Research Article

Mathematical Methods in the Applied Sciences

Received 30 September 2010

Published online 26 April 2012 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.1614
MOS subject classification: 65L05; 76D45

Extended scaling invariance of one-dimensional models of liquid dynamics in a horizontal capillary

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Communicated by T. Simos

In this paper, we consider the adaptive numerical solution of one-dimensional models of liquid dynamics in a horizontal capillary. The bulk liquid is assumed to be initially at rest and is put into motion by capillarity: the smaller is the capillary radius, the steeper becomes the initial transitory of the meniscus location derivative, and as a consequence, the numerical solution to a prescribed accuracy becomes harder to achieve. Therefore, in order to solve a capillary problem effectively, it would be advisable to apply an adaptive numerical method.

Here, we show how an extended scaling invariance that can be used to define a family of solutions from a computed one. In the viscous case, the similarity transformation involves solutions of liquids with different density, surface tension, viscosity, and capillary radii, whereas in the inviscid case, we can generate a family of solutions for the same liquid and capillaries with different radii. With our study, we are able to prove that the monitor function, used in the adaptive algorithm, is invariant with respect to the considered scaling group. It follows, from this particular results, that all the solutions within the generated family verify the adaptive criteria used for the computed one. Moreover, all the solutions have the same order of accuracy even if the maximum value of the step size varies under the action of the scaling group.

Keywords: horizontal capillary; liquid penetrant testing; extended scaling invariance; adaptive numerical method

1. Introduction

The present study was motivated by the non-destructive control named "liquid penetrant testing" used in the production of airplane parts and in many industrial applications, where the detection of open defects is of interest. Liquid penetrants are used to locate surface-accessible defects in solid parts. The basic technique uses several stages. Among those stages, at least two involve capillary action, namely: application of a penetrant liquid and use of a developer, usually some kind of porous material (like an absorbing coating). These two stages can be modeled in the same way, but at a completely different space scale. The visible dimension of a defect at the penetrant stage may be of the order of millimeters or microns; on the other hand, the porous material at the developer stage should have pores of the order of nanometers or Ångstrom. The present study may be seen as a possible way to use the results obtained within the penetrant stage to get results related to the developer stage.

One-dimensional models of liquid dynamics in a horizontal capillary have an interesting history. The first simple models were introduced at the beginning of the last century by Bell and Cameron [1], Lucas [2], Washburn [3], and Rideal [4]. The main results obtained at that time was the derivation of the celebrated Washburn solution, which is a valid asymptotic long time approximation that can be derived by neglecting the inertial effects. Those inertial effects were taken into account in a more realistic model proposed by Bosanquet [5]. The Szekely-Neumann-Chuang (SNC) model introduced by Szekely *et al*. [6] takes into account also the outside flow effects, including within the inertial terms, an apparent mass parameter. Recently, Cavaccini *et al*. [7,8] pointed out that in order to solve effectively a capillary problem, it would be advisable to apply an adaptive numerical method. For further information on this subject,

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we refer to the survey papers by Kornev and Neimark [9], Zhmud *et al.* [10], the recent book by de Gennet *et al.* [11], and the references quoted therein.

In this paper, we extend the original approach developed recently by Fazio in [12], for the numerical solution of the van der Pol model, to a class of problems. In particular, we use an extended scaling invariance for one-dimensional models of liquids dynamics in a horizontal capillary. A classical similarity analysis would require the invariance of the model with respect to a group involving only the dependent and independent variables, see for instance Bluman and Cole [13] or Dresner [14]. However, we are interested, here, to an extension of the classical analysis, obtained by requiring the invariance of physical parameters, proposed first by Na [15], see also the Chapters 7–9 of the book by Na [16], and the references quoted therein.

Here, we show how this extended invariance can be used to define a family of solutions from a computed one. In the viscous case, the scaling transformation involves solutions of liquids with different density, surface tension, viscosity, and capillary radii, whereas in the inviscid case, we can generate a family of solutions for capillaries with different radii but the same liquid. The reference solution should be as accurate as possible; and, therefore, we suggest to use for it an adaptive numerical method. With our study, we are able to prove that the used monitor function is invariant with respect to the considered scaling group. As a consequence, all the solutions generated by the scaling invariance verify the adaptive criteria used for the computed one. Moreover, all the solutions have the same order of accuracy even if the maximum value of the step size, that can be considered as a discretization parameter, varies under the action of the scaling group.

