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# A NOVEL APPROACH TO THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS*

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**Abstract.** The classical numerical treatment of two-point boundary value problems on infinite intervals is based on the introduction of a truncated boundary (instead of infinity) where appropriate boundary conditions are imposed. Then, the truncated boundary allowing for a satisfactory accuracy is computed by trial. Motivated by several problems of interest in boundary layer theory, here we consider boundary value problems on infinite intervals governed by a third-order ordinary differential equation. We highlight a novel approach to define the truncated boundary. The main result is the convergence of the solution of our formulation to the solution of the original problem as a suitable parameter goes to zero. In the proposed formulation, the truncated boundary is an unknown free boundary and has to be determined as part of the solution. For the numerical solution of the free boundary formulation, a noniterative and an iterative transformation method are introduced. Furthermore, we characterize the class of free boundary value problems that can be solved noniteratively. A nonlinear flow problem involving two physical parameters and belonging to the characterized class of problems is then solved. Moreover, the Falkner–Skan equation with relevant boundary conditions is considered and representative results, obtained by the iterative transformation method, are listed for the Homann flow. All the obtained numerical results clearly indicate the effectiveness of our approach. Finally, we discuss the possible extensions of the proposed approach and for the question of a priori error analysis.

**Key words.** nonlinear boundary value problems, infinite intervals, truncated boundary, free boundary formulation

**AMS subject classifications.** 65L10, 34B15, 76D10

**1. Introduction.** This paper describes a novel approach to the numerical solution of boundary value problems (BVPs) on infinite intervals. In solving problems of this type, we have to deal with at least one condition at infinity which is not suitable for numerical use. To overcome this difficulty, some different approaches have been considered.

One approach is to replace the boundary conditions at infinity by the same conditions at a finite value $x_\infty$ (the truncated boundary). As far as the accuracy question is concerned, an appropriate truncated boundary can be found by trial as reported by Fox [14, p. 92] or by Collatz [6, pp. 150–151]. This simple approach is sufficient and efficient in many cases, although sometimes it provides good results only for very large values of $x_\infty$. Seldom a simple estimate of the error due to the truncated boundary is available; in the case of the Blasius problem, a simple analysis based upon the invariance properties of the problem is given by Rubel [34].

Another approach, performing a preliminary asymptotic analysis to find the appropriate boundary conditions to be imposed at a truncated boundary, has been proposed by de Hoog and Weiss [18], Lentini and Keller [24], and Markowich [25] (see also the analytical work of Markowich [26] and the related work of Markowich and Ringhofer [27] and Schemeiser [35]). Since the imposed conditions are related to the asymptotic behaviour of the solution, for the same values of $x_\infty$ this approach usually yields a more accurate solution than the previous approach. However, as remarked by Ockendon [33], in order to be able to define the truncated boundary conditions for a given BVP, a preliminary asymptotic analysis involving the eigenvalues of the related Jacobian matrix evaluated at infinity has to be done. Moreover, as suggested by Lentini and Keller [24], a priori estimates for a convenient truncated boundary are usually not easy to obtain and that represents a current area of research.

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Third, it is sometimes possible to formulate the original problem as a free BVP, where the unknown free boundary can be identified with a truncated boundary. This approach has already been applied to the Blasius problem in [12].

Finally, de Hoog and Weiss [17] proposed an analytical transformation of the independent variable which reduces the original problem to a BVP over a finite interval. Usually, that produces a singularity of the second kind at the origin (see Ascher and Wan [3] for an interesting application).

In the following we shall be concerned with the third approach mentioned above. Motivated by several cases of interest, we consider the two-point BVP:

$$\frac{d^3u}{dx^3} = f\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \boldsymbol{P}\right),$$

$$u(0) = g\left(\frac{d^2u}{dx^2}(0), \boldsymbol{P}\right), \qquad \frac{du}{dx}(0) = \ell\left(\frac{d^2u}{dx^2}(0), \boldsymbol{P}\right), \tag{1.1}$$

$$\frac{du}{dx}(x) \to C \quad \text{as} \quad x \to \infty,$$

where $\boldsymbol{P} \equiv (P_1, P_2, \ldots, P_n)$ is an $n$-dimensional vector with the parameters $P_i$ for $i = 1, \ldots, n$ as components; $f : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^n \to \mathbb{R}$; $g, \ell : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy suitable smoothness conditions in order (1.1) to be well-posed; and $C$ is a given constant. Moreover, we assume that the solution of (1.1) verifies the asymptotic condition

$$\frac{d^2u}{dx^2}(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{1.2}$$

A brief list of relevant problems belonging to the class (1.1) is as follows: the problem describing the flow of a Newtonian fluid past a wedge (Falkner–Skan [11], see also Weyl [41]), for which the condition (1.2) holds; the flow of power-law non-Newtonian fluids past a semi-infinite flat plate (see Acrivos, Shah, and Petersen [1] and Nath [31]); Hiemenz magnetic flow of power-law non-Newtonian fluids for which, as indicated by Djukic [9], (1.2) is fulfilled; the flow of an incompressible fluid over a slender parabola of revolution as reported by Na [29, pp. 217–221]; and the large-scale circulation in an ocean (see Ierley and Ruehr [19] and Ierley [20]).

