Automatic Determination of Optimal Systems of Lie Subalgebras: the Package SymboLie

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In memory of prof. Alexandre Vinogradov.

Abstract. Lie groups of point symmetries of partial differential equations constitute a fundamental tool for constructing group–invariant solutions. The number of subgroups is potentially infinite and so the number of group–invariant solutions. An important goal is a classification in order to have an *optimal system* of inequivalent group–invariant solutions from which all other solutions can be derived by action of the group itself. In turn, a classification of inequivalent subgroups induces a classification of inequivalent Lie subalgebras, and vice versa. A general method for classifying the Lie subalgebras of a finite–dimensional Lie algebra relies on the use of inner automorphisms. We present a novel effective algorithm that can automatically determine optimal systems of Lie subalgebras of a generic finite–dimensional Lie algebra; here, we limit the analysis to one–dimensional Lie subalgebras, though the same approach still works well for higher dimensional Lie subalgebras. The algorithm is implemented in the computer algebra system *Wolfram Mathematica*™ and illustrated by means of some examples.

1. Introduction

Lie algebras [**1, 2, 3**], intimately connected to Lie groups, play an important role in many areas of mathematics and physics. In particular, Lie algebras of infinitesimal symmetries of partial differential equations represent the main ingredient for constructing group–invariant solutions [**4, 5**], or for mapping differential equations into equivalent ones [**6**]. By considering different Lie subgroups of continuous transformations admitted by a differential equation, one is able to recover different invariant solutions. Due to the potential infinite number of subgroups, that reflects on the number of group–invariant solutions, it is desirable to classify these solutions in order to have an *optimal system* of inequivalent group–invariant solutions from which all other solutions can be derived by action of the group itself. This classification is based on some special automorphisms of the Lie algebra of the infinitesimal operators of the Lie group.

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Let $(G, \cdot)$ be a group with multiplicative notation and a^{-1} denoting the inverse of $a \in G$. An automorphism of G is a bijective map $\phi : G \to G$ such that

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) \quad \forall\, a, b \in G.$$

In the group of all automorphisms of G, there are the automorphisms

$$\phi_a : G \to G, \qquad \phi_a(b) = a^{-1} \cdot b \cdot a,$$

where $a \in G$, called *inner automorphisms* of G. The set of all inner automorphisms of G is denoted by $\mathrm{Int}(G)$.

Since $a^{-1} \cdot b \cdot a = b$ is equivalent to saying $b \cdot a = a \cdot b$, the existence and number of inner automorphisms different from the identity is a sort of measure of the failure of the commutative law in the group. A subgroup $H \subseteq G$ is *similar* to a subgroup $H' \subseteq G$ if there exists $a \in G$ such that $H' = aHa^{-1}$, *i.e.*, the subgroups H and H' are connected by inner automorphisms of the group. This relation of similarity is a relation of equivalence and the corresponding equivalence classes are said *conjugacy classes*. A Lie group is a group which is also a differential manifold, where the group operation and the operation of taking the inverse are smooth.

Given a Lie group of continuous symmetries of a differential equation, since non–essentially different invariant solutions are found from similar Lie subgroups, the problem of the classification of H–invariant solutions [**4, 5**] is reduced to the classification of subgroups of a Lie group G, up to similarity. This problem, in turn, can be reduced to the corresponding problem of classification of Lie subalgebras, that can be approached more easily from an algorithmic point of view. In fact, introducing an equivalence relation for Lie subalgebras, the latter can be partitioned in classes whose representatives give an optimal system of Lie subalgebras. Working with Lie algebras helps to implement an effective algorithm for classifying Lie subalgebras since the group of inner automorphisms of the Lie algebra, differently from the group $\mathrm{Int}(G)$, is always a group of linear transformations of the main space.

In this paper, we face this problem from a computational viewpoint, and present a novel effective algorithm that can automatically determine the optimal systems of Lie subalgebras of a generic finite–dimensional Lie algebra. The algorithm is implemented in a package, `SymboLie`, written in the computer algebra system *Wolfram Mathematica*™ [**7**]. The name `SymboLie` (due to Lucia Margheriti, a PhD student at the University of Messina who in 2008 started to face this problem from a computational viewpoint [**8**]) merges the word *Symbol* with *Lie*: the reason is that, in Sophus Lie's notation, the infinitesimal generator of a Lie group of transformations was the *symbol*. Various examples are also considered, and the results provided by `SymboLie` shortly discussed.

2. Theoretical preliminaries and notation

Let $\mathcal{L}$ be a *Lie algebra* over a ground field $\mathbb{K}$, and $[\cdot, \cdot]$ be the corresponding Lie bracket. If $\mathcal{L}$ has finite dimension r, then it is denoted by $\mathcal{L}_r$.

Every Lie algebra defines, by its own structure, the so–called *adjoint representation*,

$$\mathrm{ad} : \mathcal{L} \to \mathfrak{gl}(\mathcal{L}), \qquad x \mapsto \mathrm{ad}_x,$$

where $\mathrm{ad}_x : \mathcal{L} \to \mathcal{L}$ and $\mathrm{ad}_x(y) = [x, y]$ for all $y \in \mathcal{L}$.

The elements of $\mathrm{ad}(\mathcal{L})$ are special derivations of the Lie algebra $\mathcal{L}$ [**1, 2, 3**], and are called *inner derivations*.

Each derivation γ of $\mathcal{L}$ determines an automorphism, $\exp(\gamma)$, defined as follows:

$$\exp(\gamma)(y) = y + \gamma(y) + \frac{\gamma^2(y)}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{\gamma^j(y)}{j!}.$$

In particular, the automorphism given by an inner derivation ad_x is

$$\exp(\mathrm{ad}_x)(y) = y + [x, y] + \frac{1}{2}[x, [x, y]] + \dots \tag{2.1}$$

This relation is known as the *Baker–Campbell–Hausdorff formula* [**1, 3**]. The *inner automorphisms* of $\mathcal{L}$ consist of the smallest subgroup of $\mathrm{Aut}(\mathcal{L})$ containing all automorphisms of form (2.1), where x runs through all elements of $\mathcal{L}$. This group is denoted by $\mathrm{Int}(\mathcal{L})$.