2. Mathematical modeling

With reference to Figure 1, we consider a liquid freely flowing within a horizontal cylindrical capillary of radius R. At the left end of the capillary, we have a reservoir filled with the penetrant liquid. The bulk liquid is assumed to be initially at rest and is put into motion by capillarity: the smaller is the capillary radius, the steeper becomes the initial transitory of the meniscus location derivative, and as a consequence, the numerical solution to a prescribed accuracy becomes harder to achieve, see Cavaccini *et al.* [7, 8]. The model governing the dynamics of a liquid inside an open ended capillary is given by

$$\begin{cases} \rho \dfrac{d}{dt}\left[(\ell + cR)\dfrac{d\ell}{dt}\right] = 2\dfrac{\gamma \cos\vartheta}{R} - 8\dfrac{\mu \ell}{R^2}\dfrac{d\ell}{dt} \\ \ell(0) = 0\,, \qquad \dfrac{d\ell}{dt}(0) = 0\,, \end{cases} \tag{1}$$

where ℓ is the moving liquid–gas interface coordinate, $d\ell/dt$ can be interpreted as the average axial velocity U; ρ, γ, ϑ, and μ are liquid density, surface tension, contact angle, and viscosity, respectively, and $c = O(1)$ is the coefficient of apparent mass, introduced by Szekely *et al.* [6] to take into account the flow effects outside the capillary. Here, and in the following, we use a coefficient of apparent mass $c = 7/6$. For the mathematical derivation of the governing equation, as well as sample numerical computations, we refer to Cavaccini *et al.* [7, 8]. Further numerical results were presented at the ICIAM congress held in Zurich, July 16–20, 2007, see Fazio *et al.* [17].

For the derivation of the model, we have to assume a quasi-steady Poiseuille velocity profile, and, to simplify it, we take the dynamic contact angle equal to the static one. The above one-dimensional model is a realistic one if the Bond, Capillary, and Weber numbers are small, that is $Bo = 4\rho g R^2/\gamma \ll 1$, $Ca = \mu U/\gamma \ll 1$, and $We = 2\rho R U^2/\gamma \ll 1$, where g is the acceleration due to gravity. Despite to its simplicity, the considered model has been validated by experimental results reported in several papers by many authors.

3. Extended scaling invariance

A classical similarity analysis would require the invariance of the model (1) with respect to transformations involving only the dependent and independent variables. However, we are interested here to an extension of the classical analysis, obtained by requiring the invariance of physical parameters, proposed first by Na [15]. In particular, we are interested to consider the same model, but with different values of the capillary radius R, density ρ, surface tension γ, and viscosity parameter μ.

Here, we derive the conditions for the invariance of the model (1) with respect to the following extended scaling group

$$t^* = \lambda^{\delta} t\,,\ \ell^* = \lambda \ell\,,\ R^* = \lambda R\,, \quad \rho^* = \lambda^{\alpha_1}\rho\,,\ \gamma^* = \lambda^{\alpha_2}\gamma\,,\ \mu^* = \lambda^{\alpha_3}\mu\,, \tag{2}$$

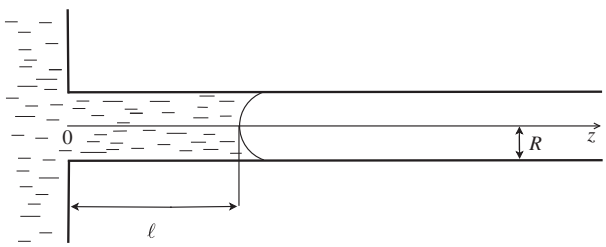

Figure 1. Sketch of a horizontal capillary section.