Existence and uniqueness results valid for a more general class of problems than (1.1), but without any dependence on parameters, have been obtained by Lentini and Keller [24]. However, we remark here that the well-posedness of (1.1) depends on the values of the components of $\boldsymbol{P}$.

In the next section we formulate (1.1) as a free BVP, where the free boundary can be considered as an unknown truncated boundary. Then, by assuming appropriate smoothness conditions, we prove the convergence of the solution of the free boundary formulation to the solution of (1.1) as a suitable parameter goes to zero. Often, free BVPs are solved numerically by an iterative method (cf., for instance, Wang [39]). However, sometimes the free boundary formulation can be solved noniteratively. Within such a context, in §3 we define a noniterative transformation method applicable to a subclass of the original class of problems. Via a transformation method the solution of a BVP is obtained by solving a sequence of related initial value problems. A noniterative transformation method was applied to the Blasius problem by Töpfer [37] and extended to a class of problems by Klamkin [22]. That method was extended to problems involving one or more physical parameters by Na [28] and by Scott, Rinschler, and Na [36]. This extension is useful when we are interested in the solution of the considered problem within a complete range of values of the parameters involved (see Na [29, Chaps. 8 and 9]). The noniterative method proposed here is a variation of the method developed by Scott, Rinschler, and Na [36] allowing for the treatment of free BVPs. For illustrative purposes, in §5 we apply our method to a problem involving two physical parameters.

In §4 an iterative extension of the transformation method is introduced. On one hand the proposed extension is iterative, but on the other hand the method is now widely applicable. As an example we consider in §5 the Falkner–Skan equation under the classical boundary conditions. This problem is of particular interest because it does not belong to the characterized class of problems that can be solved noniteratively (see Na [29, Chaps. 7–9]). Weyl [41] proved that for each value of the physical parameter involved there exists a solution for which its second derivative is positive, monotone decreasing on $[0, \infty)$ and approaching zero as $x$ goes to infinity. The uniqueness question is more complex: when the involved physical parameter is greater than one, in addition to the monotone solution characterized by Weyl, a hierarchy of solutions with reversed flow exists as shown by Coppel [7]. These solutions were the subject of a numerical study by Craven and Peletier [8]. In §5 we list the numerical results obtained for the Homann flow. This particular problem allows for a comparison of results obtained with different values of the truncated boundary.

Finally, in the last section, we make some conclusions and discuss open questions for the present work. We further point out that a free boundary formulation for a BVP on an infinite interval is an effective way to deal with the accuracy question and therefore to define the appropriate truncated boundary.

**2. The free boundary formulation and its solution convergence.** As a first step we consider the following free BVP:

$$\begin{gathered}
\frac{d^3u_\epsilon}{dx^3} = f\left(x, u_\epsilon, \frac{du_\epsilon}{dx}, \frac{d^2u_\epsilon}{dx^2}, \boldsymbol{P}\right), \\
u_\epsilon(0) = g\left(\frac{d^2u_\epsilon}{dx^2}(0), \boldsymbol{P}\right), \qquad \frac{du_\epsilon}{dx}(0) = \ell\left(\frac{d^2u_\epsilon}{dx^2}(0), \boldsymbol{P}\right), \\
\frac{du_\epsilon}{dx}(x_\epsilon) = C, \qquad \frac{d^2u_\epsilon}{dx^2}(x_\epsilon) = \epsilon,
\end{gathered} \tag{2.1}$$

where $0 < |\epsilon| \ll 1$, $\operatorname{sgn}(\epsilon) = \operatorname{sgn}(C - \frac{du}{dx}(0))$ (sgn denotes the signum function), $u_\epsilon(x) = u(x;\epsilon)$, and $x_\epsilon$ is unknown. Thus, (2.1) is a free boundary formulation for (1.1). However, we have to note here that (2.1) is a nonlinear problem, aside from the linearity of the governing differential equation and boundary conditions.

By introducing the new independent variable $x/x_\epsilon$, (2.1) can be transformed to a problem defined on a fixed domain, where the unknown free boundary can be considered as a further dependent variable by means of the equation $\frac{dx_\epsilon}{dx} = 0$. Therefore, existence and uniqueness results for two-point BVPs defined on fixed intervals, valid under suitable regularity conditions, may be extended to (2.1) (see Ascher, Mattheij, and Russell [2, pp. 87–91] and the references quoted therein on [2, pp. 541–542]). In a similar way continuous dependence and differentiability with respect to the boundary data may be available as an extension of the results obtained by Brantley and Henderson [4] or by Ehme [10].

