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Reaction-diffusion models of crimo–taxis in a street

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ABSTRACT

In this paper, two reaction-diffusion models describing the interaction among susceptible people (ordinary citizens), infective people (drug users/dealers), and law enforcement personnel are analyzed. The models here considered are a generalization of the *crimo-taxis* model originally proposed by Epstein in 1997. The modifications allow us to describe various scenarios. We analyze the equilibrium points, together with their stability, of the homogeneous system. Moreover, according to the Turing approach to reaction–diffusion models, we investigate the instability driven processes and the emergence of patterns in the complete models.

1. Introduction

Urban crime is a very common problem in today's society, and its prevention is one of the fundamental tasks of law enforcement activities [1,2]. The formation of *hot-spots* of crime suggests that the latter do not occur uniformly in space and time, but often are concentrated in relatively small places and are responsible for more than half of crime events such as drugs, robbery, theft. A mathematical model for such a phenomenon could start by dividing a given population in different subgroups, for instance law-abiding citizens, offenders and policemen, interacting each other. Among the various approaches in the literature to describe how the density of species or groups of individuals is distributed in space at different times due to local interaction and diffusion, there are reaction-diffusion models [3,4], kinetic models [5,6], and even operatorial models based on the mathematical apparatus of quantum mechanics (see [7,8], and references therein, where several operatorial models in various contexts have been studied).

In particular, reaction-diffusion equations are widely used as models for spatial effects in ecology [9–11], biology and medicine [12], epidemiology [13,14] and social sciences [15,16]. Reaction-diffusion equations can be analyzed by means of analytical and numerical methods from the theory of partial differential equations and dynamical systems. Models for urban crime dynamics based on reaction-diffusion compartmental systems have been proposed and analyzed in the last years by many authors [1,2,17–22].

In this paper, focusing on models for describing social interactions between ordinary people, drug users/dealers and policemen, we start from a one–dimensional model proposed in 1997 by Epstein [23], named crimo–taxis. In this model, the population is divided in three subgroups, whose numbers and spatial distributions evolve over time. It is assumed that events unfold on a one-dimensional region representing a "street."

Let us define $u(x,t)$, $v(x,t)$, and $w(x,t)$ as the ordinary citizens, drug users/dealers and law enforcement personnel at street position x at time t (a more sophisticated model could involve a fourth subgroup of removed, *i.e.*, arrested, but this situation will not be considered here). The structure of the model recalls that used in some epidemiological models, so that ordinary citizens can be

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thought of as susceptible individuals, drug users/dealers as infected/infectious individuals, and law enforcement personnel as a tool to fight infection [24,25].

This model, dubbed *crimo-taxis* by its creator, predicts various kinds of dynamical outcomes, with potential implications for policy makers. Starting with this model, we introduce two variants suitable to admit asymptotically stable homogeneous equilibria that lose their stability because of self- and cross-diffusive terms leading to the emergence of stable spatial patterns. The main novelty with respect to the Epstein model, besides some minor modifications in the reaction terms, relies on the introduction of a new cross-diffusive contribution describing the spread of law enforcement personnel towards the region where there is a high concentration of ordinary citizens, this circumstance occurring for instance on the occasion of social or political demonstrations.

The paper is organized as follows. Section 2 deals with the original Epstein model; after setting the governing equations, as well as briefly describing some features of the dynamical outcome [23] in a special situation, we investigate the stability of the homogeneous equilibrium configurations; no asymptotically stable equilibrium exists, and no diffusion driven instability can be found. In Section 3, we propose two variants of the Epstein model: both admit an asymptotically stable homogeneous coexistence equilibrium, so that there is the possibility to investigate the pattern formation due to Turing instability. Section 4 is devoted to analyzing Turing instability from the analytical point of view, whereas Section 5 presents the results of the numerical simulations and the rise of some patterns. Finally, Section 6 contains some concluding remarks.

2. A nonlinear reaction-diffusion model on a street

Let us consider a road (an alley) that we represent with the interval $[0, L] \subset \mathbb{R}$, where some people are distributed. We divide the entire population into three classes: ordinary citizens (susceptible individuals), drug users/dealers (infected/infectious individuals) and law enforcement personnel (policemen); let $u(x,t)$, $v(x,t)$ and $w(x,t)$ be their densities at the position x and time t, respectively. In [23], a set of reaction-diffusion equations has been proposed,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= ru - \beta uv + D_{11}\frac{\partial^2 u}{\partial x^2},\\
\frac{\partial v}{\partial t} &= \beta uv - \gamma vw - D_{21}\frac{\partial^2 u}{\partial x^2} + D_{22}\frac{\partial^2 v}{\partial x^2} + D_{23}\frac{\partial^2 w}{\partial x^2},\\
\frac{\partial w}{\partial t} &= -bw + \xi uvw - D_{32}\frac{\partial^2 v}{\partial x^2} + D_{33}\frac{\partial^2 w}{\partial x^2},
\end{aligned}
\tag{1}
$$

where all the parameters therein involved are assumed to be positive; in particular, r is the susceptible reproduction rate, β the rate of infection, γ the police arrest rate, b the police natural decay rate, ξ the police growth rate, and D_{ij} self- and cross-diffusion coefficients ($i = j$ and $i \neq j$, respectively).

As far as the *reaction* terms are concerned, ordinary citizens and drug users/dealers interact with a prey predator mechanism [26]; on the contrary, the interaction between drug users/dealers and law enforcement personnel is a little bit different: the term $-\gamma vw$ is standard, whereas the term ξuvw describes the growth in the number of police forces in parallel with the increase of the level of social alarm, due to the spread of the drug problem. The terms $D_{11}\frac{\partial^2 u}{\partial x^2}$, $D_{22}\frac{\partial^2 v}{\partial x^2}$ and $D_{33}\frac{\partial^2 w}{\partial x^2}$ model the diffusion of the three classes of individuals, and the second and third equation include cross-diffusion terms:

- the terms $-D_{21}\frac{\partial^2 u}{\partial x^2}$ and $D_{23}\frac{\partial^2 w}{\partial x^2}$ represent that drug users/dealers spread in the areas occupied by the susceptible individuals and far away from the areas occupied by the policemen, respectively;
- the term $-D_{32}\frac{\partial^2 v}{\partial x^2}$ accounts for the diffusion of the policemen towards the areas where there is a high concentration of drug users/dealers.

It is reasonable to assume $D_{23} > D_{21}$, as done in [23], at least if drug users/dealers give less importance to convert a susceptible individual with respect to the risk of being arrested; nevertheless, in this paper we also exploit some cases where $D_{23} < D_{21}$, to account for a sort of ruthlessness of drug users/dealers.

In [23], assuming the values $\beta = 0.005$, $\mu = 0.5$, $\gamma = 0.03$, $\xi = 0.0001$, $b = 1.0$, $D_{11} = 0.03$, $D_{22} = 0.01$, $D_{33} = 0.02$, $D_{23} = 0.006$, $D_{32} = 0.006$ for the parameters, and assigning the initial spatial distribution of the three subgroups, some numerical simulations have been performed. More in detail, the street was divided into 12 blocks, and the initial distribution of the various classes was assumed as follows: ordinary citizens occupy the four central blocks (1000 in each block), in blocks 8-12 there are the drug users/dealers (100 in each block), and the policemen are in blocks 1-3 (25 in each block).

The results therein presented show that the drug users/dealers diffuse toward the central blocks, where there is a large concentration of ordinary people and few cops; they peddle with many susceptible individuals, so that the latter become part of the crime. Consequently, it is observed a decrease in the population density of the susceptible individuals and an increase in that of the infected in the central blocks. This phenomenon induces a police reaction, that spread from the barracks at the end of the road to the center, the heart of the problem. Thereafter, the evolution shows a situation where the policemen wither away, and the susceptible individuals can continue their undisturbed spread.

Let us analyze the equilibrium points of the homogeneous system (no space dependence). There are three equilibria, say

$$
P_1 = (0,0,0), \quad P_2 = (0, v^\star, 0), \quad P_3 = \left(\frac{b\beta}{\xi r}, \frac{r}{\beta}, \frac{b\beta^2}{\xi\gamma r}\right),
\tag{2}
$$

where $v^\star > 0$ cannot be determined from the equilibrium conditions.

Linearizing the system around each equilibrium and computing the eigenvalues of the Jacobian matrix J of reaction terms evaluated on each equilibrium, the following results are easily deduced:

1. the equilibrium P_1 is unstable since J has the eigenvalues
$$\lambda_1 = r, \quad \lambda_2 = 0, \quad \lambda_3 = -b;$$
2. the equilibrium P_2 is not asymptotically stable or unstable depending on the sign of $\mu - \beta v^\star$ since the eigenvalues of J are
$$\lambda_1 = r - \beta v^*, \quad \lambda_2 = 0, \quad \lambda_3 = -b;$$
3. the coexistence equilibrium P_3 is not stable since J possesses a positive real eigenvalue and two complex conjugate eigenvalues with negative real part, say
$$\begin{aligned}\lambda_1 &= \ell_1 + \ell_2\\ \lambda_2 &= -\frac{1}{2}(\ell_1 + \ell_2) + \frac{\sqrt{3}}{2}(\ell_1 - \ell_2)\mathrm{i}\\ \lambda_3 &= -\frac{1}{2}(\ell_1 + \ell_2) + \frac{\sqrt{3}}{2}(\ell_2 - \ell_1)\mathrm{i},\end{aligned}$$
where
$$\begin{aligned}&\ell_1 = \sqrt[3]{\frac{b^2\beta^2}{2\xi} + \sqrt{\Delta_{III}}}, \quad \ell_2 = \sqrt[3]{\frac{b^2\beta^2}{2\xi} - \sqrt{\Delta_{III}}},\\ &\Delta_{III} = \frac{b^2\beta^4}{4\xi^2}\left(\frac{1}{4} + \frac{b\beta^2(b+r)^3}{27r^3\xi}\right).\end{aligned}$$

Due to the analysis above described, the homogeneous Epstein model does not possess asymptotically stable equilibria. This is the main reason for proposing in Section 3 two different variants of the model in order to have stable equilibria susceptible of losing their stability as a consequence of the diffusive terms; as a result, we will be able to obtain the emergence of some patterns.

3. Modified models

In this Section, we consider two variants of the Epstein model; we introduce a further cross–diffusion term and some logistic effects.