Two Lie subalgebras $\mathcal{L}'$ and $\mathcal{L}''$ of a Lie algebra $\mathcal{L}$ are *similar* if there exists an inner automorphism $\phi \in \mathrm{Int}(\mathcal{L})$ such that $\phi(\mathcal{L}') = \mathcal{L}''$. The similarity between Lie subalgebras is an equivalence relation, whereupon all subalgebras of a fixed dimension of a Lie algebra $\mathcal{L}$ are partitioned into classes of similar subalgebras. The set of the representatives of each class is called an *optimal system of subalgebras* [**4, 5**]. In the following, we will be concerned with finite–dimensional Lie algebras of dimension r, $\mathcal{L}_r$. Thus, the optimal system of subalgebras of a Lie algebra $\mathcal{L}_r$ with inner automorphisms $A = \mathrm{Int}(\mathcal{L}_r)$ is a set of subalgebras $\Theta_A(\mathcal{L}_r)$ such that:

(1) there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra $\mathcal{L}_r$;
(2) any subalgebra of the Lie algebra $\mathcal{L}_r$ can be transformed into one of subalgebras of the set $\Theta_A(\mathcal{L}_r)$.

The union of the elements of the optimal system of given dimensionality s is called *optimal system of order* s and denoted by the symbol Θ_A^s; since the dimension of an algebra is invariant under automorphisms, the solution of the classification problem for a finite-dimensional Lie algebra $\mathcal{L}_r$ yields tables of optimal systems for every $s = 1, \dots, r-1$.

For a given finite–dimensional Lie algebra $\mathcal{L}_r$, the optimal system of subalgebras is not unique and the use of different algorithms for its construction leads to different systems $\Theta_A(\mathcal{L}_r)$. All these systems are equivalent in the following sense: if $\Theta_A^1(\mathcal{L}_r)$ and $\Theta_A^2(\mathcal{L}_r)$ are two optimal systems, for any $N_1 \in \Theta_A^1(\mathcal{L}_r)$ there is an automorphism $\phi_1 \in A$ such that $\phi_1(N_1) \in \Theta_A^2(\mathcal{L}_r)$, and inversely.

3. The algorithm for optimal systems of Lie subalgebras

Let $\mathcal{L}_r$ be an r-th dimensional Lie algebra on a field $\mathbb{K}$ with basis $\{\Xi_1, \dots, \Xi_r\}$. Real ($\mathbb{K} = \mathbb{R}$) and complex ($\mathbb{K} = \mathbb{C}$) Lie algebras are of special relevance in the application of Lie groups of symmetries of differential equations; in this case, the elements of the Lie algebra are the infinitesimal generators of the Lie group leaving the differential equation at hand invariant.

By taking two elements X and Y of $\mathcal{L}_r$, they can be decomposed through the basis in the form

$$X = \sum_{\alpha=1}^{r} f^\alpha \Xi_\alpha, \qquad Y = \sum_{\alpha=1}^{r} g^\alpha \Xi_\alpha,$$

where f^α and g^α are suitable (usually real, but nothing prevents to take them in the complex domain) constants; it results

$$[X,Y] = \sum_{\alpha,\beta=1}^{r} f^\alpha g^\beta \left[\Xi_\alpha, \Xi_\beta\right] = \sum_{\alpha,\beta,\gamma=1}^{r} f^\alpha g^\beta C^\gamma_{\alpha\beta}\Xi_\gamma,$$

where $C^\gamma_{\alpha\beta}$ are the structure constants of $\mathcal{L}_r$. This implies that for the coordinates $\mathbf{f} = (f^1,\dots,f^r)$ and $\mathbf{g} = (g^1,\dots,g^r)$ of the elements X and Y with respect to the basis $\{\Xi_1,\dots,\Xi_r\}$ we may introduce an operation of commutation in $\mathbb{K}^r$:

$$[\mathbf{f},\mathbf{g}]^\gamma = \sum_{\alpha,\beta=1}^{r} f^\alpha g^\beta C^\gamma_{\alpha\beta}, \qquad \gamma = 1,\dots,r.$$

If $f^\alpha \in \mathbb{R}$, with this operation the vector space $\mathbb{R}^r$ gains the structure of a Lie algebra. Let us introduce the Lie algebra $\mathcal{L}^A_r$ spanned by the following operators:

$$E_\beta = \sum_{\alpha,\gamma=1}^{r} C^\gamma_{\alpha\beta} f^\alpha \frac{\partial}{\partial f^\gamma}, \qquad \beta = 1,2,\dots,r.$$

The algebra $\mathcal{L}^A_r$ generates through the integration of the Lie's equations,

$$\begin{aligned}&\frac{d\widetilde{f}^\gamma}{dt} = \sum_{\alpha=1}^{r} C^\gamma_{\alpha\beta}\widetilde{f}^\alpha, \qquad \beta,\gamma = 1,\dots,r,\\ &\widetilde{f}^\gamma(0) = f^\gamma,\end{aligned}$$

the group of *inner automorphisms* of the Lie algebra $\mathcal{L}_r$.

Any subalgebra of a Lie algebra $\mathcal{L}_r$ is completely defined by its basis generators. These basis generators are linear combinations of basis operators of the Lie algebra $\mathcal{L}_r$. Hence, the subalgebra is completely defined by the coefficients of these linear combinations. Actions of automorphisms can be transferred to these coefficients. Besides the automorphisms, one has also to take into account a uniform scaling of all generators: in fact, any subalgebra is transformed into a similar subalgebra under this operation. Thus, the problem of constructing an optimal system of subalgebras reduces to obtaining the maximum possible number of zero coordinates of the subalgebra basis. In fact, focusing ourselves on one–dimensional (1D) Lie subalgebras, the method usually employed in the literature for finding an optimal system takes a tuple $\{f^1, f^2,\dots,f^r\}$, and, through *judicious* applications of inner automorphisms, simplifies as many of the coefficients f^α [**5**].

We observe that this approach is difficult to be implemented in a computer, since one needs to solve algebraic equations and make suitable choices during the process; it is a relatively easy approach for small dimensionalities of the Lie algebra, but also in these cases the solution requires to do various assumptions to distinguish cases at certain stages of the work; as a consequence, the *simplicity* of the results obtained is not clear.

Our aim is to render the process of identifying similar subalgebras automatic; to achieve this result we implement a general algorithm with a bottom-up philosophy.

