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On Translation Groups and Non-iterative Transformation Methods

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Abstract

The aim of this work is to point out that, within group invariance theory, some classes of boundary value problems governed by ordinary differential equations can be transformed to initial value problems. The interest in the numerical solution of (free) boundary problems arises because these are (always) often nonlinear problems. The theoretical content of this paper is original: results already available in literature are related to the use of scaling or spiral groups of transformations; here we show how it is also possible to use the invariance with respect to two translation groups. As far as applications of the proposed approach are concerned, we solve two problems: a free boundary problem describing a rope configuration against an obstacle, where we compare the obtained numerical results with the exact solution, and a boundary problem modeling the fall of a parachutist, where we modify the classical formulation of the problem in order to prescribe the total falling time.

Keywords: `Ordinary differential equations, boundary value problems, initial value methods, translation groups, obstacle problem, military parachuting.`

1. Introduction

Boundary value problems (BVPs) or free BVPs occur in all branches of applied mathematics and science. The governing differential equations for a majority of such problems are nonlinear; since, in general, analytical solutions cannot be determined in closed form, solutions have to be approximate by numerical ones. Nonlinear BVP are usually solved by iterative numerical methods like shooting, or relaxation (finite difference, finite element, or collocation).

In the classical numerical treatment of a free BVP a preliminary reduction to a BVP is introduced by considering a new independent variable (see, Stoer and Bulirsch [34, p. 468], or Ascher, Mattheij and Russell [4, p. 471]). By rewriting a free BVP as a BVP it becomes evident that the former is always a nonlinear problem; the first to point out the nonlinearity of free BVPs was Landau [23]. Therefore, in that way free BVPs are BVPs. The first goal of this paper is to highlight the idea that free BVPs invariant with respect to a translation group can be solved non-iteratively by solving related initial value problems (IVPs) and therefore in this way those free BVPs are indeed IVPs. Moreover, by using a translation group, we are able to characterize a class of BVPs that can be solved non-iteratively by solving related IVPs.

The non-iterative numerical solution of BVPs is a subject of past and current research. Several different strategies are available in literature for the non-iterative solution of BVPs: superposition [4, pp. 135-145], chasing [30, pp. 30-51], and adjoint operators method [30, pp. 52-69] can be applied to linear models; parameter differentiation [30, pp. 233-288] and invariant imbedding [25] can be applied also to nonlinear problems and represent two alternative methods to the one based on transformation for a given Lie group. In this context transformation methods (TMs) are founded on the group invariance theory, see Bluman and Cole [7], Dresner [10], or Bluman and Kumei [8]. These methods are initial value methods because they transform BVPs to IVPs.

The first application of a non-iterative TM was defined by Töpfer in [35] for the Blasius problem, without any consideration of group invariance theory. This result is quoted in several books on fluid dynamics, see, for instance, Goldstein [20, pp. 135-136]. Acrivos, Shah and Petersen [1] first and Klamkin [21] later extended Töpfer's method respectively to a more general problem and to a class of problems. Along the lines of the work of Klamkin, for a given problem Na [27,28] showed the relation between the invariance properties, with respect to a linear group of transformation (the scaling group), and the applicability of a non-iterative TM. Moreover, Na considered the invariance with respect to a nonlinear group of transformations: the spiral group.

Belford [5] first, and Ames and Adams [2,3] later defined non-iterative TMs for eigenvalue problems.

A survey on the non-iterative TMs was written by Klamkin [22]. A survey book, written by Na [30, Chs 7-9] on the numerical solution of BVP, devoted three chapters to numerical TMs. Fazio and Evans [18] proposed non-iterative TMs for free BVPs.

However, non-iterative TMs are applicable only to particular classes of BVPs so that they have been considered as *ad hoc* methods, see Meyer [25, pp. 35-36], Na [30, p. 137] or Sachdev [31, p. 218].

The transformation of BVPs to IVPs has also a theoretical relevance. In fact, existence and uniqueness results can be obtained as a consequence of the invariance properties. For instance, a simple existence and uniqueness theorem for the Blasius problem was given by J. Serrin as reported by Meyer [26, pp. 104-105]. On this topic the application of a numerical test, defined within group invariance theory, to verify the existence and uniqueness of the solution of a free BVPs was considered by Fazio in [13], see also [16].