The first variant we propose is the following one:

$$\begin{aligned}\frac{\partial u}{\partial t} &= ru\left(1 - \frac{u}{\kappa_1}\right) - \beta uv + D_{11}\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial v}{\partial t} &= \beta uv - \gamma vw - D_{21}\frac{\partial^2 u}{\partial x^2} + D_{22}\frac{\partial^2 v}{\partial x^2} + D_{23}\frac{\partial^2 w}{\partial x^2},\\ \frac{\partial w}{\partial t} &= -bw + \xi uvw - D_{31}\frac{\partial^2 u}{\partial x^2} - D_{32}\frac{\partial^2 v}{\partial x^2} + D_{33}\frac{\partial^2 w}{\partial x^2},\end{aligned} \tag{3}$$

where the constant parameters therein involved are all positive. The rationale of this model is that we assume a maximum number of ordinary people in the street (carrying capacity, κ_1), so that we have a more realistic logistic growth for them; moreover, we introduce a cross-diffusion term in the third equation, $-D_{31}\frac{\partial^2 u}{\partial x^2}$, to account for the diffusion of the police towards the areas where there is a high concentration of ordinary citizens (for instance, this occurs when there are some events, like concerts or political demonstrations that may determine possible problems).

The second variant, still retaining the modifications above described, modifies the *crimo-taxis* term of the original model; in fact, we assume that the reaction term responsible for the growth of police tends to decrease as w approaches κ_2; this means that there is a limit, κ_2, to the number of law enforcement personnel. Therefore, we are led to consider the equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= ru\left(1 - \frac{u}{\kappa_1}\right) - \beta uv + D_{11}\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial v}{\partial t} &= \beta uv - \gamma vw - D_{21}\frac{\partial^2 u}{\partial x^2} + D_{22}\frac{\partial^2 v}{\partial x^2} + D_{23}\frac{\partial^2 w}{\partial x^2},\\ \frac{\partial w}{\partial t} &= -bw + \xi uv\left(1 - \frac{w}{\kappa_2}\right) - D_{31}\frac{\partial^2 u}{\partial x^2} - D_{32}\frac{\partial^2 v}{\partial x^2} + D_{33}\frac{\partial^2 w}{\partial x^2}.\end{aligned} \tag{4}$$

Both models will be studied in the domain $[0, L] \times \mathbb{R}^+$, where L is the length of the street.

Before analyzing these two models, let us introduce dimensionless variables:

$$\hat{t} = \frac{t}{T}, \qquad \hat{x} = \frac{x}{L}, \qquad \hat{u} = \frac{u}{\kappa_1}, \qquad \hat{v} = \frac{v}{\kappa_1}, \qquad \hat{w} = \frac{w}{\kappa_1}; \tag{5}$$

the spatial scale and the normalizations for the three subgroups is suggested by the problem; on the contrary, the time scale $T = (\beta\kappa_1)^{-1}$ is a constraint that we impose in order to have in the dimensionless equations the coefficient of the reaction term between ordinary citizens and drug users/dealers equal to 1.

In order to simplify the notation let us drop the hats, whence, introducing the vector $\mathbf{U}$ and the matrix $\mathcal{D}$, say

$$\mathbf{U} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} d_{11} & 0 & 0 \\ -d_{21} & d_{22} & d_{23} \\ -d_{31} & -d_{32} & d_{33} \end{pmatrix}, \tag{6}$$

we may write both dimensionless models in compact form as

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{R}(\mathbf{U}) + \mathcal{D}\frac{\partial^2 \mathbf{U}}{\partial x^2}, \tag{7}$$

where

$$\mathbf{R}(\mathbf{U}) = \begin{pmatrix} m_1 u(1-u) - uv \\ uv - m_2 vw \\ -m_3 w + m_4 uvw \end{pmatrix}, \tag{8}$$

for the first variant, and

$$\mathbf{R}(\mathbf{U}) = \begin{pmatrix} m_1 u(1-u) - uv \\ uv - m_2 vw \\ -m_3 w + m_5 uv(1 - m_6 w) \end{pmatrix}, \tag{9}$$

for the second variant; the parameters therein involved are

$$\begin{aligned} &m_1 = \frac{r}{\beta\kappa_1}, \quad m_2 = \frac{\gamma}{\beta}, \quad m_3 = \frac{b}{\beta\kappa_1}, \quad m_4 = \frac{\xi\kappa_1}{\beta}, \\ &m_5 = \frac{\xi}{\beta}, \quad m_6 = \frac{\kappa_1}{\kappa_2}, \quad d_{ij} = \frac{D_{ij}}{\beta\kappa_1 L^2}. \end{aligned} \tag{10}$$

3.1. Linear stability analysis of the homogeneous models

Let us consider the models where the spatial terms are neglected, say

$$\frac{d\mathbf{U}}{dt} = \mathbf{R}(\mathbf{U}). \tag{11}$$

Both variants possess the physically admissible equilibria

$$\mathbf{U_1} \equiv (0,0,0), \qquad \mathbf{U_2} \equiv (1,0,0), \qquad \mathbf{U_3} \equiv (0, v^\star, 0),$$

where $v^\star$ is a positive value not determined by the equilibrium conditions. Linearizing system (11) around an equilibrium, we easily deduce the following results:

1. $\mathbf{U}_1$ is not stable since the Jacobian matrix of $\mathbf{R}(\mathbf{U})$ evaluated on $\mathbf{U}_1$ has the eigenvalues
$$\lambda_1 = 0, \qquad \lambda_2 = m_1, \qquad \lambda_3 = -m_3;$$
2. $\mathbf{U_2}$ is not stable since the Jacobian matrix of $\mathbf{R}(\mathbf{U})$ evaluated on $\mathbf{U}_2$ has the eigenvalues
$$\lambda_1 = 1, \qquad \lambda_2 = -m_1, \qquad \lambda_3 = -m_3;$$
3. $\mathbf{U_3}$ is not asymptotically stable or not stable (according to the sign of $m_1 - v^\star$) since the eigenvalues of the Jacobian matrix of $\mathbf{R}(\mathbf{U})$ evaluated on $\mathbf{U}_3$ are
$$\lambda_1 = 0, \qquad \lambda_2 = m_1 - v^\star, \qquad \lambda_3 = -m_3.$$

Moreover, the first model admits the coexistence equilibria

$$\mathbf{U}_\pm \equiv \left(u_\pm, m_1(1-u_\pm), \frac{u_\pm}{m_2}\right),$$

where

$$u_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4m_3}{m_1 m_4}}\right),$$

provided that the parameters m_1, m_3 and m_4 satisfy the constraint

$$m_1 m_4 > 4m_3, \tag{12}$$

that collapse in the same equilibrium

$$\widetilde{\mathbf{U}} \equiv \left(\frac{1}{2}, \frac{m_1}{2}, \frac{1}{2m_2}\right)$$

when $m_1 m_4 - 4m_3 = 0$. The constraint (12) on the dimensionless parameters, expressed in terms of the original parameters, reads

$$b < \frac{\kappa_1 r \xi}{4\beta}.$$

As far as the linear stability analysis is concerned, using Hurwitz criterion, we have that

1. $\mathbf{U}_+$ is asymptotically stable because all the eigenvalues of the Jacobian matrix of $\mathbf{R}(\mathbf{U})$ evaluated on $\mathbf{U}_+$ have negative real parts;
2. $\mathbf{U}_-$ is not stable because there is at least one eigenvalue of the Jacobian matrix of $\mathbf{R}(\mathbf{U})$ evaluated on $\mathbf{U}_-$ with positive real part;
3. if $m_1 m_4 - 4m_3 = 0$, the equilibrium $\widetilde{\mathbf{U}}$ is stable, but not asymptotically stable since the eigenvalues of the Jacobian matrix are:

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \frac{-m_1 m_2 \pm \sqrt{m_1 m_2^2 (m_1 - 2m_4 - 4)}}{4m_2}.$$

As far as the second variant is concerned, we have the following equilibrium solution:

$$\overline{\mathbf{U}} \equiv \left(u^\star, \frac{m_3}{m_5(m_2 - m_6 u^\star)}, \frac{u^\star}{m_2}\right)$$

with

$$u^\star = \frac{1}{2m_6}\left(m_2 + m_6 - \sqrt{(m_2 - m_6)^2 + \frac{4m_3 m_6}{m_1 m_5}}\right).$$

The coexistence equilibrium $\overline{\mathbf{U}}$ exists if

$$m_1 m_2 m_5 - m_3 > 0, \tag{13}$$

that, in terms of the original parameters becomes

$$b < \frac{\kappa_1 r \gamma \xi}{\beta^2}.$$

Furthermore, using Hurwitz criterion, $\overline{\mathbf{U}}$ is asymptotically stable.

4. Diffusion-driven instability

The effect of self- and cross-diffusive terms in a reaction-diffusion system may imply the loss of stability of an equilibrium point and lead to the emergence of special patterns. This is a well known phenomenon described for the first time in 1952 by Alan Turing in a pioneering paper [27], and investigated in several contexts by many authors [28–35].

Let us start the analysis by considering the general system (7); we briefly sketch the required steps of computation, then we specialize the results to the two models.

Denoting with $\mathbf{U}^\star \equiv (u^\star, v^\star, w^\star)$ an asymptotic stable equilibrium of the system (7) without spatial terms, and taking $\mathbf{U}_0 \equiv (u_0, v_0, w_0)$ constant, let us consider the following perturbation:

$$\mathbf{U} = \mathbf{U}^\star + \mathbf{U}_0 \exp(\lambda t + \mathrm{i}kx). \tag{14}$$

By substituting relation (14), and linearizing the system (7), we get

$$\left(\lambda \mathcal{I} - \left(\nabla_{\mathbf{U}} \mathbf{R}(\mathbf{U})\big|_{\mathbf{U}=\mathbf{U}^\star} - \mathcal{D}k^2\right)\right)\mathbf{U}_0 = \mathbf{0}. \tag{15}$$

This is a linear homogeneous system for $\mathbf{U}_0$ possessing non zero solutions provided that

$$\det\left(\lambda_k \mathcal{I} - \left(\nabla_{\mathbf{U}} \mathbf{R}(\mathbf{U})\big|_{\mathbf{U}=\mathbf{U}^\star} - \mathcal{D}k^2\right)\right) = 0,$$

and loss of stability occurs if at least one of the eigenvalues of the matrix

$$\mathcal{A} = \nabla_{\mathbf{U}}\mathbf{R}(\mathbf{U})\big|_{\mathbf{U}=\mathbf{U}^\star} - \mathcal{D}k^2 \tag{16}$$

has positive real part.