3.1. A brief sketch. Let us illustrate our approach for recovering optimal systems of 1D Lie subalgebras. Let $\mathcal{L}_r$ be an r-dimensional Lie algebra; take the set of all possible tuples with r components (not all zero) chosen in $\{0,1\}$:

$$\mathcal{S}_r = \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(1,1,0,\dots,0),\dots,(1,1,\dots,1)\},$$

and let $\mathbf{f} \equiv \{f^1, f^2, \dots, f^r\}$ be a tuple with arbitrary and non–zero real (or complex) components. To a generic family of 1D Lie subalgebras of $\mathcal{L}_r$ spanned by

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$

where $\mathbf{s} \equiv \{s_1, s_2, \dots, s_r\} \in \mathcal{S}_r$, it corresponds the tuple

$$\{f^1 s_1, f^2 s_2, \dots, f^r s_r\} \tag{3.1}$$

which in turn can be labeled with the integer $k = \displaystyle\sum_{i=1}^{r} s_i 2^{i-1}$.

Definition 3.1. Two generic families of 1D Lie subalgebras of $\mathcal{L}_r$,

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$
$$Y = g^1 s'_1 \Xi_1 + g^2 s'_2 \Xi_2 + \cdots + g^r s'_r \Xi_r,$$

with $\mathbf{s}, \mathbf{s}' \in \mathcal{S}_r$ and $\mathbf{g} \equiv \{g^1, g^2, \dots, g^r\}$ (tuple with arbitrary and non–zero components), are equivalent if there exists some inner automorphism mapping

$$(f^1 s_1, f^2 s_2, \dots, f^r s_r) \quad \text{to} \quad (g^1 s'_1, g^2 s'_2, \dots, g^r s'_r)$$

and vice versa.

The recognition that two families of 1D Lie subalgebras are equivalent can be done in a very simple and natural way, in some cases without the need of solving equations: given a tuple with $1 \le p < r$ non–zero components, if the application of a whatever inner automorphism produces a tuple with $1 \le q \le r$ components which are functionally independent (this is simply ascertained by computing the rank of the Jacobian matrix of the components with respect to the coefficients f^α and the parameters involved in the automorphisms) and with arbitrary signs, then the two tuples correspond to equivalent families of subalgebras. In some cases, it is necessary a second scan to identify equivalent Lie subalgebras by special values of the parameters involved in the inner automorphisms, and `SymboLie` does it.

Following this approach, all possible families of 1D subalgebras can be partitioned, and the results can be graphically displayed by defining a a suitable multigraph.

Definition 3.2. Let $\mathcal{L}_r$ be an r-dimensional Lie algebra, and let

$$(f^1 s_1, f^2 s_2, \dots, f^r s_r)$$

be a generic family of 1D Lie subalgebras, where $\mathbf{s} \in \mathcal{S}_r$, $\mathbf{f} \equiv \{f^1, f^2, \dots, f^r\}$ tuple with arbitrary and non–zero components, labeled with the integer $k = \sum_{i=1}^{r} s_i 2^{i-1}$. Each family is a vertex of a multigraph, $\mathcal{G}(\mathcal{L}_r)$, having $2^r - 1$ nodes, with edges representing the inner automorphisms connecting the families of 1D Lie subalgebras. Thus, it is possible to define the adjacency matrix as the square matrix of order $2^r - 1$ whose entry a_{ij} is the list (possibly empty) of the labels of the inner automorphisms connecting the families of 1D Lie subalgebras labeled by i and j.

In our representation of the multigraph of the one–dimensional subalgebras of a Lie algebra, for the sake of simplicity, even in the case where two subalgebras are linked by different automorphisms, we connect the vertices with a single edge, and omit to show the loops (every subalgebra is trivially equivalent to itself).

Moreover, a quick look to such a picture of the multigraph clearly shows the number of optimal 1D subalgebras, simply given by the number of connected components of the multigraph, where each connected component embodies the equivalent subalgebras.

Remarkably, the same approach works fine also to determine optimal systems of Lie subalgebras of higher dimension. Albeit `SymboLie` is able to compute multi–dimensional optimal systems, this paper is devoted to the analysis of the 1D optimal systems. We will describe the full functionalities of the package `SymboLie` in a forthcoming paper [**9**]; the source code of the package `SymboLie` will be also made available.

3.2. The SymboLie package. The program `SymboLie`, developed in the CAS *Wolfram Mathematica*™ [**7**], provides an organized set of functions helping the user to investigate some properties of real or complex Lie algebras as well as determine optimal systems of Lie subalgebras.

A finite–dimensional Lie algebra can be defined by assigning its structure constants or one of its realizations in terms of matrices or vector fields; in the latter case the user has to define the set of variables involved in the realization. When the Lie algebra is given in terms of its basis generators, the program preliminarily computes the structure constants. The most important feature of the program is the automatic construction of optimal systems of Lie subalgebras: to do this the group of inner automorphisms of the Lie algebra is automatically computed as well. If a Lie algebra involves some parameters subjected to some constraints the user has to supply them.

Here is a list of some functions of the package `SymboLie` the user may call to compute optimal systems.

- `CommutatorTable[c]`: returns the table of commutators of the Lie Algebra assigned through its structure constants `c`;
- `FindAdjacency[c,pars,dim]`: returns the adjacency matrix associated with the Lie Algebra assigned through its structure constants `c` involving parameters `pars`; it contains information about similarities of the `dim`–dimensional subalgebras through inner automorphisms;
- `InnerAutomorphisms[c,t]`: returns the group of inner automorphisms of the Lie Algebra with structure constants `c` (`t` is the parameter involved in the automorphisms);
- `PrintGraph[adj]`: displays the multigraph whose vertices are all possible families of 1D Lie subalgebras, and the edges the links between equivalent subalgebras;
- `StructureConstants[gens,vars,pars]`: returns the structure constants of the Lie Algebra spanned by the vector fields `gens`, involving the variables `vars` and the parameters `pars`; if the elements of the Lie algebra are matrices then `vars` is the empty list;
- `SubAlgebra[c,pars,dim]`: returns an optimal system of `dim`–dimensional subalgebras of the Lie algebra characterized by the structure constants `c` and parameters `pars`; the procedure returns the representatives of the optimal system, the lists of equivalent subalgebras and the adjacency matrix as well.