where λ is the group parameter, and α_1, α_2, α_3 and δ are group exponents to be determined. We note that the initial conditions are invariant with respect to the action of (2). Moreover, the governing equation, under (2), becomes

$$\lambda^{2\delta-\alpha_1-2}\rho^*\frac{d}{dt^*}\left[(\ell^*+cR^*)\frac{d\ell^*}{dt^*}\right]=\lambda^{1-\alpha_2}2\frac{\gamma^*\cos\vartheta}{R^*}-\lambda^{\delta-\alpha_3}8\frac{\mu^*\ell^*}{R^{*2}}\frac{d\ell^*}{dt^*}\,,$$

therefore, its invariance is granted on condition that

$$2-2\,\delta+\alpha_1=\alpha_2-1=\alpha_3-\delta\,. \tag{3}$$

These are two linear equations for four parameters. Henceforth, our model is left invariant by a scaling group depending on two arbitrary group exponents.

Our interest, here, is to define by the similarity invariance, a family of solutions each one identified by particular values of the capillary radius R, density ρ, surface tension γ, and viscosity μ, from an approximated numerical one. In this way, it can be possible to get an approximation also for the derivatives of the field variable ℓ. In fact, as a consequence of (2), we have

$$\frac{d^n\ell^*}{dt^{*n}}(t^*)=\lambda^{1-n\,\delta}\frac{d^n\ell}{dt^n}(t)\,,\qquad n=1,2,\ldots\,. \tag{4}$$

We have shown in Cavaccini *et al*. [7, 8] that, for a capillary problem, it would be advisable to apply an adaptive numerical method. In that paper, we used for the adaptive procedure the following monitor function

$$\eta(t_k)=\frac{\left|\dfrac{d\ell}{dt}(t_k+\Delta t_k)-\dfrac{d\ell}{dt}(t_k)\right|}{\Gamma_k}, \tag{5}$$

where Δt_k is the current time-step and

$$\Gamma_k=\begin{cases}\left|\dfrac{d\ell}{dt}(t_k)\right| & \text{if } \dfrac{d\ell}{dt}\,(t_k)\neq 0\\ \epsilon & \text{otherwise}\,,\end{cases} \tag{6}$$

where $0<\epsilon\ll 1$. So that we can require that the step size is modified as needed in order to keep this monitor function between chosen tolerance bounds. More details on the adaptive strategy and the meaning of this type of monitor functions can be found in [18]. Here, we provide further evidence for the usefulness of this kind of adaptive approach, and, to this end, we apply a similarity analysis. It is straightforward to verify that the monitor function defined by (5) is invariant under (2), if and only if

$$\epsilon^*=\lambda^{1-\delta}\epsilon\,. \tag{7}$$

As a consequence, by setting

$$\eta_{\min}\le\eta(t_k)\le\eta_{\max}\,,$$

where $\eta_{\min}$ and $\eta_{\max}$ are prescribed tolerances, it follows that also

$$\eta_{\min}\le\eta^*(t_k^*)\le\eta_{\max}\,,$$

where $\eta^*(t_k^*)$ is the monitor function rewritten in the starred variables. The monitor function used in this paper is invariant with respect to (2)–(7). Therefore, all the solutions generated by the scaling group (2) from a computed one verify the same adaptive criteria.

Let us assume that the reference solution was computed in a stable way, so that a table of computed values is generated satisfying the order conditions

$$|\ell(t_k)-\ell_k|\le(\Delta t_{\max})^p\,\phi_k\quad\text{and}\quad\left|\frac{d\ell}{dt}(t_k)-\frac{d\ell}{dt}\Big|_k\right|\le(\Delta t_{\max})^p\,\psi_k\,,$$

where $\Delta t_{\max}$ is the maximum of the used time steps,

$$\phi_k=C_p\left|\frac{d^p\ell}{dt^p}(t_k)\right|,\qquad\psi_k=C_p\left|\frac{d^{p+1}\ell}{dt^{p+1}}(t_k)\right|,$$

and C_p is a positive constant related to the specific numerical method considered. It is a simple matter to verify that all the generated solutions by the scaling group (2) are of the same order p, that is

$$|\ell^*(t_k)-\ell_k^*|\le(\Delta t_{\max}^*)^p\,\phi_k^*\quad\text{and}\quad\left|\frac{d\ell^*}{dt^*}(t_k)-\frac{d\ell^*}{dt^*}\Big|_k\right|\le(\Delta t_{\max}^*)^p\,\psi_k^*\,,$$

where $\Delta t_{\max}^*=\lambda^\delta\,\Delta t_{\max}$ is the maximum of the transformed time steps, ϕ_k^* and ψ_k^* are the function ϕ_k and ψ_k rewritten in the starred variables.