4.1. First variant

Let us consider the stable equilibrium $\mathbf{U}^\star \equiv \mathbf{U}_+$, and write the characteristic polynomial of matrix $\mathcal{A}$ given in (16),

$$-\lambda^3 + a_2(k)\lambda^2 + a_1(k)\lambda + a_0(k),$$

where

$$\begin{aligned}
a_2(k) &= -(d_{11}+d_{22}+d_{33})k^2 - \frac{m_1}{2}\left(1+\sqrt{1-\frac{4m_3}{m_1m_4}}\right),\\
a_1(k) &= -\left(d_{11}d_{22}+d_{23}d_{32}+(d_{11}+d_{22})d_{33}\right)k^4\\
&\quad -\left(\left(\frac{d_{21}}{2}+\frac{m_1}{2}(d_{22}+d_{33})+\frac{m_4d_{23}}{2m_2}\right)\left(1+\sqrt{1-\frac{4m_3}{m_1m_4}}\right)\right.\\
&\qquad \left.+\frac{m_1m_2d_{32}}{2}\left(1-\sqrt{1-\frac{4m_3}{m_1m_4}}\right)+\frac{d_{23}m_3}{m_1m_2}\right)k^2\\
&\quad -\frac{m_3}{2}\left(\left(1+\frac{2}{m_4}+\sqrt{1-\frac{4m_3}{m_1m_4}}\right)\right),\\
a_0(k) &= a_{01}k^6 + a_{02}k^4 + a_{03}k^2 + a_{04},
\end{aligned}$$

along with the positions

$$\begin{aligned}
a_{01} &= -d_{11}(d_{22}d_{33}+d_{23}d_{32}),\\
a_{02} &= \frac{d_{23}}{2}\left(1+\sqrt{1-\frac{4m_3}{m_1m_4}}\right)d_{31} + \frac{m_3d_{11}d_{23}}{m_1m_2}\\
&\quad -\frac{1}{2}d_{11}d_{32}m_1m_2\left(1-\sqrt{1-\frac{4m_3}{m_1m_4}}\right)\\
&\quad -\frac{1}{2}\left(1+\sqrt{1-\frac{4m_3}{m_1m_4}}\right)\left(d_{21}d_{33}+(d_{22}d_{33}+d_{23}d_{32})m_1+d_{11}d_{23}\frac{m_4}{m_2}\right),\\
a_{03} &= \frac{m_2m_3}{m_4}d_{31} - \frac{m_3}{2}\left(1+\sqrt{1-\frac{4m_3}{m_1m_4}}\right)d_{11}\\
&\quad -\frac{1}{2m_2}\left((m_1m_4-4m_3)+(m_1m_4-2m_3)\sqrt{1-\frac{4m_3}{m_1m_4}}\right)d_{23}\\
&\quad -\frac{m_1m_2m_3}{m_4}d_{32} - \frac{m_3}{m_4}d_{33},\\
a_{04} &= -\frac{m_1m_3}{2}\left(1-\frac{4m_3}{m_1m_4}+\sqrt{1-\frac{4m_3}{m_1m_4}}\right).
\end{aligned}$$

It is easily recognized that the coefficients $a_2(k)$ and $a_1(k)$ are negative, the latter due to the constraint (12) ensuring the existence of the equilibrium point. Therefore, we may have Turing instability if $a_0(k)$ is positive for some values of k.

Remark 1. It can be easily recognized that if $d_{31} = 0$ it is not possible to have Turing instability. Therefore, in the subsequent analysis, d_{31} will be taken as the control parameter.

Since $a_0(k)$ can be expressed as a polynomial of degree 3 in k^2, we can proceed by substituting $\kappa = k^2$, whereupon we can write

$$a_0(\kappa) = a_{01}\kappa^3 + a_{02}\kappa^2 + a_{03}\kappa + a_{04}. \tag{17}$$

Though this polynomial has to be considered for $\kappa \geq 0$, let us study its trend in $\mathbb{R}$. It is $a_{01} < 0$ so that

$$\lim_{\kappa\to-\infty} a_0(\kappa) = +\infty;$$

moreover, because $a_{04} < 0$, it is $a_0(0) < 0$. Thus, there exists a value $\kappa_1 < 0$ such that $a_0(\kappa_1) = 0$. For the existence of the Turing instability we need that $a_0(\kappa)$ possesses also two distinct positive roots.

The roots of the polynomial (17) are real and distinct if the condition

$$\frac{1}{27}\left(\frac{a_{03}}{a_{01}} - \frac{1}{3}\left(\frac{a_{02}}{a_{01}}\right)^2\right)^3 + \frac{1}{4}\left(\frac{a_{04}}{a_{01}} + \frac{2}{27}\left(\frac{a_{02}}{a_{01}}\right)^3 - \frac{1}{3}\frac{a_{02}a_{03}}{a_{01}^2}\right)^2 < 0 \tag{18}$$

is satisfied.

Both a_{02} and a_{03} are linear in d_{31}; let $d_{31}^{\star}$ and $d_{31}^{\star\star}$ the (positive) values of d_{31} annihilating a_{02} and a_{03}, respectively, and let

$$\delta_1 = \min\left(d_{31}^{\star}, d_{31}^{\star\star}\right), \quad \delta_2 = \max\left(d_{31}^{\star}, d_{31}^{\star\star}\right).$$

By Descartes' rule we may have two positive roots of (17) in the following two cases:

A_1: $a_{02}a_{03} < 0$, whereupon d_{31} cannot lie outside the open interval $]\delta_1, \delta_2[$;
A_2: $a_{02} > 0$ and $a_{03} > 0$, so that it has to be $d_{31} > \delta_2$.

Nevertheless, we have to take into account the constraint coming from (18) that poses an additional restriction to the variability of d_{31}; finally, the range for the parameter d_{31} where the diffusion driven instability arises can be determined. Once we choose the parameter d_{31} such that Turing instability may occur, the real positive roots of (17) determine the interval where the wave number k can be taken. In Section 5, we will consider some reasonable values for the parameters and study pattern formation.

4.2. Second variant

For the second model, let us consider the stable equilibrium $\mathbf{U}^{\star} \equiv \overline{\mathbf{U}}$, and write the characteristic polynomial of the matrix $\mathcal{A}$ given in (16),

$$-\lambda^3 + a_2(k)\lambda^2 + a_1(k)\lambda + a_0(k),$$

whose coefficients read

$$\begin{aligned}
a_2(k) &= -2k^2(d_{11}+d_{22}+d_{33}) - \frac{1}{2}m_1(1+m_2m_5) \\
&\quad - \frac{1-m_2m_5}{2m_6}\left(m_1m_2 - \sqrt{\frac{m_1(m_5(m_2-m_6)^2+4m_3m_6)}{m_5}}\right), \\
a_1(k) &= -\frac{m_1m_2(1+m_1m_2m_5)-m_3m_6}{2m_6^2}\sqrt{(m_2-m_6)^2+\frac{4m_3m_6}{m_1m_5}} \\
&\quad + \frac{m_1m_2}{2m_6^2}(m_1m_2m_5(m_2-m_6)+m_2+m_6(2m_3-1)) \\
&\quad - \frac{m_3}{2m_5m_6}(m_5(m_2-m_6)-2) \\
&\quad - \left(\left(\frac{d_{23}}{2m_2} + \frac{m_2(d_{32}+m_5(d_{11}+d_{22}))-d_{21}-m_1(d_{22}+d_{33})}{2m_6}\right)\times\right. \\
&\qquad \times\sqrt{(m_2-m_6)^2+\frac{4m_3m_6}{m_1m_5}} + \frac{d_{21}}{2} \\
&\quad + \frac{m_1m_2(d_{22}+d_{33}-m_2d-32-m_2m_5(d_{11}+d_{22}))+m_2d_{21}}{2m_6} \\
&\quad \left.+\frac{(m_1m_5(m_2-m_6)-m_3)d_{23}}{2m_1m_2}\right)k^2 - (d_{11}d_{22}+d_{23}d_{32}+(d_{11}+d_{22})d_{33})k^4, \\
a_0(k) &= a_{01}k^6 + a_{02}k^4 + a_{03}k^2 + a_{04},
\end{aligned}$$

along with

$$
\begin{aligned}
a_0^{(1)} &= -d_{11}(d_{22}d_{33}+d_{23}d_{32}),\\
a_0^{(2)} &= \frac{m_2(d_{21}d_{33}+m_1(d_{22}d_{33}+d_{23}d_{32}))-d_{11}(m_1m_2^2(m_5d_{22}+d_{32})+m_5m_6d_{23})}{2m_2m_6}\times\\
&\quad\times\sqrt{(m_2-m_6)^2+\frac{4m_3m_6}{m_1m_5}}-\frac{1}{2m_6}(m_2+m_6)(m_1(d_{22}d_{33}+d_{23}d_{32})+d_{21}d_{33})\\
&+\frac{d_{11}(m_1^2m_2^2(m_2-m_6)d_{32}+2m_3m_6d_{23}+m_1m_5(m_2-m_6)(m_1m_2^2d_{22}-m_6d_{23}))}{2m_1m_2m_6},\\
a_0^{(3)} &= \Bigg(\frac{m_3}{2m_6}d_{11}-\frac{m_1m_2^2m_5}{2m_6^2}(d_{21}+m_1d_{22})-\frac{2m_3+m_1m_5m_6}{2m_2m_6}d_{23}\\
&+\frac{m_1m_2}{2m_6^2}(m_2d_{31}-m_1m_2d_{32}-d_{33})\Bigg)\sqrt{(m_2-m_6)^2+\frac{4m_3m_6}{m_1m_5}}\\
&-\frac{m_3(m_2+m_6)}{2m_6}d_{11}-\frac{m_2(2m_3m_6+m_1m_2m_5(m_2-m_6))}{2m_6^2}d_{21}\\
&+\frac{m_1m_2(2m_3m_6+m_1m_2m_5(m_2-m_6))}{2m_6^2}d_{22}\\
&-\frac{m_1m_5m_6(m_2-m_6)-2m_3(m_2+2m_6)}{2m_2m_6}d_{23}\\
&-\frac{m_2(2m_3m_6+m_1m_2m_5(m_2-m_6))}{2m_5m_6^2}d_{31}\\
&+\frac{m_1m_2(m_1m_2m_5(m_2-m_6)+2m_3m_6)}{2m_5m_6^2}d_{32}\\
&+\frac{m_1m_2m_5(m_2-m_6)+2m_3m_6}{2m_5m_6^2}d_{33},\\
a_0^{(4)} &= \frac{m_1((m_1m_2^2m_5+m_3m_6)(m_2-m_6)+2m_3m_6(m_2+m_6))}{2m_6^3}\times\\
&\quad\times\sqrt{(m_2-m_6)^2+\frac{4m_3m_6}{m_1m_5}}\\
&-\frac{(m_1m_5(m_2-m_6)^2+4m_3m_6)(m_1m_2^2m_5+m_3m_6)}{2m_5m_6^3}
\end{aligned}
$$

The coefficients $a_2(k)$ and $a_1(k)$ are always negative (with the constraint (13) ensuring the existence of the equilibrium point), so that Turing instability may arise if the coefficient a_0 is positive for some values of k.