4. Case studies

In this Section, we reproduce optimal systems of 1D Lie subalgebras of Lie algebras already considered in the literature; the examples serve to test the program, possibly highlighting some discrepancies between our results and those reported by other authors.

Our investigation is focused on some 3D and 4D Lie algebras investigated in [**10**], but also on higher dimensional Lie algebra studied by Meleshko [**11**], Ovsiannikov [**12**] and Olver [**5**].

4.1. 1D optimal systems of 3D Lie algebras. Optimal systems of Lie subalgebras of real 3D Lie algebras have been determined in [**10**]. Here, we select some examples.

Example 4.1. Let $\mathcal{L}_3$ be the 3D solvable Lie algebra spanned by

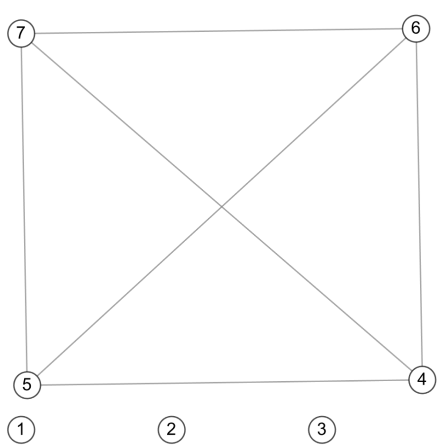
$$\{\Xi_1 = \partial_x, \quad \Xi_2 = \partial_y, \quad \Xi_3 = x\partial_x + y\partial_y\}$$

with the non–zero commutators

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2.$$

This algebra has been considered in [**13**] and corresponds to a Lie algebra characterized in [**10**] ($\mathcal{L}_3$, #5 in the classification therein). The 1D subalgebras can be labeled as in Definition 3.2 using the integers from 1 to 7.

The adjacency matrix is the square matrix of order $2^3 - 1 = 7$ whose (i, j)-entry is the list of the inner automorphisms transforming the subalgebra labeled by i to subalgebra labeled by j. This can be described through a multigraph $\mathcal{G}(\mathcal{L}_3) = \{V, E\}$, where the set of the vertices V is the set of the integers associated to all families of 1D subalgebras of $\mathcal{L}_3$, and the set of the edges E is given by the entries of the adjacency matrix. For a layout convenience, we draw only one edge between two multi–connected vertices. Here and in the sequel, we report the figures provided by the `SimboLie` method `PrintGraph[]`.



The multigraph has four connected components giving an optimal system Θ_1 of $\mathcal{L}_3$:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ f^1\Xi_1 & f^2\Xi_2 & f^1\Xi_1 + f^2\Xi_2 & f^3\Xi_3 \end{array}$$

The group $\mathrm{Int}(\mathcal{L}_3)$ is generated by

$$A_1 = \begin{pmatrix} 1 & 0 & -t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} e^{t_3} & 0 & 0 \\ 0 & e^{t_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The conjugacy classes, *i.e.*, the connected components of $\mathcal{G}(\mathcal{L}_3)$, are the orbits under the action of $\mathrm{Int}(\mathcal{L}_3)$ on the families of subalgebras of $\mathcal{L}_3$. In such a case, each $A \in \mathrm{Int}(\mathcal{L}_3)$ acts trivially on $(f^1, 0, 0)^T$, $(0, f^2, 0)^T$ and $(f^1\, f^2, 0)^T$. Moreover, we have

$$\begin{aligned}
&A_1(0,0,f^3)^T = (-f^3 t_1, 0, f^3)^T && \Rightarrow \quad f^3\Xi_1 \sim g^1\Xi_1 + g^3\Xi_3,\\
&A_2(0,0,f^3)^T = (0, -f^3 t_2, f^3)^T && \Rightarrow \quad f^3\Xi_1 \sim g^2\Xi_2 + g^3\Xi_3,\\
&A_1 \circ A_2(0,0,f^3)^T = (-f^3 t_1, -f^3 t_2, f^3)^T && \Rightarrow \quad f^3\Xi_3 \sim g^1\Xi_1 + g^2\Xi_2 + g^3\Xi_3.
\end{aligned}$$

This means that the subalgebra 4 is *similar* to the subalgebras $5, 6$ and 7, respectively. This information is sufficient to determine the optimal system Θ^1_A; anyway, we also have other relations:

$$\begin{aligned}
(g^1, g^2, g^3)^T = A_2(f^1\, 0, f^3)^T = A_1 \circ A_2(f^1, 0, f^3)^T = \\
= A_1(0, f^2, f^3)^T = A_1 \circ A_2(0, f^2, f^3)^T.
\end{aligned}$$

By rescaling the representatives, we can write

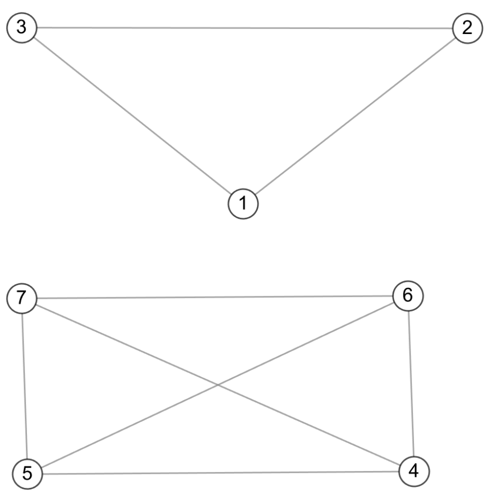
$$\Theta^1_A \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_3\}\}, \quad a_1 \neq 0,$$

in agreement with the results in [**10, 13**].

Example 4.2. Let $\mathcal{L}_3$ be the 3D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ with non–zero commutators

$$[\Xi_1, \Xi_3] = a\,\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\,\Xi_2, \quad \text{with} \quad a > 0$$

($\mathcal{L}_3$, #9 in the classification reported in [**10**]). The multigraph $\mathcal{G}(\mathcal{L}_3)$ describing the equivalences between 1D subalgebras is as follows.



