}}_{(0)1} - \mathrm{r}_{(0)1}) + m_2(t\dot{\mathrm{r}}_{(0)2} - \mathrm{r}_{(0)2})\\ &\quad + \varepsilon\left(m_1(t\dot{\mathrm{r}}_{(1)1} - \mathrm{r}_{(1)1}) + m_2(t\dot{\mathrm{r}}_{(1)2} - \mathrm{r}_{(1)2}) + m_3(t\dot{\mathrm{r}}_{(0)3} - \mathrm{r}_{(0)3})\right),\end{aligned}$$

  i.e., the approximate barycenter of the system has a uniform and rectilinear motion;

- From

$$\Xi_4 = \sum_{i=1}^{3}\left(\left(y_{(0)i} + \varepsilon y_{(1)i}\right)\frac{\partial}{\partial x_i} - \left(x_{(0)i} + \varepsilon x_{(1)i}\right)\frac{\partial}{\partial y_i}\right), \qquad \phi_{(0)} = \phi_{(1)} = 0,$$

  we have the approximate conservation of total angular momentum:

$$\begin{aligned}\Phi_4 &= m_1 \mathrm{r}_{(0)1}\wedge\dot{\mathrm{r}}_{(0)1} + m_2 \mathrm{r}_{(0)2}\wedge\dot{\mathrm{r}}_{(0)2}\\ &\quad + \varepsilon\left(m_1\left(\mathrm{r}_{(0)1}\wedge\dot{\mathrm{r}}_{(1)1} + \mathrm{r}_{(1)1}\wedge\dot{\mathrm{r}}_{(0)1}\right) + m_2\left(\mathrm{r}_{(0)2}\wedge\dot{\mathrm{r}}_{(1)2} + \mathrm{r}_{(1)2}\wedge\dot{\mathrm{r}}_{(0)2}\right) + m_3 \mathrm{r}_{(0)3}\wedge\dot{\mathrm{r}}_{(0)3}\right).\end{aligned}$$

Perturbed non variational problems

Direct method[3]: the problem

Given a system of DEs,

$$\Delta\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right) = 0,$$

we want to determine a set of non–singular multipliers (*i.e.*, non–singular when evaluated on solutions of the system) $\Lambda^k\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right)$ $(k = 1, \dots, q)$ provided that

$$\sum_{k=1}^{q}\left(\Lambda^k\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right)\Delta^k\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r)}; \varepsilon\right)\right) = \sum_{i=1}^{n} D_i\left(\Phi^i\left(\mathrm{x}, \mathrm{u}, \mathrm{u}^{(r-1)}; \varepsilon\right)\right) = O(\varepsilon^{p+1})$$

is a divergence expression, for some Φ^i, holding for all solutions of the system.

Key aspects of the direct approach

- The Euler operators annihilate any divergence expression;
- The only scalar expressions annihilated by Euler operators are divergence expressions.

The multipliers can be found algorithmically solving a linear system of determining equations.

[3] Bluman, Anco, Eur. J. Appl. Math., 2002

Approximate direct method with the consistent approach: algorithm[4]

- Expand the dependent variables in power series of ε: $\mathrm{u}(\mathrm{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathrm{u}_{(k)}(\mathrm{x}) + O(\varepsilon^{p+1})$;
- Expand in perturbation series of ε the multipliers, so obtaining:

$$\Lambda^i\left(\mathrm{x},\mathrm{u},\mathrm{u}^{(r)};\varepsilon\right) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\Lambda}^i_{(k)}\left(\mathrm{x},\mathrm{u}_{(0)},\mathrm{u}^{(r)}_{(0)},\dots,\mathrm{u}_{(k)},\mathrm{u}^{(r-1)}_{(k)}\right), \qquad i=1,\dots,q$$

- Apply the approximate Euler operators, *i.e.*,

$$E_{u_{(0)\alpha}}\left(\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{i=1}^{q}\left(\widetilde{\Lambda}^i_{(\ell)}\widetilde{\Delta}^i_{(k-\ell)}\right)\right)\right) = O(\varepsilon^{p+1}), \qquad \alpha=1,\dots,m$$

  where $E_{u_{(0)\alpha}} = \frac{\partial}{\partial u_{(0)\alpha}} - \sum_{i=1}^{n} D_i\left(\frac{\partial}{\partial u_{(0)\alpha,i}}\right) + \dots + (-1)^s \sum_{i_1=1}^{n}\dots\sum_{i_s=i_{s-1}}^{n} D_{i_1}\dots D_{i_s}\left(\frac{\partial}{\partial u_{(0)\alpha,i_1\dots i_s}}\right)$;
- Separate the resulting conditions at each order in ε, and split into an overdetermined system for the unknown approximate multiplers;
- Insert the recovered approximate multipliers into the identity

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{\ell=0}^{k}\sum_{i=1}^{q}\left(\widetilde{\Lambda}^i_{(\ell)}\widetilde{\Delta}^i_{(k-\ell)}\right) - \sum_{j=1}^{n} D_j\widetilde{\Phi}^j_{(k)}\right) = O(\varepsilon^{p+1})$$

  and, if possible, find the approximate fluxes.