The same analysis done in the previous subsection applies to this case, the only difference being that we have to impose the following additional constraints:

$$
m_2>\frac{m_6}{2},\qquad m_1\ge\frac{4m_3}{m_5(2m_2-m_6)};
$$

moreover, taking d_{31} as the control parameter, we can characterize its range such that the diffusion driven instability arises. Once we choose the parameter d_{31} in such a way Turing instability may occur, the range where the wave number can be chosen is determined as well. In Section 5, we will consider some reasonable values for the parameters yielding the formation of some patterns.

5. Numerical results

In this Section, we present some numerical computations, and the solutions clearly exhibit the formation of well recognized patterns. Most of the values of the parameters are chosen close enough to those used in [23], except for the new control parameter d_{31} that is essential for Turing instability. For both variants of (7) with (8) or (9), we take the initial condition

$$
\mathbf{U}(x,0)=\mathbf{U}^*+\mathbf{U_0}\cos(kx),\qquad x\in[0,1],
$$

$\mathbf{U}^*$ being the asymptotically stable equilibrium of the homogeneous model and $\mathbf{U_0}=(0.01,0.01,0.01)^T$; moreover, Neumann boundary conditions at $x=0$ and $x=1$ are used; we use Neumann boundary conditions because we assume that the three groups are confined in a limited region and there is not diffusion neither towards nor from the outer world. Numerical integrations are performed by using an implicit central finite difference scheme [36] with steps $\delta x = 0.01$ and $\delta t = 0.01$. We also considered smaller steps and the accuracy of the results has not been affected.

5.1. First variant

Here, we present the numerical solutions of the system (7) along with (8) in two different scenarios characterized by different values of some of the parameters and possessing qualitatively different equilibrium solutions.

In the first scenario, the equilibrium values for drug users/dealers and law enforcement personnel are very close, so that we can describe a situation where the street is mainly occupied by ordinary people. Moreover, we take the coefficient d_{23} larger than the coefficient d_{21}; this means that criminals prefer to avoid arrest rather than convert ordinary citizens. More in detail, let us choose the set of parameters

$$\begin{aligned} &m_1 = 1, \qquad m_2 = 8, \qquad m_3 = 0.2, \qquad m_4 = 2,\\ &d_{11} = 0.055, \qquad d_{21} = 0.0018, \qquad d_{22} = 0.004,\\ &d_{23} = 0.011, \qquad d_{32} = 0.011, \qquad d_{33} = 0.036, \end{aligned} \tag{19}$$

whereupon the stable homogeneous equilibrium is

$$\mathbf{U}^\star \equiv (0.887298, 0.112782, 0.110912);$$

to ensure Turing instability it is required $d_{31} > 0.0532081$.

Figs. 1 and 2 show the contour plots of the solution in correspondence of the parameters (19) for two different values of d_{31} greater than the critical value. Each figure displays on the left the evolution in the time interval needed for the emergence of the pattern, whereas on the right the asymptotic solution after the transient evolution. For large values of time it can be observed the formation of *strips*. There is a sort of segregation of the three subgroups: ordinary citizens and law enforcement personnel are more abundant far from the center of the domain, whereas drug users/dealers remain concentrated (and isolated) in the central part of the domain.

In the second scenario, let us choose the set of parameters

$$\begin{aligned} &m_1 = 1, \qquad m_2 = 11, \qquad m_3 = 0.25, \qquad m_4 = 1.5,\\ &d_{11} = 0.055, \qquad d_{21} = 0.03, \qquad d_{22} = 0.004,\\ &d_{23} = 0.011, \qquad d_{32} = 0.011, \qquad d_{33} = 0.036, \end{aligned} \tag{20}$$

whence the stable homogeneous equilibrium turns out to be

$$\mathbf{U}^\star \equiv (0.788675, 0.211325, 0.0716977);$$

the constraint $d_{31} > 0.0378693$ guarantees the rise of diffusion driven instability. In this scenario, at the equilibrium, the value for drug users/dealers is about three times the value of law enforcement personnel. Because of this, to mimic the situation where drug users/dealers do not care about the risk of being arrested in trying to convert ordinary citizens, we take a value of d_{23} smaller than that of d_{21}.

Figs. 3 and 4 show the contour plots of the solution in correspondence of the parameters (20) for two different values of d_{31} greater than the critical value. As done above, each figure displays on the left the evolution in the time interval needed for the emergence of the pattern, whereas on the right the asymptotic solution after the transient evolution. Also in this scenario, for large values of time we observe the formation of *strips*. Nevertheless, differently from the first scenario, for large values of time, ordinary citizens and law enforcement personnel concentrate in the left part of the street, whereas drug dealers are concentrated in the right part of the domain.

Fig. 5 displays the time evolution of averages and variances of the densities of the three subgroups all over the spatial domain in all the cases considered. It can be observed that, after the transient phase, as one expects, the variances stay constant in time, and this corresponds to the fact that the strips remain stationary even for a long time. In both scenarios, we notice that, increasing the value of d_{31} the duration of the transient phase is shortened.

5.2. Second variant

For the system (7), along with (9), we consider two different scenarios too. The first scenario uses the following set of parameters

$$\begin{aligned} &m_1 = 1, \qquad m_2 = 10, \qquad m_3 = 0.05, \qquad m_5 = 0.3, \qquad m_6 = 9,\\ &d_{11} = 0.06, \qquad d_{21} = 0.001, \qquad d_{22} = 0.02,\\ &d_{23} = 0.012, \qquad d_{32} = 0.012, \qquad d_{33} = 0.04; \end{aligned} \tag{21}$$

the stable homogeneous equilibrium is

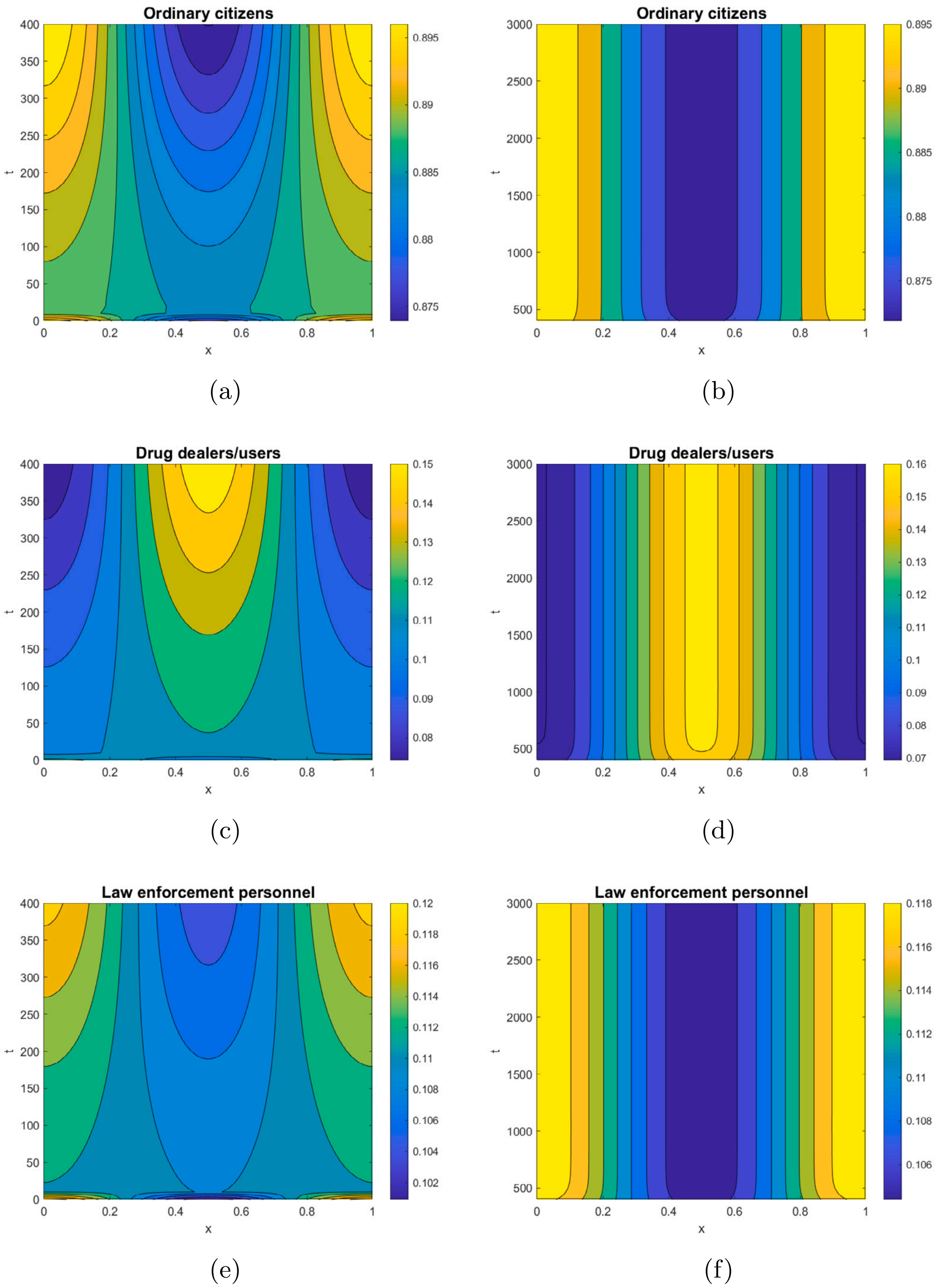

Fig. 1. Contour plots of the solution for $t \in [0, 400]$ in (a)-(c)-(e) and for $t \in [400, 3000]$ in (b)-(d)-(f) with the parameters given in (19); $d_{31} = 0.075$, $k = 2\pi$.

$$\mathbf{U}^\star \equiv (0.908569, 0.0914306, 0.0908569),$$

and, in order to have Turing instability, it is required $d_{31} > 0.0503979$.

Figs. 6 and 7 show the contour plots of the solution according to the parameters in (21) for two different values of d_{31} greater than the critical value. Each figure exhibits on the left the evolution in the time interval needed for the emergence of the pattern, whereas on the right the stationary solution. Also in this case hot *strips* originate for large values of time. It is highlighted the formation of aggregations of the three subgroups on the sides of the domain. In particular, citizens and policemen are shifted on the left side of the domain while drug users/dealers concentrate on the right side. The formation of pattern takes longer with respect to what observed in the first variant.