Here we consider two-point BVPs and define non-iterative TMs by using the invariance properties of translation groups. The paper is organized as follows. In the next section we consider the group of translations in the independent variable and define a related non-iterative TM. In section 3 we use the group of translations in the dependent variable. Within each of these two sections an application, belonging to the characterized classes of BVPs, is reported in order to show the performance of the TMs. The numerical results were obtained by means of the ODE45 integrator, with default tolerance parameter set up, from the MATLAB ODE suite written by Samphine and Reichelt [33] and available with the latest releases of MATLAB. The last section concerns with concluding remarks pointing out limitations and possible extensions of the proposed approach. There we indicate how the present approach is by no means the only way to extend non-iterative TMs.

2. Translation of the independent variable

Let us consider the class of free BVPs given by

$$\begin{aligned} &\frac{d^2u}{dx^2} = \Omega\left(u, \frac{du}{dx}\right) \\ &u(0) = A \\ &u(s) = B \quad , \quad \frac{du}{dx}(s) = C \, , \end{aligned} \tag{1}$$

where A, B and C are arbitrary constants, and $s > 0$ is the unknown free boundary. Free BVPs represent a numerical challenge because they are always nonlinear as pointed out first by Landau [23]. The governing differential equation in (1) is invariant with regard to the translation Lie group

$$x^* = x + \mu \quad ; \quad s^* = s + \mu \quad ; \quad u^* = u \, . \tag{2}$$

The non-iterative numerical solution of (1) can be obtained by the following steps:

- we fix freely a value of s^*;
- we integrate backwards from s^* to x_A^* the following auxiliary IVP

$$\begin{aligned} &\frac{d^2u^*}{dx^{*2}} = \Omega\left(u^*, \frac{du^*}{dx^*}\right) \\ &u^*(s^*) = B \\ &\frac{du^*}{dx^*}(s^*) = C \, , \end{aligned} \tag{3}$$

  using an event locator in order to find x_A^* such that $u^*(x_A^*) = A$;

- finally, through the invariance property, we can deduce the similarity parameter
$$\mu = x_A^*$$
from which we get the unknown free boundary
$$s = s^* - \mu \ .$$
The missing initial condition is given by
$$\frac{du}{dx}(0) = \frac{du^*}{dx^*}(x_A^*) \ .$$

2.1. *The obstacle problem on a string*

The obstacle problem on a string is depicted in the left frame of figure 1 within the (x,u)-plane where the x axis is taken overlying to the obstacle. In this problem we have to find the position of a uniform string of finite length L under the action of gravity. The string has fixed end points, say $(0,u_0)$ and $(b,0)$, where $u_0 > 0$ and $b > 0$. Furthermore, we assume that the condition $L^2 > \left(u_0^2 + b^2\right)$ is fulfilled; this condition allows us to define a free boundary s for this problem: s is the detached rope position from the obstacle. The mathematical model is given by

$$\begin{aligned}
&\frac{d^2u}{dx^2} = \theta \left[1 + \left(\frac{du}{dx}\right)^2\right]^{1/2} \ , \qquad x \in (0,s) \\
&u(0) = u_0 \ , \\
&u(s) = \frac{du}{dx}(s) = 0 \ ,
\end{aligned} \tag{4}$$

where the positive value of θ depends on the string properties, see Collatz [9] or Glashoff and Werner [19].

The free BVP (4) was solved by the first author in [12] by iterative methods, namely a shooting method and the iterative extension of the TM derived by using the invariance with respect to a scaling group.

The exact solution of the free BVP (4) is given by

$$u(x) = \theta^{-1}\left[\cosh\left(\theta\left(x-s\right)\right) - 1\right] \ , \quad s = \theta^{-1} \ln\left[\theta u_0 + 1 + \left(\left(\theta u_0 + 1\right)^2 - 1\right)^{1/2}\right]$$

from which we get

$$\frac{du}{dx}(0) = \sinh(-\theta s) = \frac{1}{2}\left(e^{-\theta s} - e^{\theta s}\right) \ .$$

The obtained numerical results, for increasing values of the θ parameter, are shown in figure 1. In order to highlight the free boundary variation with respect

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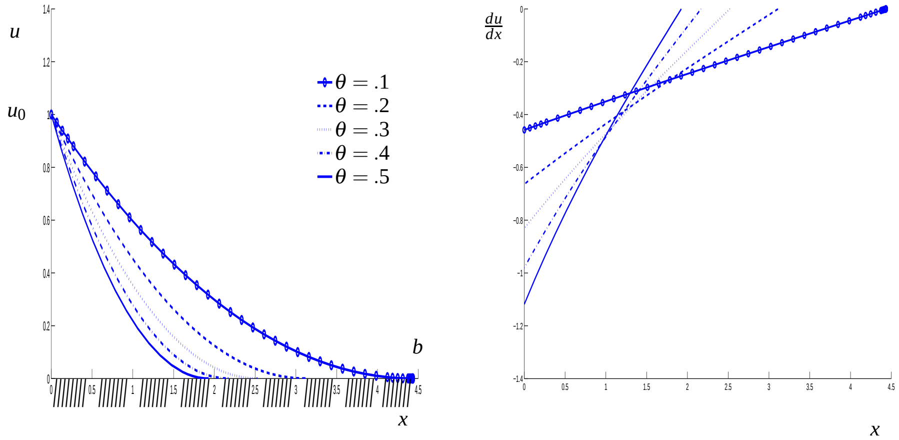