As shown by Shampine and Witt [19], the application of an adaptive method stabilizes the computation. Our advise is to use a sufficiently small value for $\Delta t_{\max}$, so that, taking into account, the range of values for λ used to produce the family of transformed solution, it turn out that $\Delta t^*_{\max}$ has a suitable value with respect to the time scale $[0, t^*_{\max}]$. For the adaptive method used, we always define and apply values of $\Delta t_{\min}$ and $\Delta t_{\max}$ for the allowed minimum and maximum of the step sizes.

4. Numerical results

In this section, we report on sample numerical results for viscous or inviscid capillary flows. We consider first the case of a class of silicone oils (the so called polydimethylsiloxane [PDMS] series) according to the parameter values listed in Table I.

Here, we apply the scaling group (2) with group exponents

$$\delta = 3/2\,, \qquad \alpha_1 = \alpha_2 = -1\,, \qquad \alpha_3 = -1/2\,, \tag{8}$$

that satisfy the conditions (3).

Table I. Parameters for the series of PDMS silicone oils. For the sake of simplicity, we assume that $\vartheta = 0$ in all cases.

Liquid	ρ [Kg/m^3]	γ [mN/m]	μ [mPa · s]
PDMS5	918	19.7	5
PDMS10	935	20.1	10
PDMS20	950	20.3	20
PDMS50	960	20.8	50
PDMS500	971	21.5	500

PDMS, polydimethylsiloxane.

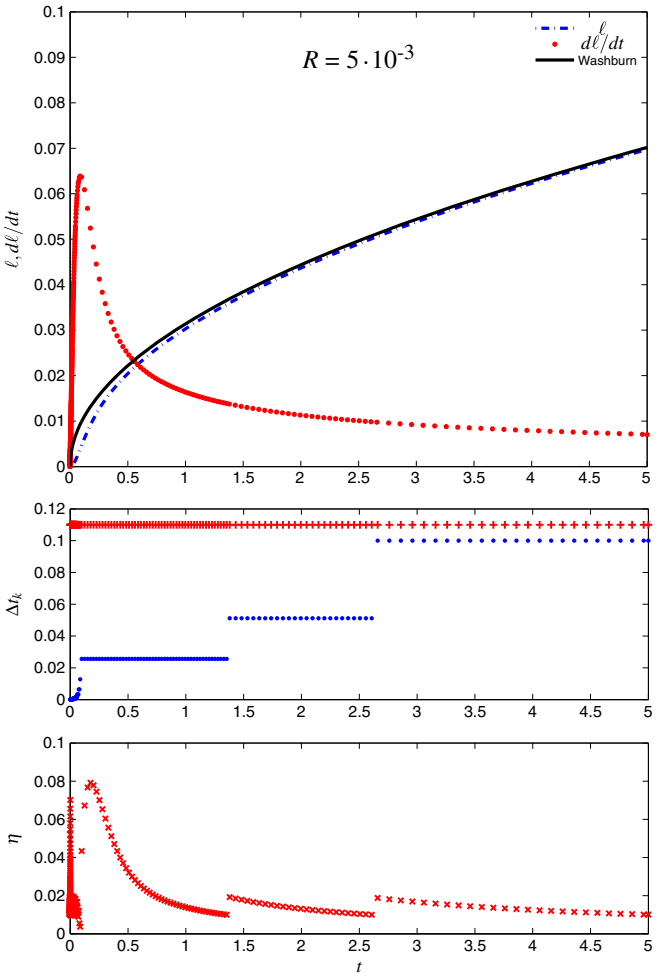

Figure 2. Adaptive step-size results. Here, by setting $R = 5$ mm, we get: $\lambda = 10^{-2}$, $\mu = 0.5$ mPa · s, $\rho = 9.18$ Kg/m^3, $\gamma = 0.197$ mN/m. From top to bottom: numerical solution, adaptive step-size selection Δt_k, and monitor function η.