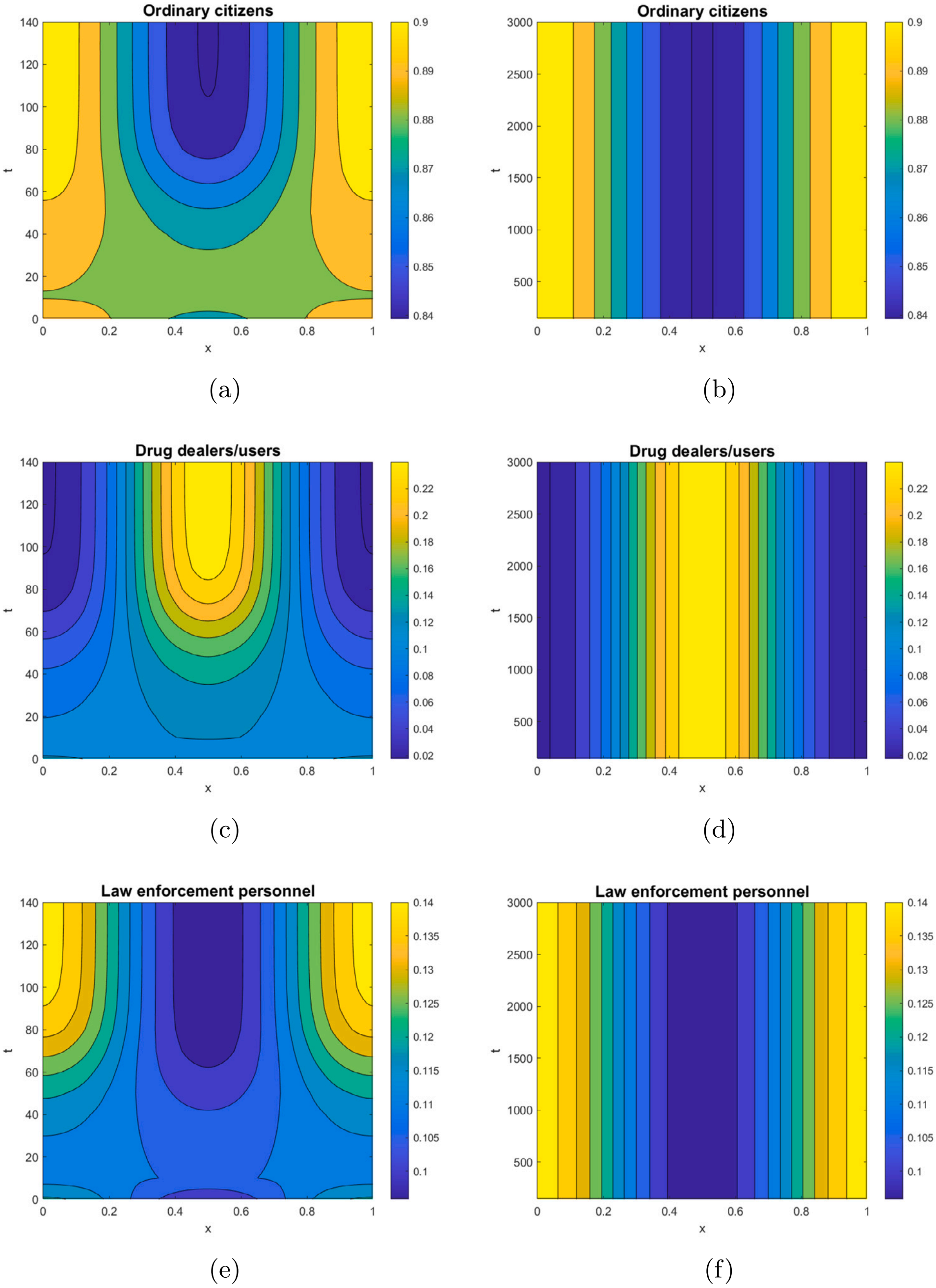

Fig. 2. Contour plots of the solution for $t \in [0, 140]$ in (a)-(c)-(e) and for $t \in [140, 3000]$ in (b)-(d)-(f) with the parameters given in (19); $d_{31} = 0.08$, $k = 2\pi$.

On the contrary, the second scenario uses the set of parameters

$$
\begin{aligned}
& m_1 = 1, \qquad m_2 = 10, \qquad m_3 = 0.07, \qquad m_5 = 0.2, \qquad m_6 = 10, \\
& d_{11} = 0.06, \qquad d_{21} = 0.05, \qquad d_{22} = 0.02, \\
& d_{23} = 0.012, \qquad d_{32} = 0.012, \qquad d_{33} = 0.04;
\end{aligned}
\tag{22}
$$

the stable homogeneous equilibrium is

$$\mathbf{U}^\star \equiv (0.687868, 0.312132, 0.0859835),$$

and, in order to have Turing instability, it is required $d_{31} > 0.0595941$.

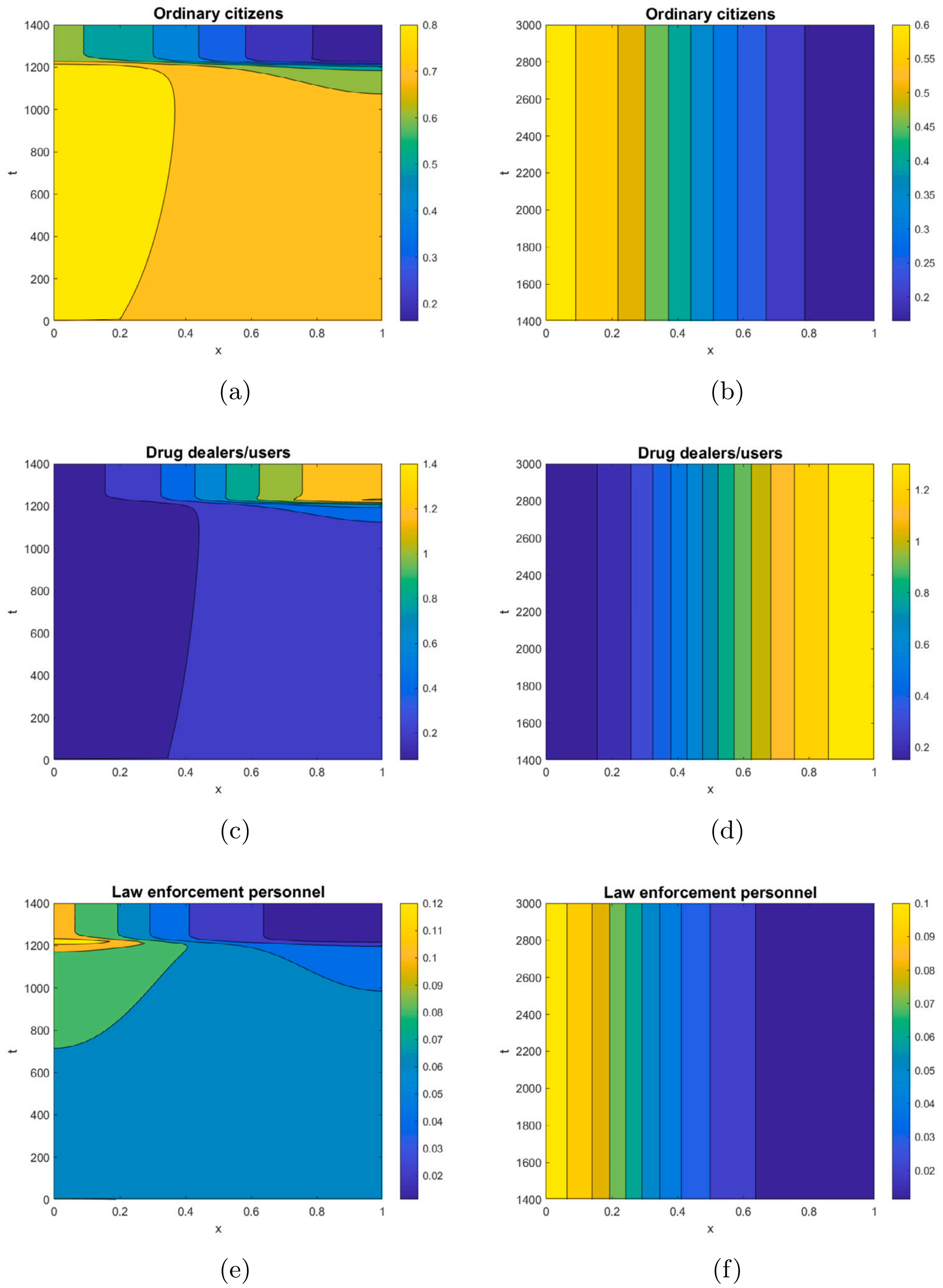

Fig. 3. Contour plots of the solution for $t \in [0, 1400]$ in (a)-(c)-(e) and for $t \in [1400, 3000]$ in (b)-(d)-(f) with the parameters given in (20) and $d_{31} = 0.039$, $k = \pi$.

In this scenario, at the equilibrium, the value for drug users/dealers is about four times the value of law enforcement personnel. Therefore, by assigning a value of d_{23} less than d_{21}, this allows the drug users/dealers to move fearless towards ordinary citizens.

Figs. 8 and 9 depict the contour plots of the solution in correspondence of the parameters (22) for two different values of d_{31} greater than the critical value. As in the previous cases, on the left it is represented the evolution of the solution up to the formation of the pattern, while on the right is shown the stationary state exhibiting the pattern with strips. Ordinary citizens and law enforcement personnel definitely concentrate on the same side (left), while drug users/dealers are more abundant on the opposite part (right) of the street.

Fig. 10 illustrates the time evolution of averages and variances of the densities of the three subgroups all over the spatial domain in all the cases considered. As seen in the first variant, once the transient is over, the variance remains constant in time, thence the strips are stationary. Finally, the time required for the formation of Turing patterns decreases as the diffusion coefficient d_{31} increases.