Let us assume the existence and uniqueness of a solution $u(x)$ of (1.1). Since for $\epsilon = 0$ (2.1) has the same boundary conditions as (1.1)–(1.2), then $(u(x), \infty)$ is a solution of (2.1) for $\epsilon = 0$. Let us now assume the existence and uniqueness of a solution $u(x;\epsilon)$ of (2.1) and its continuous dependence on $\epsilon$ on a nonempty interval including $\epsilon = 0$, right or left neighbourhood of $\epsilon = 0$ depending on the value of $\mathrm{sgn}(\epsilon)$; hence $u(x; 0) \equiv u(x)$, $u(x; \epsilon) \to u(x; 0)$ on $[0, x_\epsilon]$ and $x_\epsilon \to \infty$ as $\epsilon \to 0$. This seems to be a convergence result, but in this form it does not provide any information about the order of convergence. However, our formulation is effective from a numerical viewpoint when $u_\epsilon \to u$ on $[0, x_\epsilon]$ at least linearly in $|\epsilon|$. This is the result of the following analysis.

LEMMA 1. *Let us assume that $x_\epsilon$ is a differentiable function of $\epsilon$ on a nonempty interval including $\epsilon = 0$ and that* $\lim_{\epsilon\to 0}\frac{dx_\epsilon}{d\epsilon}$ *exists. Then there exists a subinterval $I_0$, where for all $\epsilon_1, \epsilon_2 \in I_0$ such that $|\epsilon_1| < |\epsilon_2|$, it follows that* $[0, x_{\epsilon_2}] \subset [0, x_{\epsilon_1}]$.

*Proof.* The proof of this lemma can be split into two parts. First let us suppose $\mathrm{sgn}(\epsilon) = +1$, so that we are considering a right interval containing $\epsilon = 0$, where $x_\epsilon \to \infty$ as $\epsilon \to 0$. As a simple consequence of the limit concept there exists an interval of $\epsilon = 0$ where $1/x_\epsilon > 0$, whereupon $\lim_{\epsilon\to 0}\frac{d(1/x_\epsilon)}{d\epsilon} > 0$. As a result we get $\lim_{\epsilon\to 0}\frac{dx_\epsilon}{d\epsilon} = -\infty$. By means of a similar argument we find that there exists an interval $I_0$ including $\epsilon = 0$ where $\frac{dx_\epsilon}{d\epsilon} < 0$, so that, on $I_0$, $x_\epsilon$ is a decreasing function of $\epsilon$.

In the case $\mathrm{sgn}(\epsilon) = -1$, the proof can be carried out along the same lines as above to show that there exists a left interval including $\epsilon = 0$, where $x_\epsilon$ is an increasing function of $\epsilon$. $\square$

In the following theorem we prove, under suitable smoothness conditions, the uniform convergence of the solution of (2.1) to the solution of (1.1) on every arbitrarily large finite interval.

THEOREM 1. *Provided Lemma* 1 *holds, a sufficient condition for the uniform convergence of the solution of* (2.1) *to the solution of* (1.1) *on every arbitrarily large finite interval is that there exists an interval ${I_0}^*$ including $\epsilon = 0$ such that $u(x; \epsilon)$ and $\frac{\partial u}{\partial \epsilon}(x; \epsilon)$ are continuous functions on* $[0, x_\epsilon] \times [{I_0}^*]$.

*Proof.* Here and in the following the notation $[\cdot]$ indicates the closure of the indicated interval. Let us fix $\epsilon^* \in I_0 \cap {I_0}^*$. For each $\epsilon$ such that $|\epsilon| < |\epsilon^*|$, owing to the mean value theorem, we have

$$u(x; \epsilon) - u(x; 0) = \epsilon \frac{\partial u}{\partial \epsilon}(x; \vartheta),$$

where $0 < |\vartheta| < |\epsilon|$. As a consequence

$$|u_\epsilon(x) - u(x)| = |\epsilon| \left| \frac{\partial u}{\partial \epsilon}(x; \vartheta) \right|,$$

whereupon

$$\|u_\epsilon(x) - u(x)\| \le M|\epsilon|,$$

where $\|\cdot\| = \max_{x\in[0,x_{\epsilon^*}]} |\cdot|$ and $M$ is a bound of $\left|\frac{\partial u}{\partial \epsilon}(x; \epsilon)\right|$ on $[0, x_{\epsilon^*}] \times [I_0 \cap {I_0}^*]$ which is finite by our hypotheses. Because $M$ is independent on $\epsilon$, it is obvious that $u_\epsilon \to u$ uniformly on $[0, x_{\epsilon^*}]$ as $\epsilon \to 0$. With $\epsilon^*$ arbitrarily small, as a consequence of Lemma 1, $[0, x_{\epsilon^*}]$ is an arbitrarily large finite interval. $\square$

Note that, as a further consequence of Lemma 1, $u_\epsilon(x)$ is always defined on $[0, x_{\epsilon^*}]$ because $|\epsilon| < |\epsilon^*|$. Provided that we are able to define the value of $M$, the inequality above gives an estimate of the approximation error introduced by solving (2.1) instead of (1.1).