Fig. 1. Picture of the numerical solution obtained for values of θ in the legend, $u_0 = 1$ and $b = 4.5$; slashes in the left frame stand for the obstacle.

to θ we omitted to display the rope ahead the free boundary. Table 1 reports the obtained numerical results for the free boundary and the missing initial condition as well as a comparison to the exact solution. It is easily seen that the reported

θ	exact s	numerical s	exact $\frac{du}{dx}(0)$	numerical $\frac{du}{dx}(0)$
0.1	4.435682543851	4.435682543331	-0.458257569438	-0.458257569455
0.2	3.111812518574	3.111812501510	-0.663324953976	-0.663324956263
0.3	2.521443036190	2.521442990630	-0.830662368524	-0.830662372717
0.4	2.167536816226	2.167536535889	-0.979795740124	-0.979795851446
0.5	1.924847300238	1.924847181887	-1.118033899987	-1.118033873655

results are correct to the sixth decimal place.

3. Translation of the dependent variable

Let us consider the class of BVPs represented by

$$
\begin{aligned}
& \frac{d^2 w}{dx^2} = \Theta\left(x, \frac{dw}{dx}\right) \\
& \frac{dw}{dx}(0) = C \\
& w(b) = B \,,
\end{aligned}
\tag{5}
$$

where B and C are arbitrary constants, and $b > 0$. In (5) the governing differential equation and the boundary condition at zero are invariant with regard to the translation Lie group

$$x^* = x \quad ; \quad w^* = w + \mu \,, \tag{6}$$

The non-iterative numerical solution of (5) can be obtained by the following steps:

- we fix freely a value of $w^*(0) = A$;
- we integrate up to b the following auxiliary IVP

$$\begin{aligned} & \frac{d^2 w^*}{dx^{*2}} = \Theta\left(x^*, \frac{dw^*}{dx^*}\right) \\ & w^*(0) = A \\ & \frac{dw^*}{dx^*}(0) = C \,; \end{aligned} \tag{7}$$

- finally, through the invariance property $w^*(b) = w(b) + \mu$, we can deduce the similarity parameter

$$\mu = w^*(b) - B \,,$$

and the missing initial condition

$$w(0) = w^*(0) - \mu \,.$$

3.1. *An application to a parachute model*

Among the differential equations invariant to the group (6), there is the one used for the well-known model describing the vertical motion of a parachutist through the atmosphere. In particular, instead of studying such a model as an IVP according to its usual treatment, see for instance [24], we are interested in looking at it as a BVP where a zero initial velocity is prescribed and the parachutist is expected to touch the ground at a prefixed time.

Such a model is expressed by

$$\begin{aligned} & \frac{d^2 z}{dt^2} = \frac{K(t)}{m}\left(\frac{dz}{dt}\right)^2 - g \\ & \frac{dz}{dt}(0) = 0 \\ & z(b) = 0 \,, \end{aligned} \tag{8}$$

where z is the vertical coordinate taken positive going outward the ground that is also taken as the origin of the axes, t is the time variable, m is the total mass of parachutist's body plus his equipment, g is the acceleration of gravity, $K(t)$ is the

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damp coefficient of air, and b is a given final time when the parachutist is expected to land.

It is worth saying that such a problem, mathematically expressed in the BVP (8), has a practical importance for all those applications where the total time of fall must be kept under a given safety value. For instance, we can consider the military tactical throw of paratroopers behind the enemy lines where the less is the duration of the fall the lower is the probability to be targeted.

According to [24], because of a very high Reynolds numbers ($Re \approx 10^5 \div 10^6$), the air resistance against the motion of a parachutist through the atmosphere can be better modeled by means of a force proportional to the square of velocity, i.e.

$$F_{drag} = K(t)\left(\frac{dz}{dt}\right)^2 ,$$

instead of a force proportional to the velocity itself, which is more commonly used within an elementary modeling.