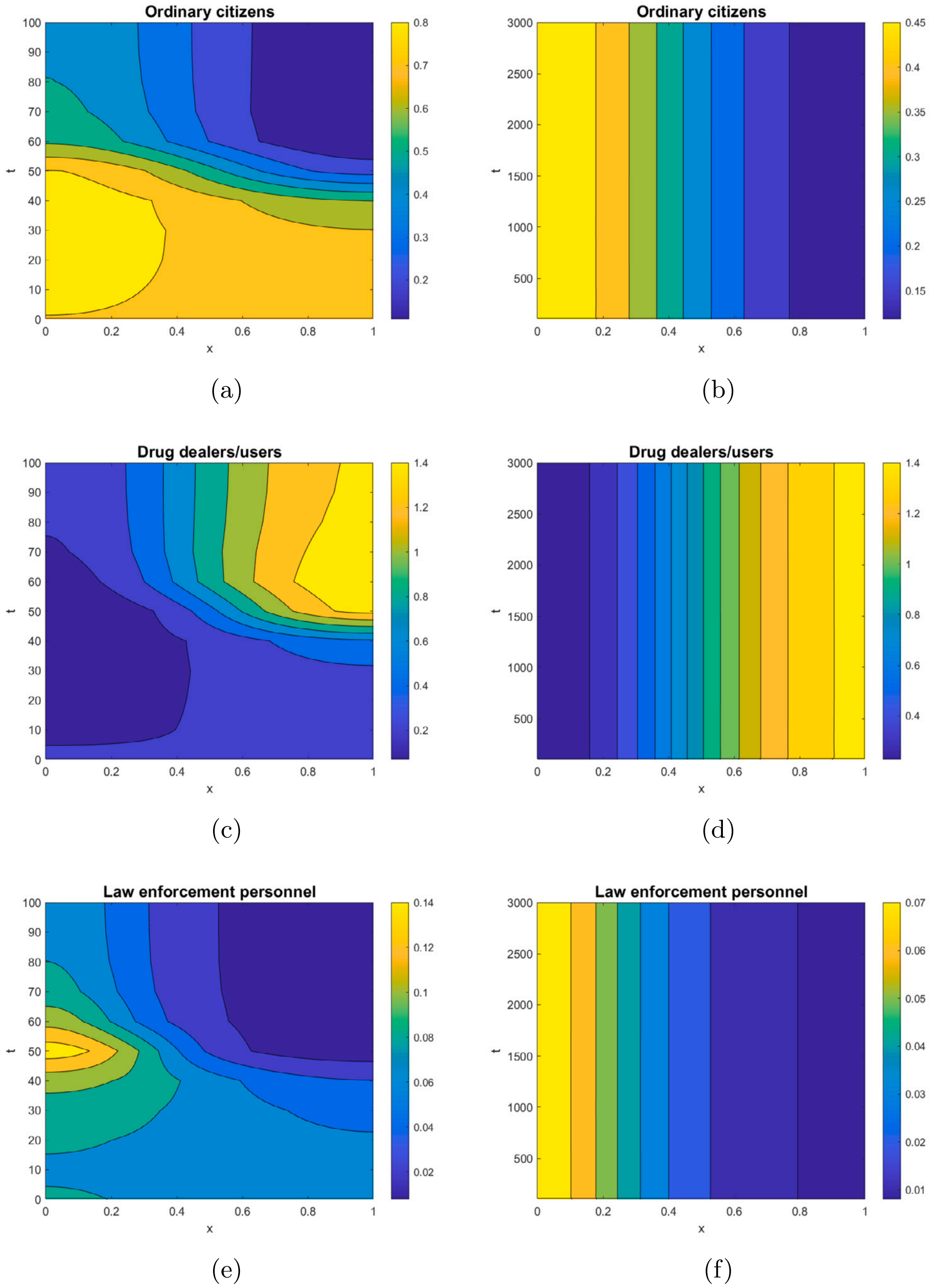

Fig. 4. Contour plots of the solution for $t \in [0, 100]$ in (a)-(c)-(e) and for $t \in [100, 3000]$ in (b)-(d)-(f) with the parameters given in (20) and $d_{31} = 0.043$, $k = \pi$.

6. Conclusions

The pattern formation in reaction-diffusion systems is discussed in areas such as ecology and social science.

In this paper, we implemented and investigated two variants of a model originally proposed by Epstein, named crimo-taxis. The models consist of three coupled reaction-diffusion equations involving self- and cross-diffusion coefficients. The modifications to the Epstein model consist in adding a logistic effect in the susceptible population and a cross-diffusion coefficient in order to describe a sort of *citizens' protection* made by the policemen. The second variant modifies also the original term accounting for the growth in the number of police forces in parallel with the increase of the level of social alarm.

Both models here studied admit asymptotically stable homogeneous coexistence equilibria, susceptible of losing their stability due to the self- and cross-diffusive terms. We analytically prove that the models may experience Turing instability depending on the value of a control parameter. The numerical simulations show the emergence of some characteristic patterns that remain stationary

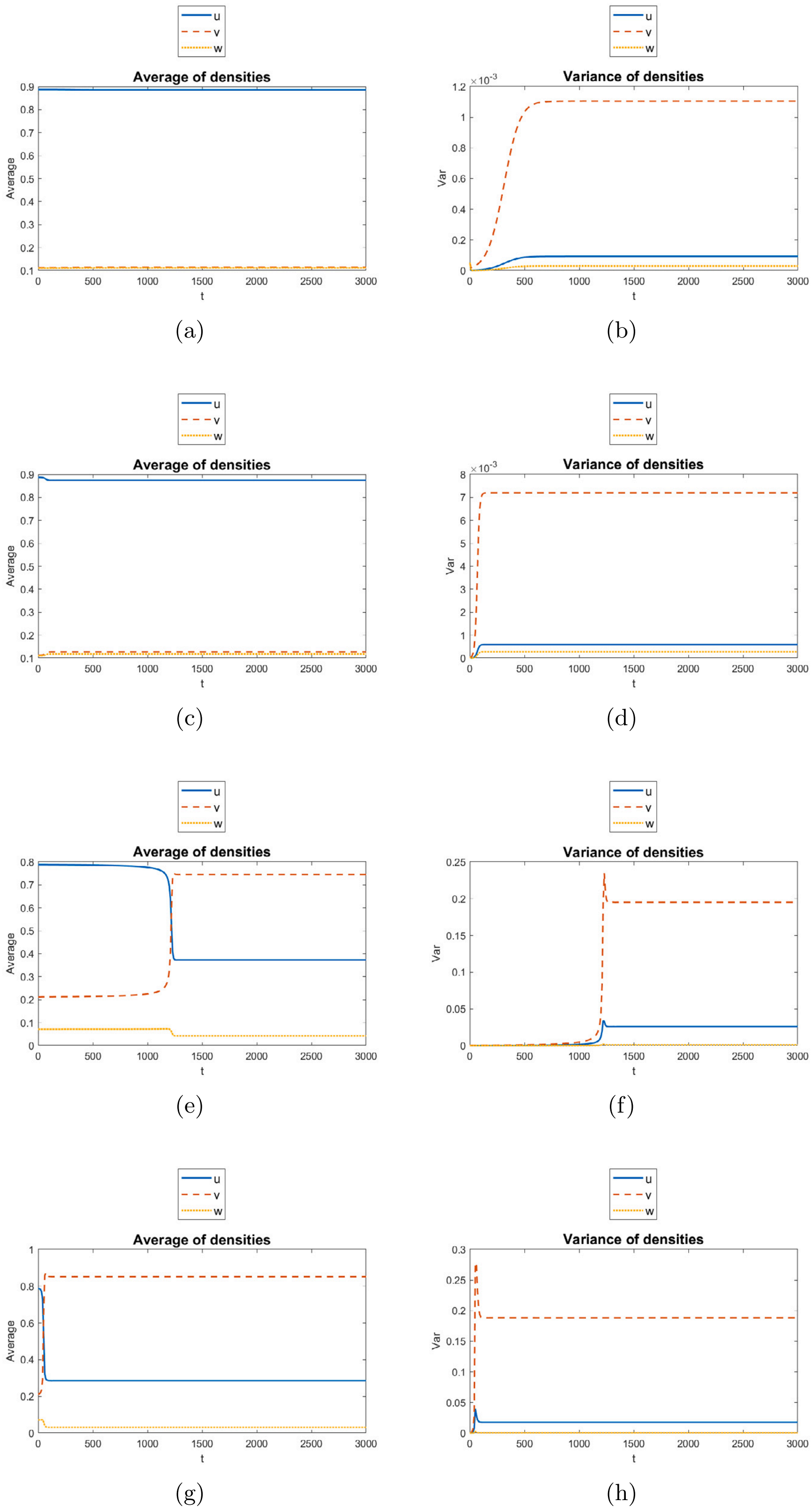

Fig. 5. Plots of the averages (left) and variances (right) of the densities; subfigures (a)-(d) refer to the first scenario with $d_{31} = 0.075$ ((a), (b)), $d_{31} = 0.08$ ((c), (d)); subfigures (e)-(h) refer to the second scenario with $d_{31} = 0.039$ ((e), (f)), $d_{31} = 0.043$ ((g), (h)).