**3. A noniterative transformation method.** Let us consider here the class of problems

$$
\begin{aligned}
&\frac{d^3u_\epsilon}{dx^3} = u_\epsilon^{1-3\delta}\Phi\left(u_\epsilon{}^{-\delta}x, u_\epsilon{}^{\delta-1}\frac{du_\epsilon}{dx}, u_\epsilon{}^{2\delta-1}\frac{d^2u_\epsilon}{dx^2}, u_\epsilon{}^{-\pi_1}P_1, \ldots, u_\epsilon{}^{-\pi_n}P_n\right),\\
&u_\epsilon(0) = P_1{}^{1/\pi_1}\Psi\left(P_1{}^{(2\delta-1)/\pi_1}\frac{d^2u_\epsilon}{dx^2}(0), P_1{}^{-\pi_2/\pi_1}P_2, \ldots, P_1{}^{-\pi_n/\pi_1}P_n\right),\\
&\frac{du_\epsilon}{dx}(0) = P_1{}^{(1-\delta)/\pi_1}\Sigma\left(P_1{}^{(2\delta-1)/\pi_1}\frac{d^2u_\epsilon}{dx^2}(0), P_1{}^{-\pi_2/\pi_1}P_2, \ldots, P_1{}^{-\pi_n/\pi_1}P_n\right),\\
&\frac{du_\epsilon}{dx}(x_\epsilon) = C, \qquad \frac{d^2u_\epsilon}{dx^2}(x_\epsilon) = \epsilon,
\end{aligned} \tag{3.1}
$$

where $\delta \neq 1/2$ and $\delta \neq 1$, $\Phi(\cdot, \cdot, \cdot, \cdot, \ldots, \cdot)$, $\Psi(\cdot, \ldots, \cdot)$, and $\Sigma(\cdot, \ldots, \cdot)$ are arbitrary functions of their arguments and $\pi_i$ for $i = 1, \ldots, n$ are constants fulfilling the further condition $\pi_1 \neq 0$. We can state the following result.

THEOREM 2 (noniterative transformation method). *The class of free BVPs* (3.1) *with the limitation indicated above can be solved by a noniterative transformation method.*

*Proof.* The proof of this theorem can be obtained within group invariance theory and it is completely constructive. Here we would like to indicate the general outline of the proof. Let us consider the free BVP (2.1) and require the invariance of the governing differential equation and of the two boundary conditions at the origin with respect to the stretching group

$$
x^* = \lambda^\delta x, \quad u_\epsilon{}^* = \lambda u_\epsilon, \quad P_i{}^* = \lambda^{\pi_i} P_i \quad \text{for} \quad i = 1, \ldots, n, \tag{3.2}
$$

where $\lambda$ is the exponential of the group parameter. By making use of a well-established procedure (see Klamkin [23]), it is then possible to characterize the functional forms and the constraints listed above.

The noniterative method can be defined as follows. Let $\delta$, $\pi_i$ for $i = 1, \ldots, n$ be fixed by the particular problem under consideration. Then we can set values of $P_i{}^*$ for $i = 1, \ldots, n$ and a value of $\frac{d^2u_\epsilon{}^*}{dx^{*2}}(0)$. As a consequence, the values of $u_\epsilon{}^*(0)$ and $\frac{du_\epsilon{}^*}{dx^*}(0)$ will be defined according to the boundary conditions at the origin. Therefore, taking into account the obtained initial conditions, the governing differential equation can be integrated in $[0, x_\epsilon{}^*]$. Owing to the invariance properties, we can find the relations

$$
\begin{aligned}
&\lambda = \left[C^{-1}\frac{du_\epsilon{}^*}{dx^*}(x_\epsilon{}^*)\right]^{1/(1-\delta)},\\
&\frac{d^2u_\epsilon}{dx^2}(0) = \lambda^{2\delta-1}\frac{d^2u_\epsilon{}^*}{dx^{*2}}(0),\\
&x_\epsilon = \lambda^{-\delta}x_\epsilon{}^*,\\
&P_i = \lambda^{-\pi_i}P_i{}^* \quad \text{for} \quad i = 1, \ldots, n,\\
&\frac{d^2u_\epsilon}{dx^2}(x_\epsilon) = \lambda^{2\delta-1}\frac{d^2u_\epsilon{}^*}{dx^{*2}}(x_\epsilon{}^*).
\end{aligned} \tag{3.3}
$$

