Perpetual American Put Option: an Error Estimator for a Second Order Non-Standard Finite Difference Scheme

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Abstract

In this paper, we present the numerical result obtained by a MATLAB version of a second order non-standard finite difference scheme for the numerical solution of the perpetual American put option models of financial markets. These models can be derived from the celebrated Black-Scholes models letting the time go to infinity. The considered problem is a free boundary problem defined on a semi-infinite interval so that it is a non-linear problem complicated by a boundary condition at infinity.

By using non-uniform maps and non-standard finite difference formulae, we show how it is possible to apply the boundary condition at infinity exactly. Moreover, we define a posteriori error estimator that is based on the Richardson classical extrapolation theory. Our finite difference scheme

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and error estimator are favourably tested for a simple problem with a known exact analytical solution, the scheme is found to be of second order accuracy.

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1 Introduction

In the market of financial derivatives, the most important problem is the so-called *option valuation problem* or in a few words: the problem of computing a fair price for a given option. Analytical solutions of models of American option problems are seldom available, so such derivatives of financial markets must be priced by numerical methods (Amin and Khanna [2], Barraquand and Pudet [5], Broadie and Detemple [9], Nielsen et al. [27], Barone-Adesi [4], Düring and Fournié [12], Milev and Tagliani [26]), Fazio [15] or Fazio et al. [16]). In this paper, we present a MATLAB version of a non-standard finite difference scheme for the numerical solution of the perpetual American put option models of financial markets. These models can be derived from the celebrated Black-Scholes models (Leland [25], Avellaneda and Parás [3], Frey and Patie [20] and Jandačka and Ševčovič [22]) letting the time go to infinity (Bensoussan [6] or Elliot and Kopp [13, pp. 196-199]).Fazio:2020:AFF The considered problem is a free boundary problem defined on a semi-infinite interval so that it is a non-linear problem complicated by a boundary condition at infinity.

By using non-uniform maps, we show how it is possible to apply the boundary condition at infinity exactly. Non-uniform maps have been applied to the numerical solution of ordinary and partial differential equations on unbounded domains (van de Vooren and Dijkstra [31], Botta et al. [7], Davis [11], Grosch and Orszag [21], Boyd [8], Alshina et al. [1], Koleva [23] or Fazio and Jannelli [17, 18]). Moreover, we deduce an a posteriori error estimator within Richardson classical

extrapolation theory, see Fazio and Jannelli [19]. Our finite difference scheme and error estimator are favourably tested for a simple problem with a known exact analytical solution. From the obtained numerical results we can asses that: the finite difference method is second-order accurate, the numerical solution can be improved by repeated Richardson extrapolations and the error estimator provides upper bounds for the exact error.

As this author's knowledge is concerned this is the first time that the perpetual American put model is solved by a quasi-uniform non-standard finite difference scheme. Naturally, it would be possible to use the defined error estimator in order to find a numerical solution within a given prefixed error tolerance. To this end, we should modify our numerical algorithm in the same way it has been done by this author for the classical American put model, see Fazio [14, 15] for details.

2 Perpetual American put option

In order to test our error estimator, in this section, we consider a test problem with known exact analytical solution. This problem is a free boundary problem arising as a simple toy model in the study of financial markets [6]. A mathematical model describing the perpetual American put option is given by

$$\begin{aligned} &\frac{1}{2}\sigma^2 S^2 \frac{d^2P}{dS^2} + rS\frac{dP}{dS} - rP = 0\,, \qquad \text{on } \; R \le S < \infty\,, \\ &P(R) = \max\{E-R,0\}\,, \qquad \frac{dP}{dS}(R) = -1\,, \\ &\lim_{S\to\infty} P(S) = 0\,, \end{aligned} \tag{2.1}$$

where S is the price of a given asset, $P(S)$ is the price of the perpetual American put option to sell the asset, R is the unknown free boundary, σ, r and E are the volatility, interest rate and exercise price of the asset, respectively. This problem (2.1) has the exact solution

$$P(S) = (E-R)\, R^{2\,r/\sigma^2}\, S^{-2\,r/\sigma^2}\,, \qquad R = \frac{2\,r\,E}{2\,r+\sigma^2}\,, \tag{2.2}$$

see [13, pp. 196-199]. In order to fix the domain, see Crank [10, pp. 187-192], we can apply Landau's transformation of variables

$$x = S/R\ , \qquad u(x) = P(x\,R)\ .$$

In the new variables, the put option problem (2.1) can be rewritten as follows

$$\begin{aligned}
&\frac{1}{2}\sigma^2 x^2 \frac{d^2u}{dx^2} + rx\frac{du}{dx} - ru = 0\ , \qquad \text{on } \ 1 \le x < \infty\ , \\
&u(1) = \max\{E-R,0\}\ , \qquad \frac{du}{dx}(1) = -R\ , \\
&\lim_{x\to\infty} u(x) = 0\ ,
\end{aligned} \tag{2.3}$$