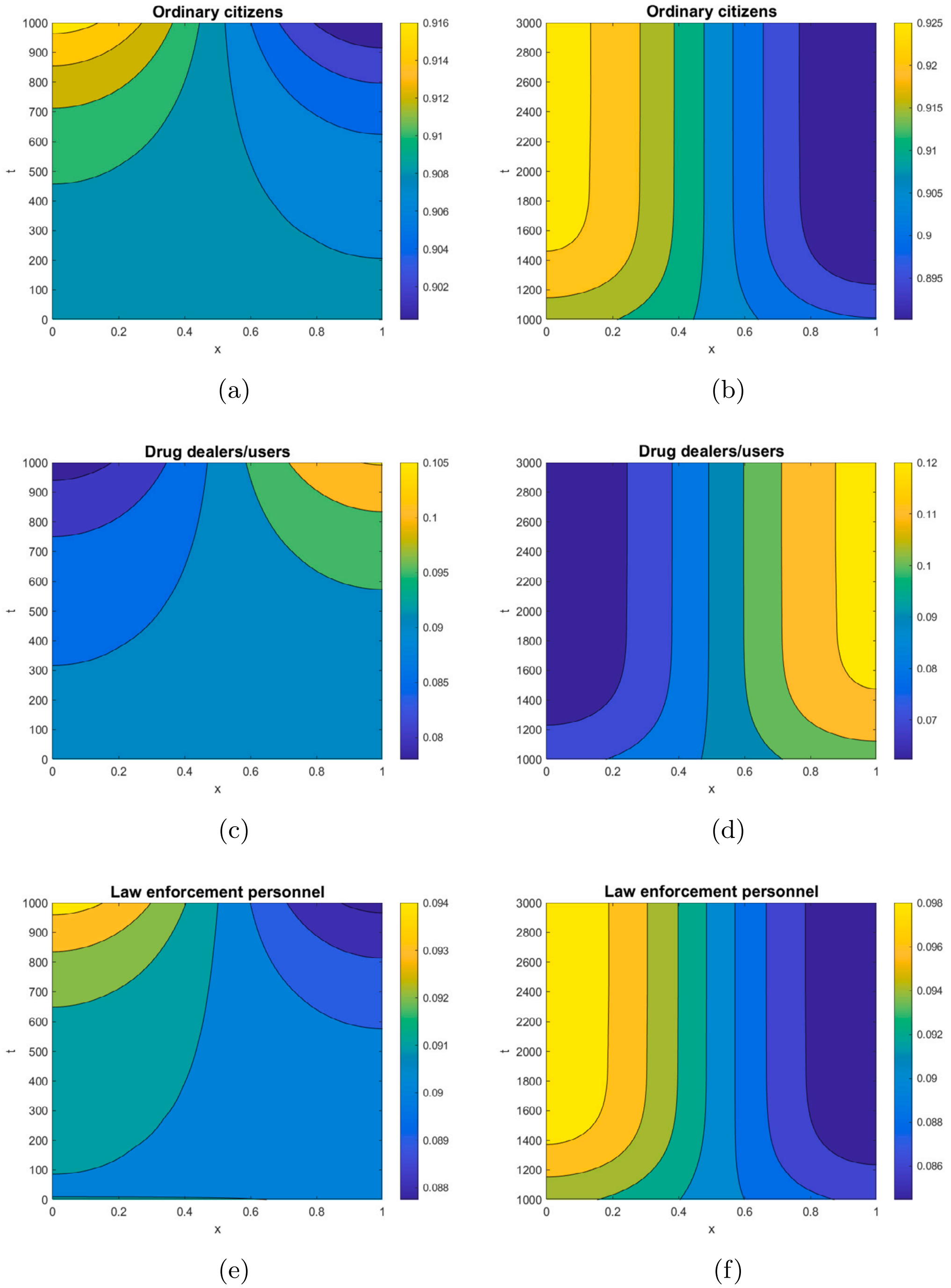

Fig. 6. Contour plots of the solution for $t \in [0, 1000]$ in (a)-(c)-(e) and for $t \in [1000, 3000]$ in (b)-(d)-(f) with the parameters given in (21) and $d_{31} = 0.055$, $k = 0.7\pi$.

over time. The stationary nonhomogeneous solutions show that the models definitely allow for a distribution of the three subgroups in such a way law enforcement personnel is able to protect ordinary citizens, whereas drug users/dealers are isolated.

The two models proposed in this paper share a property not exhibited by the original Epstein's crimo-taxis model [23], that is the existence of non trivial asymptotically stable homogeneous equilibria suitable to lose their stability because of self- and cross-diffusive terms. Moreover, the inclusion in both models of a new cross-diffusive term allows to model situations, such as concerts or social or political demonstrations, where the diffusion of law enforcement personnel is driven also by higher concentrations of ordinary citizens.

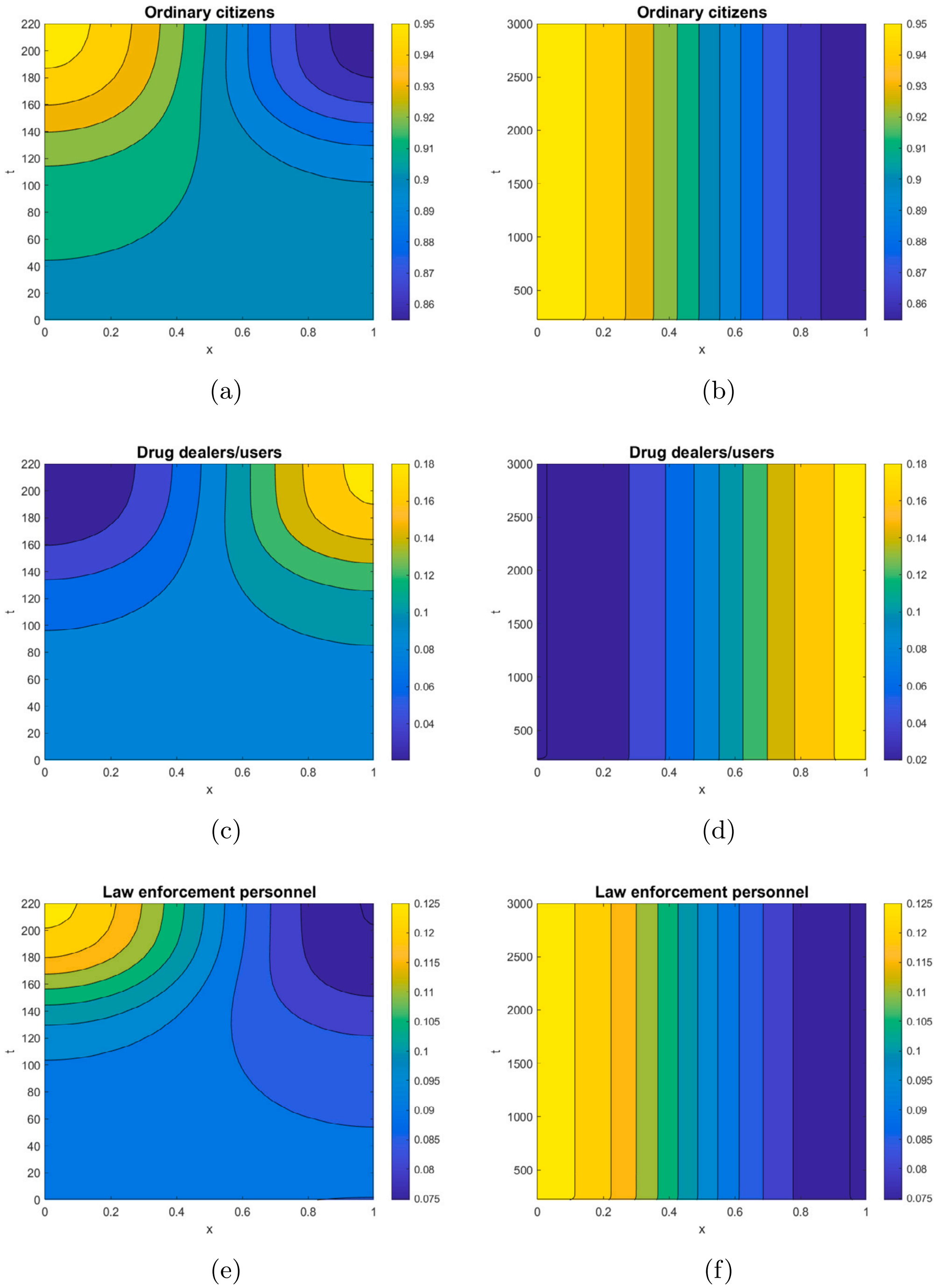

Fig. 7. Contour plots of the solution for $t \in [0, 220]$ in (a)-(c)-(e) and for $t \in [220, 3000]$ in (b)-(d)-(f) with the parameters given in (21) and $d_{31} = 0.06$, $k = \pi$.

The patterns exhibited by the two variants heretofore analyzed are somewhat similar, although they exhibit a different distribution of the three subgroups. In the first variant, ordinary citizens and law enforcement personnel are more abundant far from the center of the domain, whereas drug users/dealers are primarily isolated in the central part of the street. On the contrary, as far as the second variant is concerned, the pattern exhibited after the transient phase show that ordinary citizens and law enforcement personnel definitely concentrate on the same side (left) of the street, while drug users/dealers are more abundant on the opposite part (right) of the street. The formation of both kinds of patterns is a consequence of the introduction in the model of the new cross-diffusive term in the third equation able to model the diffusion of the law enforcement personnel towards the region where ordinary citizens

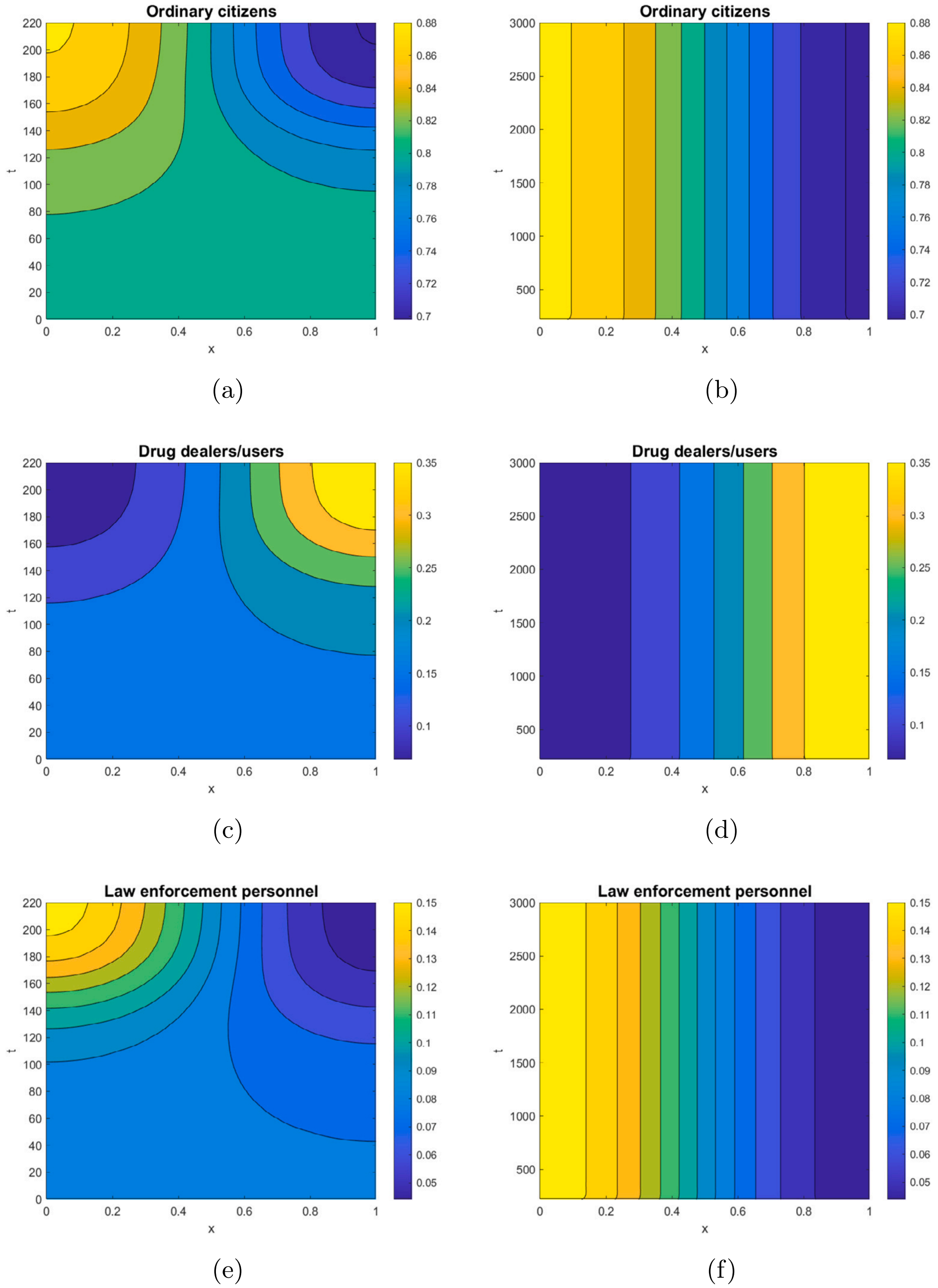

Fig. 8. Contour plots of the solution for $t \in [0, 220]$ in (a)-(c)-(e) and for $t \in [220, 3000]$ in (b)-(d)-(f) with the parameters given in (22) and $d_{31} = 0.07, k = \pi$.

move in order to avoid the interaction with drug users/dealers. The models investigated, possibly supported by real data in order to tune the values of the involved parameters, could be useful for policy makers in their decisional processes.