In order to define the parameter $K(t)$, we must take into account that the parachutist motion is essentially divided into a first phase relative to a free fall, a second one relative to the deployment of the parachute, and finally a third one relative to the dampened fall. Furthermore, instead of its usual piecewise definition, here we preferred to model it as the continuous function of time

$$K(t) = \frac{K_2 - K_1}{\pi}\arctan\left(P(t-t_c)\right) + \frac{K_1 + K_2}{2} .$$

The parameters involved can be defined as follows: according to the values reported in [24] for a parachute with cross sectional area of $43.8\,\mathrm{m}^2$, we can take $K_1 = 0.49\,\mathrm{Kg/m}$ and $K_2 = 29.16\,\mathrm{Kg/m}$ as the damp coefficient for the free fall and the dampened fall, respectively. With regard to the other parameters, by knowing that after $10\,\mathrm{s}$ the parachute activation is triggered and that the time spell for the complete deployment of the parachute is about $\Delta t_d = 3.2\,\mathrm{s}$, we chose to fix $t_c = 10 + \Delta t_d/2 = 11.6\,\mathrm{s}$. Finally, the value of P was determined by taking the average slope between the value of K_1 and K_2 over the time spell deployment Δt_d. For the reader convenience $K(t)$ is shown in the left frame of figure 2.

Having fixed $g = 9.81\,\mathrm{m/s}^2$, a final time $b = 40\,\mathrm{s}$ and $z^*(0) = 100\,\mathrm{m}$ as the initial missing condition for the auxiliary problem, for discrete increasing values of m ranging from $70\,\mathrm{Kg}$ to $120\,\mathrm{Kg}$, it was carried out a numerical experiment for the model (8). The missing initial condition found is reported in table 2. In figure 2, for a few values of m they are depicted the corresponding numerical solutions.

At a first glance, it is crystal clear that the higher is value of m the higher must be the drop altitude $z(0)$ in order to comply with the assigned boundary conditions.

Table 2. Parachute model: numerical results for increasing m.

m	$z(0)$	$\frac{dz}{dt}(b)$
70	425.1710	-4.8553
80	449.8336	-5.1899
90	472.4850	-5.5049
100	493.4470	-5.8031
110	512.9900	-6.0886
120	531.3088	-6.3569

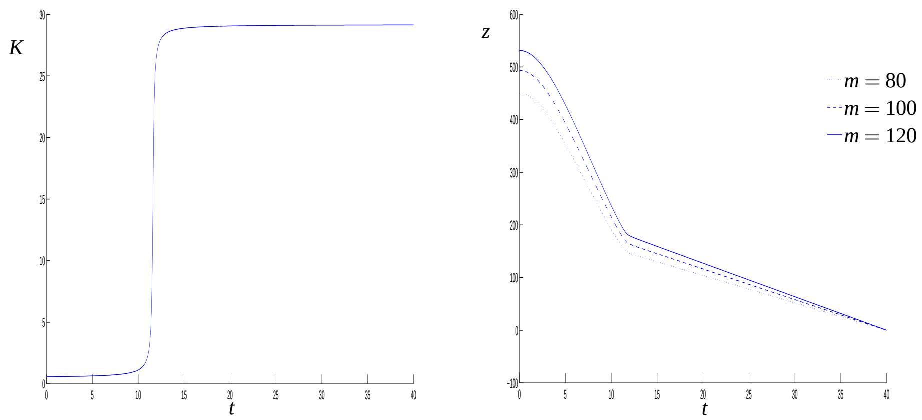

Fig. 2. On the left frame $K(t)$, on right one sample numerical results for the parachute model.

4. Conclusion

In closing, we can outline some further implications coming out from this work. Extensions of non-iterative TMs, by requiring the invariance of one and of two or more physical parameters when they are involved in the mathematical model, were respectively proposed by Na [29] and by Scott, Rinschler and Na [32], cf. also Na [30, Chapters 8 and 9]. Moreover, the introduction of a variable transformation linking two different invariant groups is a different way to extend the applicability of non-iterative TMs, as shown by Fazio [11]. Here, by considering the invariance with respect to translation groups, we have investigated a further way to extend non-iterative TMs.

However, it is a simple matter to show a differential equation not admitting any group of transformations: e.g. the differential equation considered by Bianchi [6, pp. 470-475]. Consequently, it is easy to realize that non-iterative TMs cannot be extended to every BVPs, because their applicability depends upon the invariance properties of the governing differential equation and the given boundary conditions.

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On the other hand, free BVPs governed by the most general second order differential equation, in normal form, can be solved iteratively by extending a scaling group via the introduction of a numerical parameter h so as to recover the original problem as h goes to one, see Fazio [14,15,16]. The application of this iterative TM to moving boundary problems governed by parabolic equations has been considered in [17].

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