In (3.3) we have used the boundary condition $\frac{du_\epsilon}{dx}(x_\epsilon) = C$. Next, we have to apply the remaining boundary condition at the free boundary; however, the value of $x_\epsilon{}^*$ is not defined yet. Therefore, the condition $\frac{d^2u_\epsilon}{dx^2}(x_\epsilon) = \epsilon$ defines the value of $x_\epsilon{}^*$. $\square$

The noniterative method described so far allows us to obtain an approximate numerical solution for the particular values of the parameters given by (3.3). Therefore, we cannot choose a priori the values of $P_i$ for $i = 1, \dots, n$ and it will be difficult to plot in a figure or to list in a table the obtained numerical results. However, within group invariance theory, it is possible to overcome this difficulty. To this end we introduce the $n-1$ invariants defined by

$$\omega_k = P_k P_1^{-\pi_k/\pi_1} \qquad \left(\omega_k{}^* = \omega_k\right) \quad \text{for} \quad k = 2, \dots, n. \tag{3.4}$$

By means of these invariants the problem (3.1) becomes

$$\begin{aligned}
&\frac{d^3 u_\epsilon}{dx^3} = u_\epsilon^{1-3\delta} \Phi \left( u_\epsilon{}^{-\delta} x, u_\epsilon{}^{\delta-1} \frac{du_\epsilon}{dx}, u_\epsilon{}^{2\delta-1} \frac{d^2 u_\epsilon}{dx^2}, u_\epsilon{}^{-\pi_1} P_1, \right. \\
&\qquad\qquad \left. u_\epsilon{}^{-\pi_2} P_1{}^{\pi_2/\pi_1} \omega_2, \dots, u_\epsilon{}^{-\pi_n} P_1{}^{\pi_n/\pi_1} \omega_n \right), \\
&u_\epsilon(0) = P_1{}^{1/\pi_1} \Psi \left( P_1{}^{(2\delta-1)/\pi_1} \frac{d^2 u_\epsilon}{dx^2}(0), \omega_2, \dots, \omega_n \right), \\
&\frac{du_\epsilon}{dx}(0) = P_1{}^{(1-\delta)/\pi_1} \Sigma \left( P_1{}^{(2\delta-1)/\pi_1} \frac{d^2 u_\epsilon}{dx^2}(0), \omega_2, \dots, \omega_n \right), \\
&\frac{du_\epsilon}{dx}(x_\epsilon) = C, \qquad \frac{d^2 u_\epsilon}{dx^2}(x_\epsilon) = \epsilon .
\end{aligned} \tag{3.5}$$

Since the only parameter to be varied in (3.5) is $P_1$, we can fix its value along with the values of the $n-1$ invariants $\omega_k$ for $k = 2, \dots, n$ and look for a value $P_1{}^*$ that will transform to the chosen value of $P_1$ through (3.3). This is equivalent to locating a root of an implicit function of $P_1{}^*$ so that we can apply a root-finding method for scalar equations.

## 4. An iterative extension of the transformation method.

In this section we define the iterative transformation method for the numerical solution of free BVPs in the class (2.1). To this end let us state the following result.

THEOREM 3 (iterative transformation method). *The class of free BVPs* (2.1) *can be solved numerically by an iterative transformation method.*

*Proof.* In the proof of this theorem the leading idea is that (2.1) can be imbedded into a family of problems by introducing a numerical parameter. Hence, we consider the family of problems

$$\begin{aligned}
&\frac{d^3 u_\epsilon}{dx^3} = h^{(1-3\delta)/\sigma} f \left( h^{-\delta/\sigma} x, h^{-1/\sigma} u_\epsilon, h^{(\delta-1)/\sigma} \frac{du_\epsilon}{dx}, h^{(2\delta-1)/\sigma} \frac{d^2 u_\epsilon}{dx^2}, \boldsymbol{P} \right), \\
&u_\epsilon(0) = h^{1/\sigma} g \left( h^{(2\delta-1)/\sigma} \frac{d^2 u_\epsilon}{dx^2}(0), \boldsymbol{P} \right), \\
&\frac{du_\epsilon}{dx}(0) = h^{(1-\delta)/\sigma} \ell \left( h^{(2\delta-1)/\sigma} \frac{d^2 u_\epsilon}{dx^2}(0), \boldsymbol{P} \right), \\
&\frac{du_\epsilon}{dx}(x_\epsilon) = C, \qquad \frac{d^2 u_\epsilon}{dx^2}(x_\epsilon) = \epsilon,
\end{aligned} \tag{4.1}$$

where $h$ is the introduced numerical parameter. By setting $h = 1$ in (4.1) we recover (2.1). On the other hand, in (4.1) the governing differential equation and the two boundary conditions at the origin are invariant with respect to the transformation (3.2) where $\pi_i = 0$ for $i = 1, \ldots, n$ ($\boldsymbol{P}$ is now considered as a vector of $n$ invariants) but extended by the relation

$$h^* = \lambda^\sigma h, \tag{4.2}$$

where $\sigma \neq 0$.