There are two connected components giving the 1D optimal system

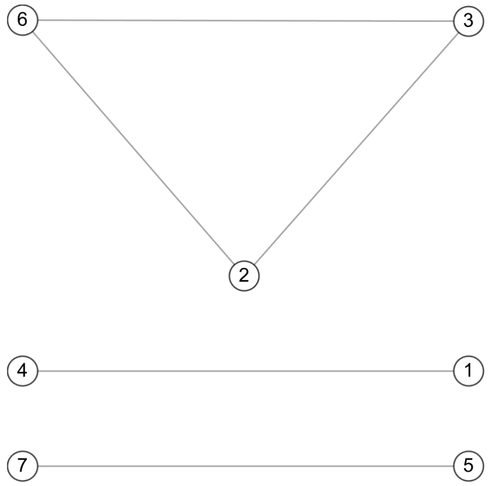
$$\Theta^1_A \equiv \{\{\Xi_1\}, \{\Xi_3\}\}.$$

The same result can be found in [**10**].

Example 4.3. Let $\mathcal{L}_3$ be the 3D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ with the non–zero commutators:

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3, \quad [\Xi_3, \Xi_1] = 2\Xi_2$$

($\mathcal{L}_3$, #10 in [**10**]). The multigraph is represented as follows.



There are three connected components giving

$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\},$
$\alpha_1 = \pm 1$

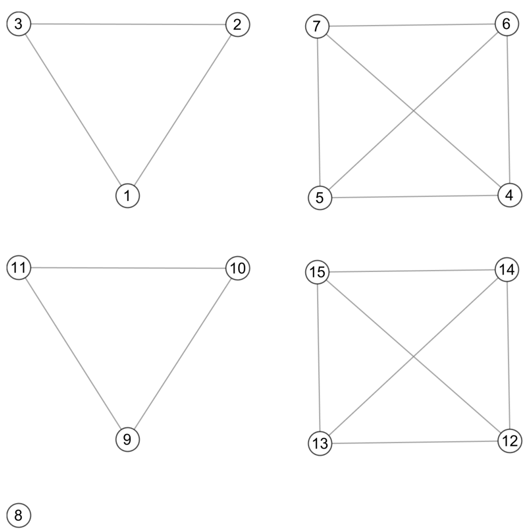
Also in this case, the result is the same as that exhibited in [**10**].

4.2. 1D optimal systems of 4D Lie algebras. In [**10**], 4D real Lie algebras have been classified, and the list of optimal Lie subalgebras characterized. Here we consider some selected examples.

Example 4.4. Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with non–zero commutators

$$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2, \quad \text{with } a > 0$$

($\mathcal{L}_4$, #10 in [**10**]). The multigraph is



There are five connected components whereupon

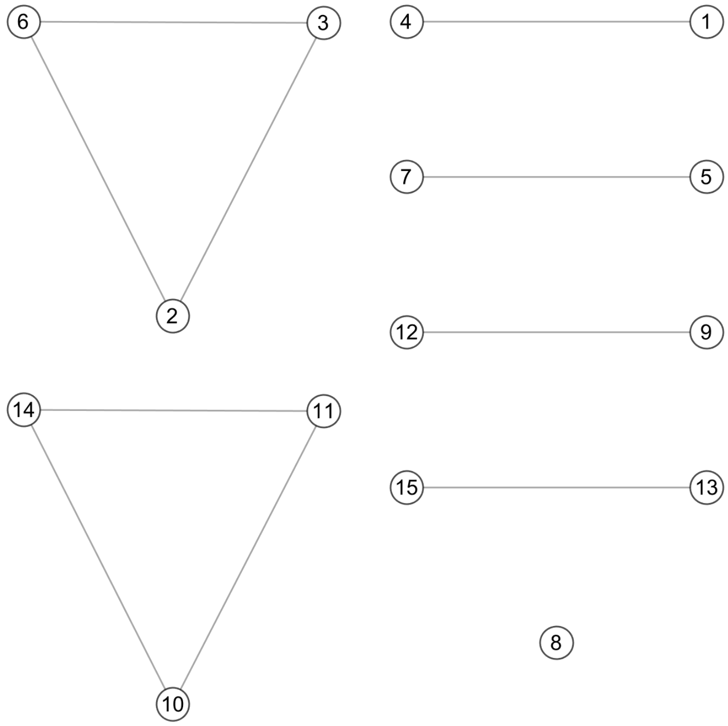
$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}\},$$
$a_1 \neq 0.$

Θ_A^1 is the same as the optimal system in [**10**].

Example 4.5. Let $\mathcal{L}_4$ be the Lie algebra spanned $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the following non–zero commutators:

$$[\Xi_3, \Xi_1] = 2\Xi_2, \quad [\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3$$

($\mathcal{L}_4$, #11 in [**10**]). The multigraph $\mathcal{G}(\mathcal{L}_4)$ is



There are seven connected components, so that

$$\Theta^1_A \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{\Xi_1+\alpha_1\Xi_3\}, \{\Xi_1+\alpha_1\Xi_4\}, \{\Xi_2+a_1\Xi_4\}, \{\Xi_1+\alpha_1\Xi_3+a_1\Xi_4\}\},$$

with $\alpha_1 = \pm 1$ and $a_1 \neq 0$; the result coincides with that in [**10**].

EXAMPLE 4.6. Let $\mathcal{L}_4$ be the 4D Lie algebra of symmetries of KdV equation spanned by

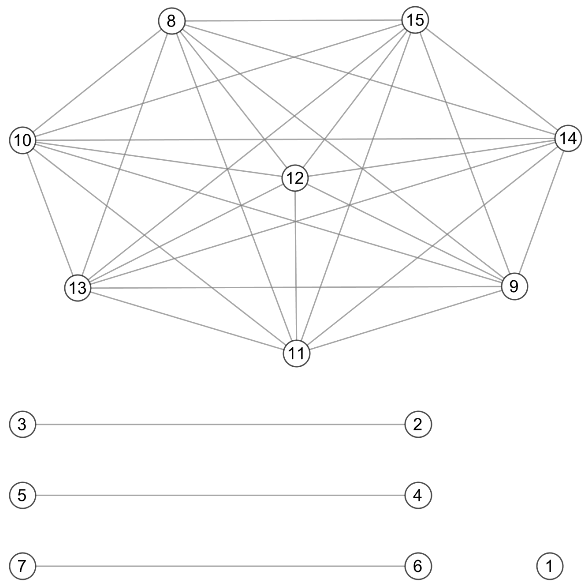
$$\{\Xi_1 = \partial_x, \quad \Xi_2 = \partial_t, \quad \Xi_3 = t\partial_x + \partial_u, \quad \Xi_4 = x\partial_x + 3t\partial_t - 2u\partial_u\}$$

($\mathcal{L}_4$, #25 in [**10**]). Olver [**5**] shows the following optimal system of 1D subalgebras:

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 - \Xi_2\}, \{\Xi_3 + \Xi_2\}\}.$$

Nevertheless, he adds a consideration: the list can be reduced slightly if we admit the discrete symmetry $(x, t, u) \mapsto (-x, -t, u)$, which maps $\Xi_3 - \Xi_2$ to $\Xi_3 + \Xi_2$, thereby reducing the number of inequivalent subalgebras to five.