[4] Gorgone, Inferrera, 2021

Approximate direct method – Results

By means of the approximate direct method, approximate conservation laws have been determined for:

- Perturbed Van der Pool equation:
$$\ddot{u} + u - \varepsilon\left(1 - u^2\right)\dot{u} = 0;$$
- Perturbed KdV–Burgers equation:
$$u_{,t} + uu_{,x} + u_{,xxx} - \varepsilon u_{,xx} = 0;$$
- A perturbed nonlinear wave equation:
$$u_{,xx} - \frac{1}{c^2}u_{,tt} - \lambda u^3 - \varepsilon f(u) = 0;$$
- Two perturbed nonlinear Schrödinger equations:
$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon|p|^4 p = 0;$$
$$\mathrm{i}p_{,t} + \frac{1}{2}p_{,xx} + |p|^2 p + \mathrm{i}\varepsilon\left(\beta_1 p_{,xxx} + \beta_2|p|^2 p_{,x} + \beta_3 p(|p|^2)_{,x}\right) = 0;$$
- The generalized Kaup–Newell equation:
$$u_{,t} - \frac{1}{2}u_{,xx} + uvu_{,x} + \frac{1}{2}u^2 v_{,x} + 2\varepsilon uu_{,x} = 0,$$
$$v_{,t} + \frac{1}{2}v_{,xx} + uvv_{,x} + \frac{1}{2}v^2 u_{,x} + 2\varepsilon(vu_{,x} + uv_{,x}) = 0.$$

Perturbed nonlinear second order Schrödinger equation:

$$\mathrm{i}p_{,t} + p_{,xx} + 2|p|^2 p - \varepsilon |p|^4 p = 0,$$

with $p \equiv p(t, x; \varepsilon)$ the complex-valued envelope of the wave. By decomposing into real and imaginary parts:

$$\begin{aligned}\Delta^1 =& u_{,t} + v_{,xx} + 2v(u^2 + v^2) - \varepsilon v \left(u^2 + v^2\right)^2 = 0,\\ \Delta^2 =& v_{,t} - u_{,xx} - 2u(u^2 + v^2) + \varepsilon u \left(u^2 + v^2\right)^2 = 0.\end{aligned}$$

Expand $u(t, x; \varepsilon)$ and $v(t, x; \varepsilon)$ at first order in ε and look for the approximate multipliers of the form

$$\Lambda^j = \Lambda^j_{(0)} + \varepsilon \left(\Lambda^j_{(1)} + \frac{\partial \Lambda_{(0)}}{\partial u_{(0)}} u_{(1)} + \frac{\partial \Lambda_{(0)}}{\partial v_{(0)}} v_{(1)} + \frac{\partial \Lambda_{(0)}}{\partial u_{(0),x}} u_{(1),x} + \frac{\partial \Lambda_{(0)}}{\partial v_{(0),x}} v_{(1),x} + \frac{\partial \Lambda_{(0)}}{\partial u_{(0),xx}} u_{(1),xx} + \frac{\partial \Lambda_{(0)}}{\partial v_{(0),xx}} v_{(1),xx} \right),$$

where $\Lambda^j_{(i)} \equiv \Lambda^j_{(i)} \left(t, x, u_{(0)}, v_{(0)}, u_{(0),x}, v_{(0),x}, u_{(0),xx}, v_{(0),xx}\right)$ $(i = 0, 1$ and $j = 1, 2)$.
By solving the approximate determining equations

$$E_{u_{(0)}} \left(\Lambda^1 \Delta^1 + \Lambda^2 \Delta^2\right) = 0, \qquad E_{v_{(0)}} \left(\Lambda^1 \Delta^1 + \Lambda^2 \Delta^2\right) = 0,$$

where $E_{w_{(0)}} = \frac{\partial}{\partial w_{(0)\alpha}} - D_t \left(\frac{\partial}{\partial w_{(0),t}}\right) - D_x \left(\frac{\partial}{\partial w_{(0),x}}\right) + D_t D_t \left(\frac{\partial}{\partial w_{(0),tt}}\right) + D_t D_x \left(\frac{\partial}{\partial w_{(0),tx}}\right) + D_x D_x \left(\frac{\partial}{\partial w_{(0),xx}}\right)$,

we obtain the sets of approximate multipliers with the corresponding approximate conservation laws.