To define a transformation method, we can proceed as in the previous section. However, the values of $\delta$ and $\sigma$ are not defined in (4.1) and consequently they can be chosen at our convenience. Moreover, we have to iterate for different values of $h^*$ until we get $h = 1$ from (4.2). For fixed values of $\delta$ and $\sigma$ the above request is equivalent to finding a root of the implicit function

$$\Gamma(h^*) = [\lambda(h^*)]^{-\sigma} h^* - 1,$$

and consequently a root-finding method can be applied. □

As a conclusion, we can solve (2.1) via an iterative transformation method. A sequence $h_j$ for $j = 0, 1, 2, \ldots$, is defined within the iteration. If $\Gamma({h^*}_j) \to 0$ as $j \to \infty$, then the related values obtained by (3.3) approach the correct values.

**5. Numerical results.** Let us consider here some applications of our approach. From (3.1), by setting

$$\Phi = -{u_\epsilon}^{2\delta-1}\frac{d^2 u_\epsilon}{dx^2}, \quad \delta = -1, \quad \Psi = 0, \quad \Sigma = 0, \quad \text{and} \quad C = 1,$$

we recover a free boundary formulation for the Blasius problem that was solved in [12].

Next, we consider the mathematical model related to the study of the flow of an incompressible fluid over a slender parabola of revolution (see Na [29, pp. 217–221]). By introducing the Levy–Lees transformation to the relevant governing equations in boundary layer theory, we get

$$\begin{gathered} (1 + P_1 x)\frac{d^3 u}{dx^3} + \left(\frac{1}{2}u + P_1\right)\frac{d^2 u}{dx^2} = 0, \\ u(0) = P_2, \quad \frac{du}{dx}(0) = 0, \quad \frac{du}{dx}(x) \to 1 \quad \text{as} \quad x \to \infty, \end{gathered} \tag{5.1}$$

where $P_1$ is the transverse curvature parameter and $P_2$ represents suction ($P_2 < 0$) or blowing ($P_2 > 0$). By setting $P_1 = 0$ in (5.1), we recover the Blasius problem with suction or blowing.

A free boundary formulation for the problem (5.1) belongs to the class (3.1) where

$$\Phi = -\frac{\left(\frac{1}{2}{u_\epsilon}^{2\delta-1} + P_1 {u_\epsilon}^{2\delta-1-\pi_1}\right)\dfrac{d^2 u_\epsilon}{dx^2}}{1 + P_1 x},$$

$$\delta = -1, \quad \Psi = 1, \quad \Sigma = 0, \quad \pi_1 = 1, \quad \pi_2 = 1, \quad \text{and} \quad C = 1.$$

In Table 1 we list representative numerical results. Our goal was to solve the problem (5.1) where $P_1 = 2$ and $P_2 = 2$, that is $\omega_2 = 1$. The secant method was used with initial guesses ${P_1}^{*(0)} = 3$ and ${P_1}^{*(1)} = 4$. After an evaluation of the numerical results, the simple termination criterion

$$|{P_1}^{*(k)} - {P_1}^{*(k-1)}| \leq \text{TOL}$$

was adopted where TOL (a fixed tolerance) was chosen equal to 1.D−06. The above condition was always verified within six iterations.

TABLE 1
*Numerical results for* (5.1) *where* $P_1 = 2, \omega_2 = 1$. *Obtained with* $\frac{d^2u_\epsilon{}^*}{dx^{*2}}(0) = 1.D+01$.

| $\epsilon \leq$ | $P_1{}^*$ | $\lambda^{-1}P_1{}^* - P_1$ | $x_\epsilon{}^*$ | $x_\epsilon$ | $\frac{d^2u_\epsilon}{dx^2}(0)$ |
|---|---|---|---|---|---|
| $10^{-6}$ | 3.814497997 | $-2.0D-13$ | 19.52 | 37.229499817 | 1.441377749 |
| $10^{-7}$ | 3.814502704 | $1.5D-13$ | 23.92 | 45.621452332 | 1.441372413 |
| $10^{-8}$ | 3.814503183 | $-1.8D-14$ | 28.39 | 54.146873474 | 1.441371870 |
| $10^{-9}$ | 3.814503231 | $5.0D-14$ | 32.90 | 62.748577118 | 1.441371815 |

TABLE 2
*A convergence numerical test for the results reported in Table* 1.

| $\frac{d^2u_\epsilon}{dx^2}(0)$ | $x_\epsilon$ | $\frac{du_\epsilon}{dx}(x_\epsilon)$ | $\frac{d^2u_\epsilon}{dx^2}(x_\epsilon)$ |
|---|---|---|---|
| 1.441377749 | 37.229499817 | 1.000000000010 | $9.97226D-07$ |
| 1.441372413 | 45.621452332 | 1.000000000016 | $9.9747D-08$ |
| 1.441371870 | 54.146873474 | 0.999999999825 | $9.961D-09$ |
| 1.441371815 | 62.748577118 | 0.999999999686 | $1.000D-09$ |

As far as the missing initial condition is concerned, the last column of Table 1 seems to indicate the achievement of six decimal digits of accuracy. This is of relevant interest because we are solving our problem via an initial value method. The obtained values of $\frac{d^2u_\epsilon}{dx^2}(0)$ and $x_\epsilon$ were used in a further integration to verify the numerical results for $\frac{du_\epsilon}{dx}(x_\epsilon)$ and $\frac{d^2u_\epsilon}{dx^2}(x_\epsilon)$. This validation is reported in Table 2.