Our target is a capillary with $R^* = 0.05$ mm for the liquid parameters marked by PDMS5:

$$\rho^* = 918\ \text{Kg/m}^3\,, \quad \gamma^* = 19.7\ \text{mN/m}\,, \quad \mu^* = 5\ \text{mPa}\cdot\text{s}\,, \quad \vartheta = 0^\circ\,. \tag{9}$$

The top frame of Figure 2 displays the numerical results obtained by setting a reference value $R = 5$ mm and computing all the other non-starred parameters from (9) according to the scaling group (2)–(8). Therefore, we end up with parameters not listed in Table I, that is to say that our reference computation might not correspond to a real penetrant liquid. This computation, using the adaptive procedure with criteria specified later, required 1362 steps with 19 rejections for $t \in [0, 5]$; the used limits for the step size were $\Delta t_{\min} \approx 5 \cdot 10^{-11}$ and $\Delta t_{\max} = 10^{-1}$.

The Washburn asymptotic approximation

$$\ell(t) = \left(\frac{\gamma R \cos\vartheta}{2\mu} t\right)^{1/2}, \tag{10}$$

shown in all figures as a solid line, is reported for comparison. This approximation is valid only for $t >> t_\mu$, where $t_\mu = \rho R^2/\mu$ is a viscous time scale. We note from the middle and bottom frame of Figure 2 that the adaptive procedure concentrates most of the computational effort within the initial transient.

In the top frame of Figure 3, we plot one solution obtained by applying the scaling invariance to the numerical solution shown in Figure 2. The reader can compare the top frame of Figure 3 with its bottom frame, the bottom frame was obtained by plotting the numerical results of a second computation with the parameters in (9). In this case, the computation, using the same adaptive criteria as in the first one, required 1520 steps with 32 rejections with $t \in [0, 5 \cdot 10^{-3}]$; the used limits for the step size were $\Delta t_{\min} \approx 5.82 \cdot 10^{-15}$ and $\Delta t_{\max} = 10^{-4}$.

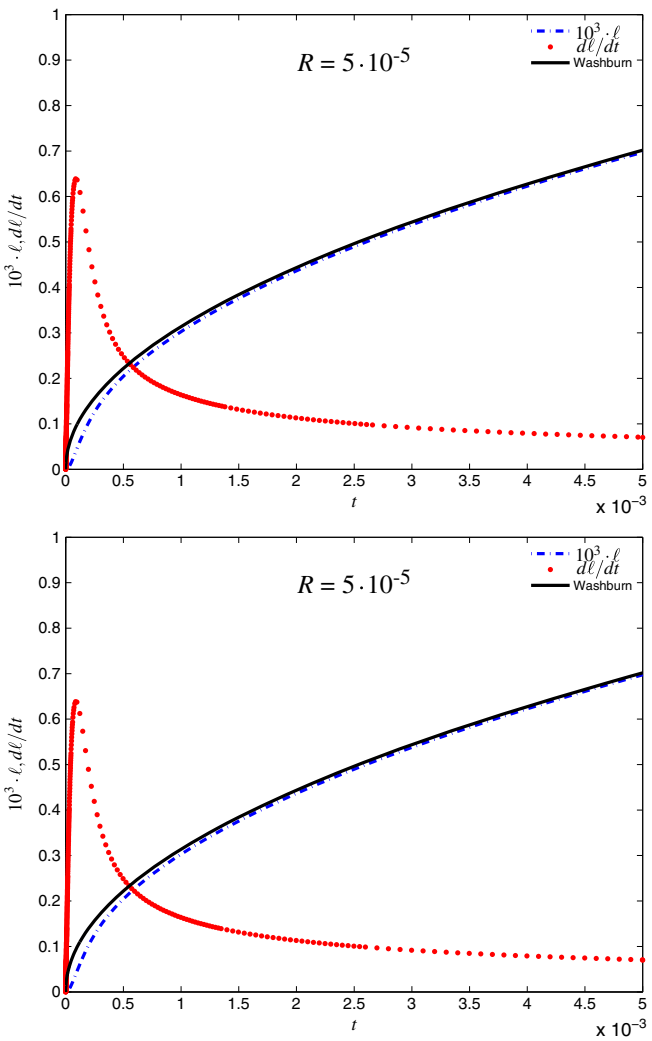

Figure 3. Viscous case $\mu = 5$ mPa · s with $R = 0.05$ mm. Top frame: results found by invariance. Bottom frame: direct numerical results. Washburn solution and $\ell(t)$ are magnified by a factor 10^3.