- $$\Lambda_1^1 = v_{(0),xx} + 2v_{(0)}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon\left(v_{(1),xx} - v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(u_{(0)}^2 v_{(1)} + 2u_{(0)}u_{(1)}v_{(0)} + 3v_{(0)}^2 v_{(1)})\right),$$
$$\Lambda_1^2 = -u_{(0),xx} - 2u_{(0)}(u_{(0)}^2 + v_{(0)}^2) - \varepsilon\left(u_{(1),xx} - u_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2(v_{(0)}^2 u_{(1)} + 2u_{(0)}v_{(0)}v_{(1)} + 3u_{(0)}^2 u_{(1)})\right),$$
with
$$\begin{aligned} &D_t\left(\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 - (u_{(0)}^2 + v_{(0)}^2)^2\right) + \varepsilon\left(\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),x}\right.\right.\\ &\left.\left.+\left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),x} - 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right)\\ &+D_x\left(-(u_{(0),t}u_{(0),x} + v_{(0),t}v_{(0),x}) - \varepsilon\left(u_{(0),x}u_{(1),t} + v_{(0),x}v_{(1),t}\right.\right.\\ &\left.\left.+\left(u_{(1),x} - xu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)u_{(0),t} + \left(v_{(1),x} - xv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2\right)v_{(0),t}\right)\right) = 0 \end{aligned}$$

- $$\Lambda_2^1 = u_{(0),x} + \varepsilon u_{(1),x}, \qquad \Lambda_2^2 = v_{(0),x} + \varepsilon v_{(1),x},$$
with
$$\begin{aligned} &D_t\left(v_{(0)}u_{(0),x} + \varepsilon\left((tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),x} + tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),x} + v_{(0)}u_{(1),x}\right)\right)\\ &+D_x\left(-\frac{1}{2}\left(u_{(0),x}^2 + v_{(0),x}^2 + (u_{(0)}^2 + v_{(0)}^2)^2\right) - v_{(0)}u_{(0),t}\right.\\ &-\varepsilon\left(u_{(0),x}u_{(1),x} + v_{(0),x}v_{(1),x} + (tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + v_{(1)})u_{(0),t}\right.\\ &\left.\left.+\,tv_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 v_{(0),t} + v_{(0)}u_{(1),t} + 2(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right) = 0 \end{aligned}$$

- $$\Lambda_3^1 = 2tu_{(0),x} + xv_{(0)} + \varepsilon\left(2tu_{(1),x} + xv_{(1)}\right), \qquad \Lambda_3^2 = 2tv_{(0),x} - xu_{(0)} + \varepsilon\left(2tv_{(1),x} - xu_{(1)}\right),$$

  with

$$\begin{aligned}
&D_t\left(2tv_{(0)}u_{(0),x} + \frac{x}{2}(u_{(0)}^2 + v_{(0)}^2)\right.\\
&\left.+\,\varepsilon\left(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),x} + t^2v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2v_{(0),x} + 2tv_{(0)}u_{(1),x} + x(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\right)\\
&+D_t\left(-t\left(u_{(0),x}^2 + v_{(0),x}^2 + 2v_{(0)}u_{(0),t} + (u_{(0)}^2 + v_{(0)}^2)^2\right) - x(v_{(0)}u_{(0),x} - u_{(0)}v_{(0),x}) - u_{(0)}v_{(0)}\right.\\
&-\varepsilon\left(t(tu_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2 + 2v_{(1)})u_{(0),t} + (2tu_{(1),x} + xv_{(1)})u_{(0),x} + (2tv_{(1),x} - xu_{(1)})v_{(0),x}\right.\\
&+t^2v_{(0)}(u_{(0)}^2 + v_{(0)}^2)^2v_{(0),t} + x(v_{(0)}u_{(1),x} - u_{(0)}v_{(1),x})\\
&\left.\left.+\,2tv_{(0)}u_{(1),t} + 4t(u_{(0)}^2 + v_{(0)}^2)(u_{(0)}u_{(1)} + v_{(0)}v_{(1)}) + u_{(0)}v_{(1)} + v_{(0)}u_{(1)}\right)\right) = 0
\end{aligned}$$

- $$\Lambda_4^1 = v_{(0)} + \varepsilon v_{(1)}, \qquad \Lambda_4^2 = -u_{(0)} - \varepsilon u_{(1)},$$

  with

$$\begin{aligned}
&D_t\left(\frac{1}{2}(u_{(0)}^2 + v_{(0)}^2) + \varepsilon(u_{(0)}u_{(1)} + v_{(0)}v_{(1)})\right)\\
&+D_x\left(u_{(0)}v_{(0),x} - v_{(0)}u_{(0),x} + \varepsilon(u_{(1)}v_{(0),x} - v_{(1)}u_{(0),x} + u_{(0)}v_{(1),x} - v_{(0)}u_{(1),x})\right) = 0
\end{aligned}$$

Thanks