A comparison for the values of $\frac{d^2u_\epsilon}{dx^2}(0)$ can be made with the value of 1.4426 reported by Na [29]. Unfortunately, Na does not give any indication about the value of the truncated boundary used by him. The problem (5.1) where $P_1 = 2$ and $P_2 = 2$ can be indicated as a possible test problem for numerical methods.

Let us consider now the Falkner–Skan equation with relevant boundary conditions

$$\frac{d^3u}{dx^3} + u\frac{d^2u}{dx^2} + P\left[1 - \left(\frac{du}{dx}\right)^2\right] = 0, \tag{5.2}$$
$$u(0) = \frac{du}{dx}(0) = 0, \quad \frac{du}{dx}(x) \to 1 \quad \text{as} \quad x \to \infty.$$

A free boundary formulation for the problem (5.2), when $P \neq 0$, does not belong to the class (3.1). Therefore, the proposed noniterative method cannot be applied here. However, the iterative extension of the transformation method is available. A free boundary formulation for (5.2) which belongs to (4.1) is given, for instance, by setting $\delta = -1$ and $\sigma = 8$ to characterize the stretching group (3.2)–(4.2), so that

$$f(\cdot, \ldots, \cdot) = -h^{-1/2}u_\epsilon\frac{d^2u_\epsilon}{dx^2} - P_1\left[1 - h^{-1/2}\left(\frac{du_\epsilon}{dx}\right)^2\right],$$
$$g(\cdot, \ldots, \cdot) = 0, \quad \ell(\cdot, \ldots, \cdot) = 0, \quad P_1 = P, \quad \text{and} \quad C = 1.$$

The numerical results obtained for the Homann flow, corresponding to the value of $P = 1/2$, are listed in Table 3.

The secant method with an appropriate termination criterion (see Nickel and Ritter [32]) was used in all iterations. As discussed above, a direct validation of the numerical results for $\frac{d^2u_\epsilon}{dx^2}(0) = 0.927680$ and $x_\epsilon = 5.571160$ gives the values $\frac{du_\epsilon}{dx}(x_\epsilon) = 0.999999522$ and $\frac{d^2u_\epsilon}{dx^2}(x_\epsilon) = 8.92D-07$ to nine decimal digits.

TABLE 3
*Results for the Homann flow obtained with* $\frac{d^2u_\epsilon{}^*}{dx^{*2}}(0) = 1.D+01$.

| $\epsilon \leq$ | $h^*$ | $\lambda^{-8}h^* - 1$ | $x_\epsilon{}^*$ | $x_\epsilon$ | $\frac{d^2u_\epsilon}{dx^2}(0)$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 542.670752 | 1.3D − 09 | 1.009 | 2.216707 | 0.943081 |
| $10^{-2}$ | 565.733143 | 2.7D − 09 | 1.442 | 3.184503 | 0.928476 |
| $10^{-3}$ | 566.942904 | 4.0D − 09 | 1.777 | 3.925363 | 0.927733 |
| $10^{-4}$ | 567.022195 | 3.9D − 10 | 2.060 | 4.550585 | 0.927684 |
| $10^{-5}$ | 567.027843 | 4.7D − 09 | 2.302 | 5.085175 | 0.927680 |
| $10^{-6}$ | 567.028301 | 1.5D − 09 | 2.522 | 5.571160 | 0.927680 |

In the application of the proposed iterative method a difficulty arises: the interval of integration may not be constant in the iteration. Nevertheless, a skilled operator can obtain meaningful numerical results. On that matter our experience is as follows. Even if an accuracy of 12 decimal places is sought, one can start anyway by selecting $|\epsilon| = 10^{-1}$. Then, by reducing the absolute value of $\epsilon$ gradually, one can take advantage of the "continuation" behaviour of the numerical results. This is also pertinent to the applications of the noniterative method with a target value for the selected parameter. As far as the problem (5.1) is concerned, the choice $\epsilon = 10^{-1}$ allowed us to get, for the related value of $\frac{d^2u_\epsilon}{dx^2}(0)$, only one digit of accuracy. It can be easily understood that the numerical results reported in Table 1 are the most meaningful among those obtained. The application of our approach to the solution of relevant BVPs may not only be interesting per se, but it can also provide a guide to the solution of other problems of interest.