Using `SymboLie`, we obtain the following multigraph



There are five connected components so giving the optimal system

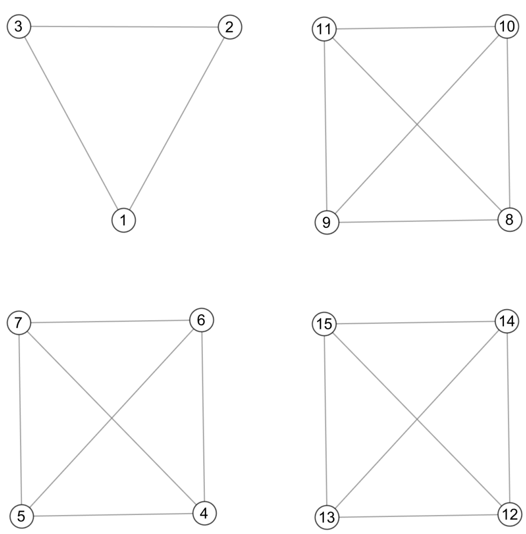
$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_2 + \alpha_1 \Xi_3\}, \{\Xi_4\}\}, \quad \alpha_1 = \pm 1.$$

We note that $\{\Xi_2 + \alpha_1 \Xi_3\}$ is obtained using the rescaling automorphisms, in such a way $\alpha_1 = \pm 1$ without the need of considering the discrete symmetry. The same result can be found also in [**10**].

EXAMPLE 4.7. Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with non–zero commutators

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2, \quad [\Xi_1, \Xi_4] = -\Xi_2, \quad [\Xi_2, \Xi_4] = \Xi_1$$

($\mathcal{L}_4$, #30 in [**10**]). The multigraph is



There are four connected components, whereupon the optimal system of 1D subalgebras is

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}\},$$

with $a_1 \neq 0$.

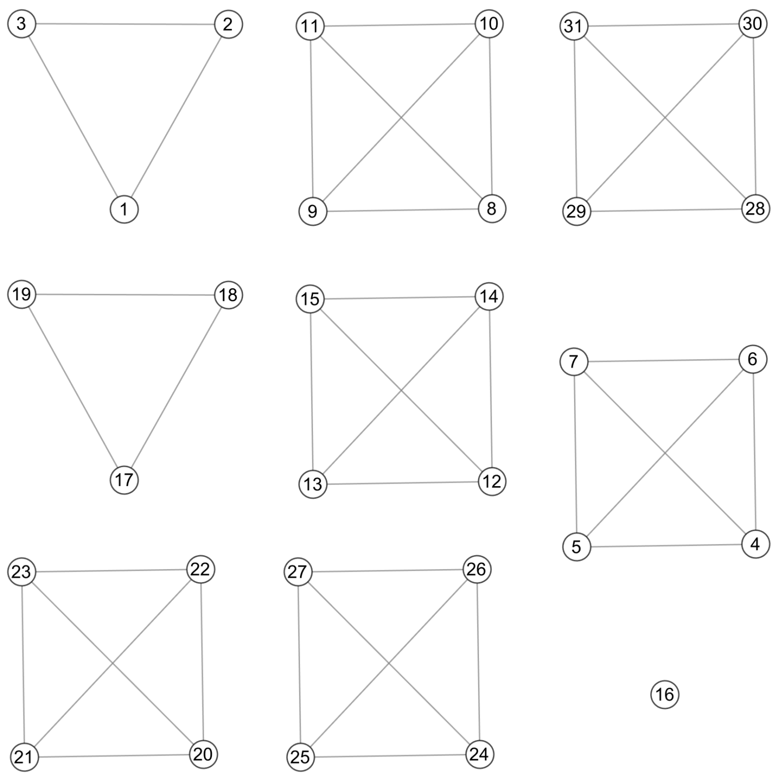
The optimal system coincides with the one in [**10**].

4.3. 1D optimal systems of higher dimensional Lie algebras. In this subsection, some Lie algebras with dimension higher than 4 are considered, and the optimal system of 1D Lie subalgebras are given and compared with the results in the literature.

EXAMPLE 4.8. Let $\mathcal{L}_5$ be the 5D Lie algebra spanned by:

$$\{\,\Xi_1=\partial_x,\ \Xi_2=\partial_y,\ \Xi_3=-y\partial_x+x\partial_y-v\partial_u+u\partial_v,\\ \Xi_4=-x\partial_x-y\partial_y+\rho\partial_\rho+p\partial_p,\ \Xi_5=u\partial_u+v\partial_v+\rho\partial_\rho+p\partial_p\,\}\,.$$

These generators give the symmetries of the 2D steady ideal gas dynamics equations [**11**]. The multigraph is