4.1. The classical scaling invariance strategy

Finally, we report other results obtained by ignoring low viscosity values of the considered liquids. The model should be modified as follows

$$\begin{cases} \rho \dfrac{d}{dt}\left[(\ell + cR)\dfrac{d\ell}{dt}\right] = 2\dfrac{\gamma \cos\vartheta}{R} \\ \ell(0) = 0\,, \qquad \dfrac{d\ell}{dt}(0) = 0\,, \end{cases} \tag{11}$$

This simplified model is invariant with respect to the scaling group

$$t^* = \lambda^{3/2} t\,, \quad \ell^* = \lambda \ell\,, \quad R^* = \lambda R\,. \tag{12}$$

As a consequence, for a given liquid, we can generate a family of similarity solutions from a computed reference one. Figure 4 displays the numerical results for the computation with parameters

$$\rho = 815\ \text{Kg/m}^3\,, \quad \gamma = 21\ \text{mN/m}\,, \quad \mu = 0\,, \quad \vartheta = 0^\circ\,, \tag{13}$$

and $R = 0.0025$ mm.

These results can be contrasted with those reported in Figure 5 for the parameters, listed by Clanet and Quéré [20],

$$\rho = 815\ \text{Kg/m}^3\,, \quad \gamma = 21\ \text{mN/m}\,, \quad \mu = 1\ \text{mPa}\cdot\text{s}\,, \quad \vartheta = 0^\circ \tag{14}$$

and $R = 0.0025$ mm.

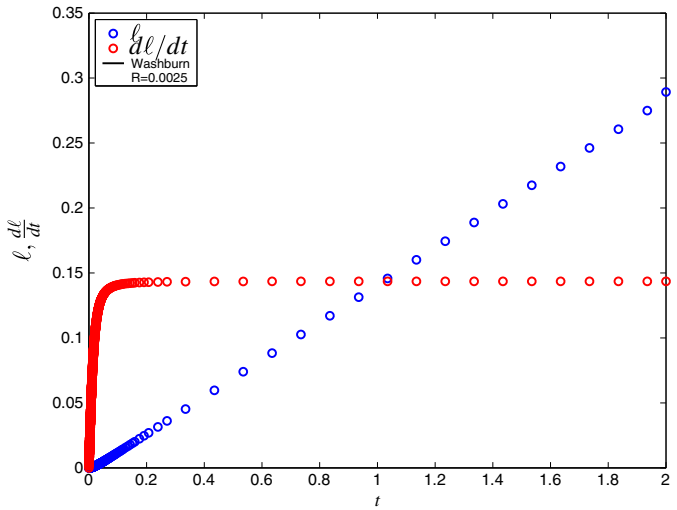

Figure 4. Adaptive step-size results: $\ell(t)$ and its first derivative. Solution computed by rescaling. Inviscid case $\mu = 0$ with $R = 0.0025$ mm.