Although we do not have at the present rigorous results establishing the conditions for the positivity of the solutions, the parameters we used and the numerical solutions above presented describe physically meaningful situations.

The models here analyzed, as in [23], involve linear diffusive or cross diffusion terms. Nevertheless, possible extensions we plan to investigate in the near future could include nonlinear diffusion terms, that is to say analyze systems like

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{R}(\mathbf{U}) + \frac{\partial}{\partial x}\left(\mathcal{D}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x}\right), \tag{23}$$

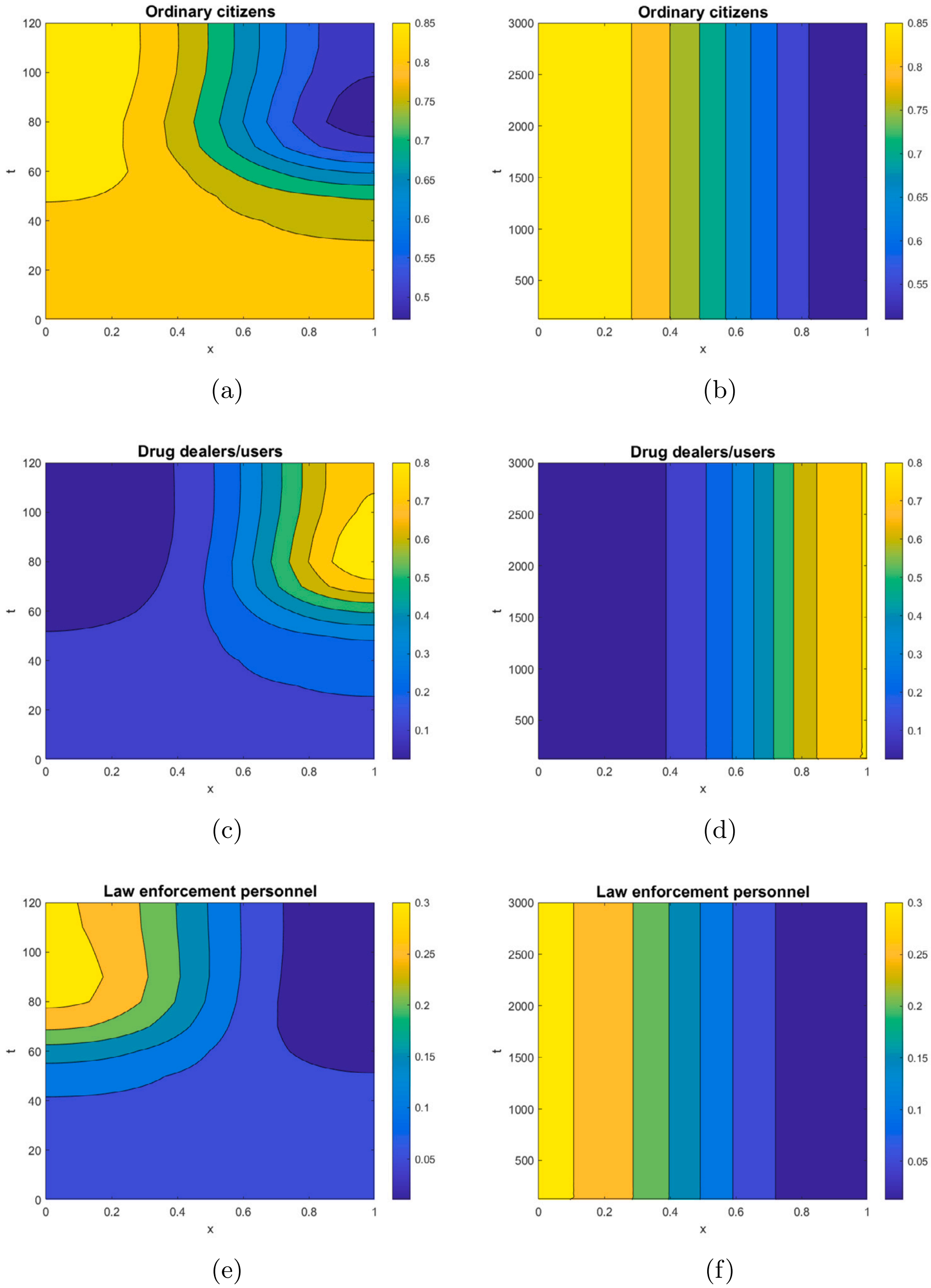

Fig. 9. Contour plots of the solution for $t \in [0, 120]$ in (a)-(c)-(e) and for $t \in [120, 3000]$ in (b)-(d)-(f) with the parameters given in (22) and $d_{31} = 0.08$, $k = \pi$.

where $\mathbf{U}$ is the vector of unknown variables, and $\mathcal{D}$ is a matrix with entries depending on $\mathbf{U}$. Moreover, it could be interesting an extension of the models to a two–dimensional spatial setting modeling a neighborhood of a town, or a further generalization including an additional subgroup (the arrested individuals) so that the action of the law enforcement personnel is not limited to protect ordinary citizens but also to actively crack down on illegal behavior. Last but not the least, we are implementing an operatorial model where the actors of the system are represented by annihilation, creation and number fermionic operators whose evolution is ruled by a time-independent Hermitian Hamiltonian $\mathcal{H}$ within the recently introduced framework of $(\mathcal{H}, \rho)$–induced dynamics [37].

Data availability

No data was used for the research described in the article.

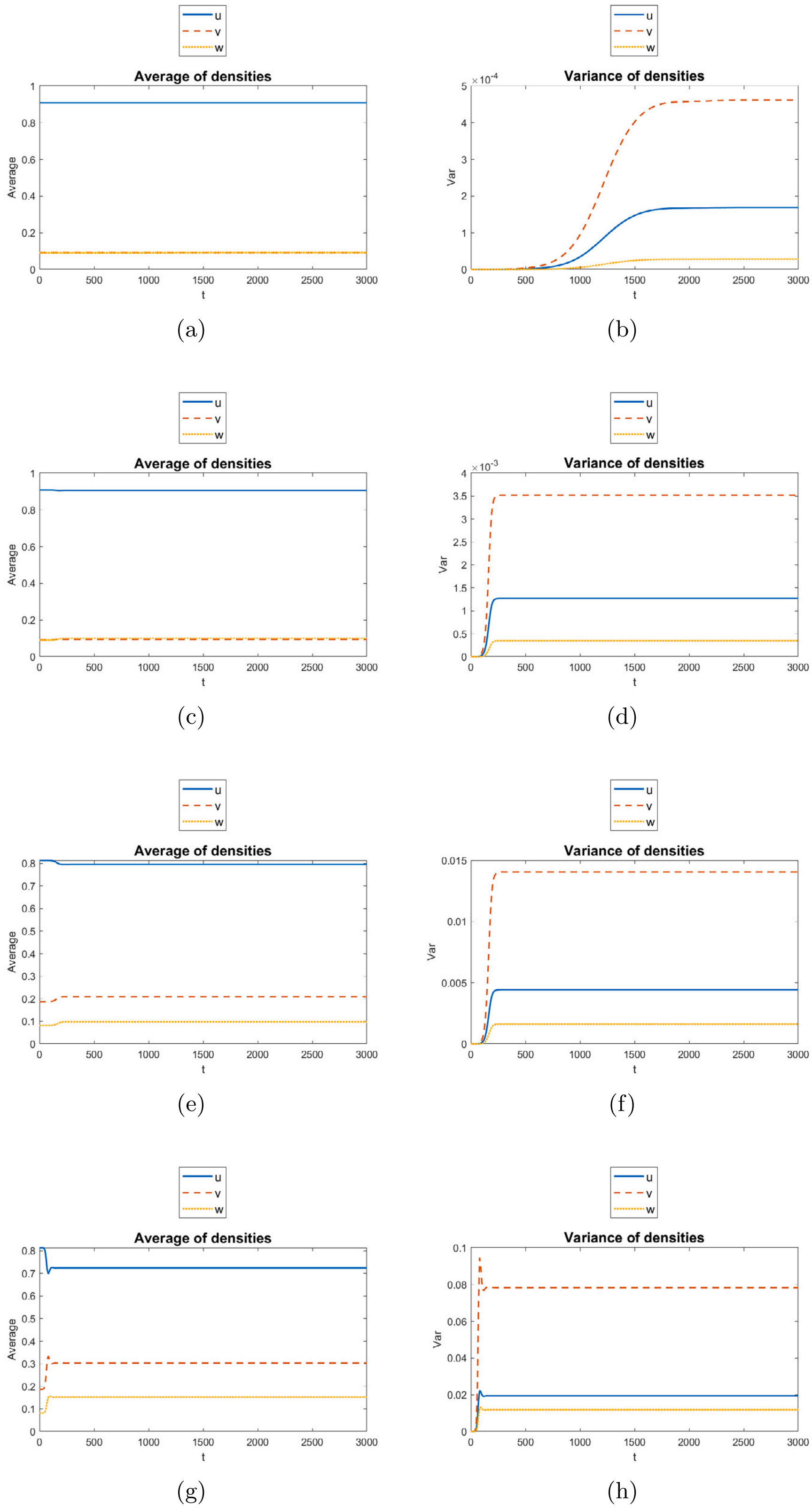

Fig. 10. Plots of the averages (left) and variances (right) of the densities; subfigures (a)-(d) refer to the first scenario with $d_{31} = 0.06$ ((a), (b)), $d_{31} = 0.07$ ((c), (d)); subfigures (e)-(h) refer to the second scenario with $d_{31} = 0.07$ ((e), (f)), $d_{31} = 0.08$ ((g), (h)).

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