All computations were performed in FORTRAN on a RISC System/6000 IBM computer with the DIVPAG routine in the IMSL Math/Library [21]. The DIVPAG allows us to apply step size and local error control. A user-supplied Jacobian and a value of 1.D − 12 for the error control were used within the DIVPAG.

**6. Conclusions and final remarks.** Some motivations for the present work were given in §1. Therein we quoted several problems of interest in boundary layer theory defined on infinite intervals. Indeed, two-point BVPs on infinite intervals arise in several branches of science. From the numerical viewpoint such a kind of problem has been replaced by a finite interval problem, whose solution approximates reasonably well the true solution on that interval. In particular, often the original problem is solved by comparing the numerical results obtained for several truncated boundaries. Hence, two main questions arise: (1) how does one select the truncated boundary and (2) what boundary conditions have to be imposed there? For the latter question an asymptotic analysis has been proposed by several authors (see de Hoog and Weiss [18], Lentini and Keller [24], or Markowich [25]). Lentini and Keller [24] proposed a priori estimates for the truncated boundary as an important area of research.

Here, we suggested formulating BVPs on infinite intervals as free BVPs. Actually, in the new formulation the free boundary is not known and has to be found as part of the solution. Moreover, we have proved that the solution of the free boundary formulation converges uniformly to the solution of the original problem as $\epsilon \to 0$ on every arbitrarily large finite interval. On the other hand, we can verify the numerical convergence of the missing initial conditions as $\epsilon \to 0$ (cf. the results reported in Tables 1 and 3). A first application of this idea was given for the Blasius problem in [12].

In §3 we defined a noniterative transformation method applicable to the class of problems characterized by (3.1). However, from a theoretical viewpoint, the noniterative method is not widely applicable and this is an obvious limit of it. In §4 we introduced an iterative extension of the transformation method. This extension is of theoretical value because it is widely applicable. We remark that the Falkner–Skan equation was indicated as a caveat for transformation methods by Na [29, pp. 146–147]. Here we solved the Homann problem for illustrative pur-

TABLE 4
*Truncated boundary and wall shear stress for the Blasius problem and the Homann flow.*

| | Fazio [12] and this work Initial value method | | Nasr et al. [30] Chebyshev method | | Beckett [5] Finite difference method | |
|---|---|---|---|---|---|---|
| $P$ | $x_\epsilon$ | $\frac{d^2u_\epsilon}{dx^2}(0)$ | $x_\infty$ | $\frac{d^2u_\epsilon}{dx^2}(0)$ | $x_\infty$ | $\frac{d^2u_\epsilon}{dx^2}(0)$ |
| 0.0 | 8.500195503 | 0.469599988 | 6.9 | 0.4696000 | 5. | 0.4697 |
| 0.5 | | | 3.7 | 0.9278050 | 5. | 0.9277 |
| 0.5 | 5.085175 | 0.927680 | 7.4 | 0.9276800 | | |

poses. Further details and numerical results can be found in [13]. Table 4 compares our results with others obtained by choosing different values of the truncated boundary.

As far as Table 4 is concerned, it is evident that, besides the numerical method used, the choice of the truncated boundary also influences the obtained numerical accuracy. This was already recognized by several authors. In particular, in relation to a finite difference method, Wadia and Payne [38] suggested using a first value for the truncated boundary and then repeating the computation by doubling its value. The first choice could prove to be valid if the two computed solutions differ up to a prefixed tolerance. As a further example Warsi and Koomullil [40], introducing an improved shooting method, allowed a modification of the truncated boundary in each iteration to verify the asymptotic condition (1.2). Unfortunately, they quoted the value of $x_e$ where $\frac{du}{dx}(x_e) = 0.9999$ instead of the value of the related truncated boundary.

It is evident that our approach can be extended to problems on infinite intervals governed by higher-order or systems of differential equations. On the other hand, a further possible extension to partial differential equations on unbounded domains seems not so obvious (see Hagstrom and Keller [15]–[16] for the extension of the asymptotic analysis approach to elliptic problems on unbounded domains).

In closing we point out an open problem. As we have seen at the end of §2, the error due to the truncated boundary is difficult to obtain (cf. Lentini and Keller [24]). Sometimes, the error estimates involve an upper bound for the solution of the original problem on the infinite interval that in general may be not available (see Markowich and Ringhofer [27] for such a case). The simple analysis due to Rubel [34] for the Blasius problem relies on the invariance properties of the governing differential equation and therefore cannot be extended to the Falkner–Skan equation with relevant boundary conditions. Hence, until a reliable error analysis will be available we sometimes have to rely exclusively on computational practice. In this context, the numerical results reported herein point out that the accuracy question for BVPs with a boundary condition at infinity can be adequately treated by formulating the original problem as a free BVP.

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