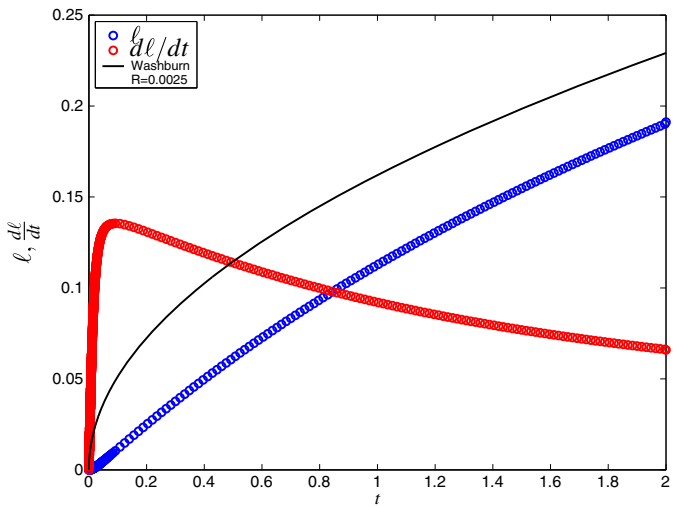

Figure 5. Adaptive step-size results: $\ell(t)$ and its first derivative. Solution obtained by a direct computation. Viscous case $\mu = 1$ mPa · s with $R = 0.0025$ mm.

We can easily realize that this strategy would be useless for the considered class of problems. Indeed, the generated family of similarity solution provides an upper bound for the real dynamics, but this is more expensive to get than the Washburn asymptotic solution that provides a more accurate and inexpensive upper bound.

4.2. Numerical method and adaptive criteria

The numerical method used for all the reported results was the classical fourth order Runge–Kutta's method implemented with the adaptive procedure developed by Jannelli and Fazio [18]. The adaptive procedure has been applied to: a scalar flame propagation problem [18], a reduced (four components only) air pollution model [18], and a 2 months start-up simulation for an aquarium model [21].

For the adaptive procedure, we usually enforced the following conditions: $\eta_{\min} \leq \eta(t_k) \leq \eta_{\max}$ with $\eta_{\min} = 10^{-2}$ and $\eta_{\max} = 10\ \eta_{\min}$, $\epsilon = 10^{-9}$, and $\Delta t_{\min} \leq \Delta t_k \leq \Delta t_{\max}$ with, if not otherwise specified, $\Delta t_{\min} = 10^{-12}$, $\Delta t_{\max} = 10^{-1}$. We used a smaller value of $\Delta t_{\min}$ only in the direct solution of the test case with parameters denoted by PDMS5 and shown in the bottom frame of Figure 3. Moreover, the time step was modified in two cases: when $\eta(t_k) < \eta_{\min}$ we used $\Delta t_{k+1} = 2\ \Delta t_k$ as the next time step, or if $\eta(t_k) > \eta_{\max}$, then we repeated the same step using $\Delta t_k = \Delta t_k/2$. Of course, a posteriori we have verified that also ϵ^* verifies the condition $0 < \epsilon^* \ll 1$.

5. Concluding remarks

In this study, we have proposed an extended scaling invariance for the dynamics of a liquid penetrating inside a horizontal capillary. By requiring the invariance of all the physical parameters involved in the mathematical model, we were able to show that the model itself is left invariant by a scaling group depending on two arbitrary group exponents. As a consequence, we can choose a particular group that allows us to compute by the scaling transformation a target solution from a computed simpler one. Indeed, several choices are available and, for instance, we have performed some successful computations, for the PDMS series of silicone oils, with the group exponents

$$\delta = 3/2\,, \qquad \alpha_1 = \alpha_2 = 1\,, \qquad \alpha_3 = 3/2\,, \tag{15}$$

that satisfy the conditions (3). These results, omitted here for the sake of brevity, are reported in the note by Fazio *et al*. [22].

The approach proposed in this paper may also be used to get transformed numerical results for problems where the capillary radius is at the levels of nanometers or in any case when the resolution of the initial transient would require a minimum step-size really small. We have not pursued in investigating these specific cases, here, because the limit for the nondestructive liquid penetrant testing, used in the industry, is a fracture of about 10 micron.

We have to remark that in the case of a vertical capillary, it is necessary to add to the right-hand side of the model (1), the term $-\rho g \ell$ where g represents the acceleration of gravity. Moreover, further terms should be added to the model in order to take into account the entrapped gas action in the case of a closed capillary. All these extended models require an extended scaling group that can be defined following the analysis outlined in the present study.

The invariance properties of the considered monitor function are valid in more general settings and will be extended to different monitor functions elsewhere.

Acknowledgements

This work was partially supported by Alenia Aeronautics company and the University of Messina.

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