$\mathcal{G}(\mathcal{L}_5)$ has nine connected components giving

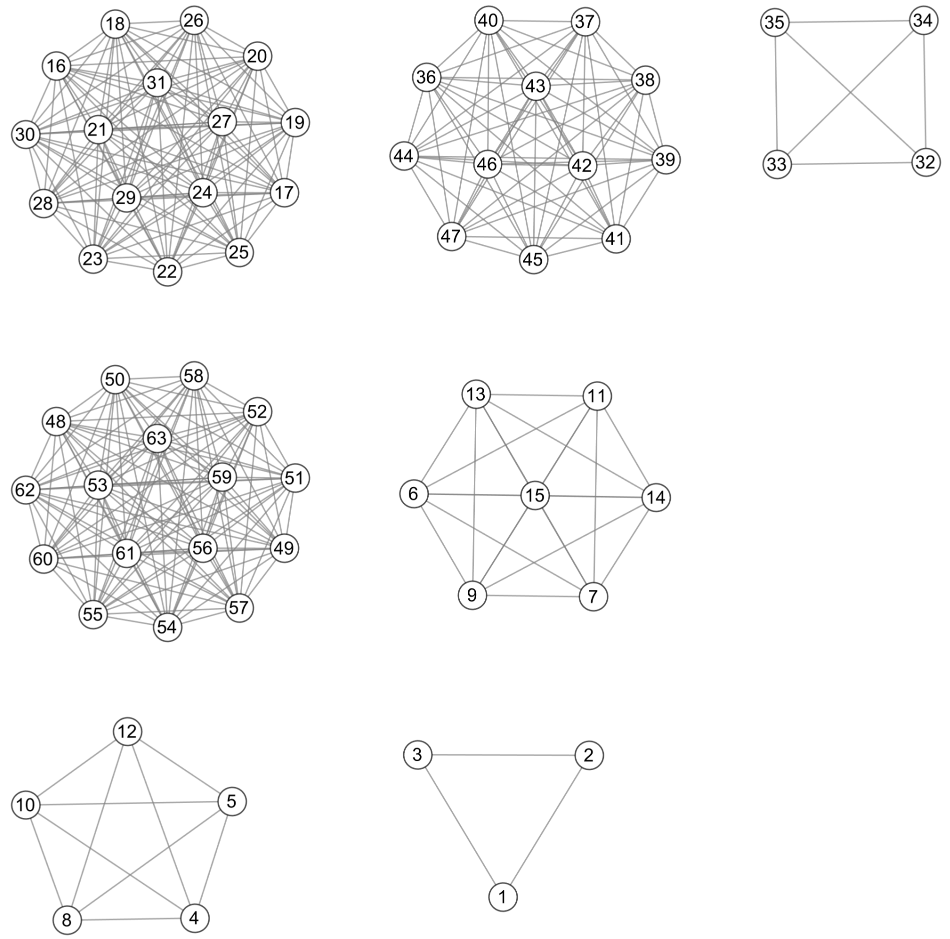
$$\Theta_1\equiv\{\{\Xi_1\},\{\Xi_3\},\{\Xi_4\},\{\Xi_3+a_1\Xi_4\},\{\Xi_5\},\{\Xi_1+\alpha_1\Xi_5\},\\ \{\Xi_3+a_1\Xi_5\},\{\Xi_4+a_1\Xi_5\},\{\Xi_3+a_1\Xi_4+a_2\Xi_5\}\},$$

with $\alpha_1=\pm1$ and $a_1,a_2\neq 0$. The result agrees with the one given in [**11**].

EXAMPLE 4.9. Let $\mathcal{L}_6$ be a Lie algebra spanned by

$$\{\Xi_1=\partial_x,\ \Xi_2=\partial_y,\ \Xi_3=t\partial_x,\ \Xi_4=t\partial_y,\ \Xi_5=y\partial_x-x\partial_y,\ \Xi_6=\partial_t\}.$$

The multigraph has seven connected components representing an optimal system of 1D subalgebras of $\mathcal{L}_6$.



Therefore, we have the 1D optimal system

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{\Xi_3 + a_1\Xi_6\}, \{\Xi_5 + a_1\Xi_6\}\},$$

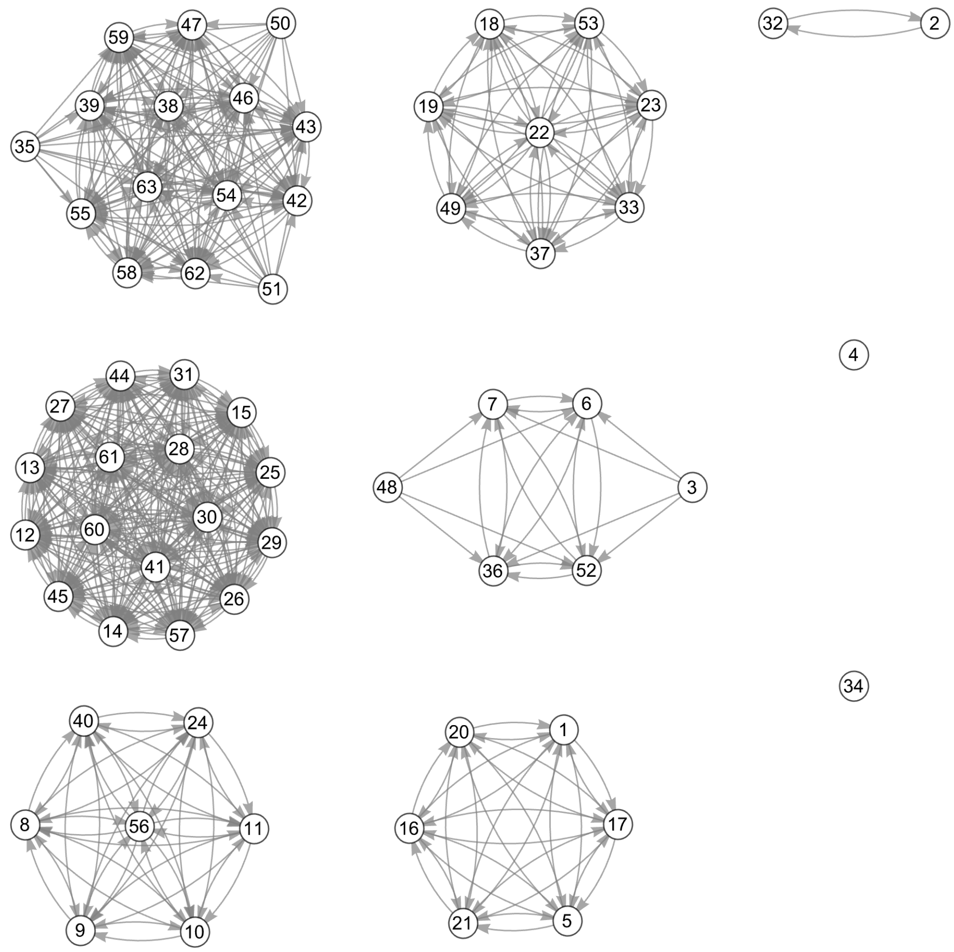
with $a_1 \neq 0$.

This result is in agreement with the one given by Ovsiannikov [**12**], except for the choice of some representatives of the equivalence classes.

Example 4.10. Consider the finite–dimensional Lie algebra of symmetries of $u_t - u_{xx} = 0$ (linear heat equation) spanned by

$$\begin{aligned} &\Xi_1 = \partial_x, \quad \Xi_2 = \partial_t, \quad \Xi_3 = u\partial_u, \quad \Xi_4 = x\partial_x + 2t\partial_t, \\ &\Xi_5 = 2t\partial_x - xu\partial_u, \quad \Xi_6 = 4tx\partial_x, +4t^2\partial_t - (x^2 + 2t)u\partial_u. \end{aligned}$$

The multigraph given by `SymboLie` is



`SymboLie` recovers an optimal system of nine 1D Lie subalgebras:

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_3\},$$
$$\{\Xi_3 + a_1 \Xi_4\}, \{\Xi_2 + \alpha_1 \Xi_5\}, \{\Xi_2 + \alpha_1 \Xi_6\}, \{\Xi_2 + \alpha_1 \Xi_3 + a_1 \Xi_6\}\}$$

with $\alpha_1 = \pm 1$ and $a_1 \neq 0$. Other authors [**5, 14, 15, 16**] found similar optimal 1D subalgebras for the finite-dimensional Lie algebra of symmetries of linear heat equation. Notice in the representation of the multigraph that it is oriented: in fact, some families of Lie subalgebras can be mapped to other families but it is not granted the vice versa; nevertheless, the representative chosen by `SymboLie` is such that all the other families of the connected component of the multigraph are mapped to it by some automorphisms.

5. Conclusions

In this paper, we faced the problem of finding optimal systems of Lie subalgebras by means of a bottom–up approach that can be implemented in a computer program written in the CAS *Wolfram Mathematica*™. We limited ourselves to illustrate the derivation of optimal systems of 1D Lie subalgebras; nevertheless, our package works also when searching optimal systems of multi–dimensional Lie subalgebras, and a multigraph representation is still possible..

We can show, *e.g.*, the multi–dimensional optimal systems of the Lie algebra analyzed in Example 4.4:

$$\begin{aligned}\Theta^2_A &\equiv \{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_4\},\{\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_4,\Xi_2+a_2\Xi_4\}\} \quad \text{and}\\ \Theta^3_A &\equiv \{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_2,\Xi_3+a_1\Xi_4\}\},\end{aligned}$$

where $a_1, a_2 \neq 0$. We observe that `SymboLie` found the optimal systems Θ^2_A and Θ^3_A listed by Patera and Winternitz in [**10**].

As a further example, let us consider the four–dimensional real Lie algebra spanned by $\{\Xi_1,\Xi_2,\Xi_3,\Xi_4\}$ with non-zero Lie brackets

$$[\Xi_1,\Xi_4]=\Xi_1,\quad [\Xi_2,\Xi_4]=a\Xi_2,\quad [\Xi_3,\Xi_4]=\Xi_3,$$

where $-1 \leq a < 1$, $a \neq 0$ ($\mathcal{L}_4$, #20 in [**10**]). `SymboLie` determines the optimal system

$$\begin{aligned}\Theta^1_A \equiv\ &\{\{\Xi_1\},\{\Xi_2\},\{\Xi_1+\alpha_1\Xi_2\},\{\Xi_3\},\{\Xi_1+a_1\Xi_3\},\{\Xi_2+\alpha_1\Xi_3\},\\ &\{\Xi_1+\alpha_1\Xi_2+a_1\Xi_3\},\{\Xi_4\}\},\\ \Theta^2_A \equiv\ &\{\{\Xi_1,\Xi_2\},\{\Xi_1,\Xi_3\},\{\Xi_1,\Xi_4\},\{\Xi_2,\Xi_3\},\{\Xi_2,\Xi_4\},\{\Xi_3,\Xi_4\},\\ &\{\Xi_1,\Xi_2+\alpha_1\Xi_3\},\{\Xi_1+\alpha_1\Xi_2,\Xi_3\},\{\Xi_1+a_1\Xi_3,\Xi_2\},\\ &\{\Xi_1+a_1\Xi_3,\Xi_4\},\{\Xi_1+a_1\Xi_3,\Xi_2+a_2\Xi_3\}\},\\ \Theta^3_A \equiv\ &\{\{\Xi_1,\Xi_2,\Xi_3\},\{\Xi_1,\Xi_2,\Xi_4\},\{\Xi_1,\Xi_3,\Xi_4\},\{\Xi_2,\Xi_3,\Xi_4\},\\ &\{\Xi_1+a_1\Xi_3,\Xi_2,\Xi_4\}\},\end{aligned}$$

with a_1, a_2 arbitrary non vanishing real coefficients, and $\alpha_1 = \pm 1$; these results are in agreement with those derived in [**10**].

A complete analysis of the optimal systems of three and four–dimensional real Lie algebras can be found in [**17**]. We limit ourselves to observe that `SymboLie` classifies the optimal systems of all the 3D and 4D real Lie algebras classified in [**10**] in few minutes on a laptop with Intel i5 processor.

A complete description of the package `SymboLie` will be the subject of a forthcoming paper [**9**]. In parallel, we plan to extend the program by introducing additional features. In particular, it would be desirable to allow the program to suitably use special properties of the Lie algebra to be analyzed (for instance, its decomposition as direct sum of an ideal and a subalgebra), as well as to implement some routines allowing us to construct the submodels [**18**] once the optimal system of Lie subalgebras of a Lie algebra of symmetries of a differential equation is obtained (see [**19**] for a computer algebra program automatically computing the Lie symmetries of a differential equation).

Finally, it would be interesting to study the possible connections between optimal systems of one–dimensional Lie subalgebras of a Lie algebra and the elementary constituents (*dyons* and *triadons*), introduced by Vinogradov [**20, 21**], who proved that any finite–dimensional Lie algebra over an algebraically closed field of characteristic zero or over $\mathbb{R}$ can be assembled in a finite number of steps from these elementary constituents.

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