Moreover, this model can be rewritten in standard form as a first-order system of ordinary differential equations. The model (2.1) is a special instance of the American put option obtained formally by letting the time variable to go to infinity. In recent years several generalizations, ranging from the introduction of further relevant market parameters to non-constants volatility and the like, of this model have been proposed in the literature. In particular, one can take into account: the presence of transaction costs (see e.g. Leland [25], Avellaneda and Parás [3]), feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Frey and Patie [20]), risk from unprotected portfolio (Jandačka and Ševčovič [22]). In order to take into account also those different models, and using the fixed boundary formulation (2.3), we study here the following class of problems

$$\begin{aligned}
&\frac{du}{dx} = \phi(x,u,v)\ , \qquad \text{on } \ 1 \le x < \infty\ , \\
&\frac{dv}{dx} = \psi(x,u,v)\ , \\
&\frac{dR}{dx} = 0\ , \\
&u(1) = \max\{E-R,0\}\ , \qquad v(1) = -R\ , \qquad \lim_{x\to\infty} u(x) = 0\ ,
\end{aligned} \tag{2.4}$$

where R is treated as a supplementary variable because its value is unknown and has to be found as part of the solution. Of course, our benchmark problem (2.3)

belongs to (2.4) for a suitable change of variables and suitable functional form of ϕ and ψ.

3 Quasi-uniform grids

Let us consider the smooth strict monotone quasi-uniform maps $x = x(\xi)$, the so-called grid generating functions,

$$x = -c \cdot \ln(1-\xi) + 1 \ , \tag{3.1}$$

and

$$x = c\frac{\xi}{1-\xi} + 1 \ , \tag{3.2}$$

where $\xi \in [0,1]$, $x \in [1,\infty]$, and $c > 0$ is a control parameter. So that, a family of uniform grids $\xi_n = n/N$ defined on interval $[0,1]$ generates one parameter family of quasi-uniform grids $x_n = x(\xi_n)$ on the interval $[1,\infty]$. The two maps (3.1) and (3.2) are referred to as a logarithmic and an algebraic maps, respectively. The logarithmic map (3.1) gives slightly better resolution near $x = 1$ than the algebraic map (3.2), while the algebraic map gives much better resolution than the logarithmic map as $x \to \infty$. In fact, it is easily verified that

$$-c \cdot \ln(1-\xi) + 1 < c\frac{\xi}{1-\xi} + 1 \ ,$$

for all ξ but $\xi = 0$.

The problem under consideration can be discretized by introducing a uniform grid ξ_n of $N+1$ nodes in $[0,1]$ with $\xi_0 = 0$ and $\xi_{n+1} = \xi_n + h$ with $h = 1/N$, so that x_n is a quasi-uniform grid in $[1,\infty]$. The last interval in (3.1) and (3.2), namely $[x_{N-1}, x_N]$, is infinite but the point $x_{N-1/2}$ is finite, because the non integer nodes are defined by

$$x_{n+\alpha} = x\left(\xi = \frac{n+\alpha}{N}\right) \ ,$$

with $n \in \{0,1,\ldots,N-1\}$ and $0 < \alpha < 1$. These maps allow us to describe the infinite domain by a finite number of intervals. The last node of such grid is placed on infinity so right boundary conditions are taken into account correctly.

4 A non-standard finite difference scheme

We can approximate the values of $u(x)$ on the mid-points of the grid

$$u(x_{n+1/2}) \approx \frac{x_{n+3/4}-x_{n+1/2}}{x_{n+3/4}-x_{n+1/4}}u_n + \frac{x_{n+1/2}-x_{n+1/4}}{x_{n+3/4}-x_{n+1/4}}u_{n+1} \ . \tag{4.1}$$

that is a non-standard central difference formula. Taking into account the results by Veldam and Rinzema [32], for the first derivative at the mid-points of the grid we can apply the following approximation

$$\frac{du}{dx}(x_{n+1/2}) \approx \frac{u_{n+1}-u_n}{2\left(x_{n+3/4}-x_{n+1/4}\right)} \ , \tag{4.2}$$

that is, again, a non-standard central difference formula. These finite difference formulae use the value $u_N = u_\infty$, but not $x_N = \infty$. The two finite difference approximations (4.1) and (4.2) have order of accuracy $O(N^{-2})$. For a system of differential equations, formulae (4.1) and (4.2) can be applied component-wise. The approximation (4.1) is a variant of the formula used by Fazio and Jannelli [18] allowing for a simpler definition of the finite difference scheme. A non-standard finite difference scheme on a quasi-uniform grid for our financial problem (2.1) can be defined by using the approximations given by (4.1) and (4.2) above.

We denote by the $3-$dimensional vector $\mathbf{U}_n = (U_n, V_n, R)^T$ the numerical approximation to the solution $\mathbf{u}(x_n) = (u(x_n), v(x_n), R)^T$ of (2.4) at the points of the mesh, that is for $n = 0, 1, \ldots, N$. A finite difference scheme for (2.4) can be written as follows:

$$\begin{gathered} \mathbf{U}_{n+1} - \mathbf{U}_n - a_{n+1/2}\boldsymbol{\phi}\left(x_{n+1/2}, b_{n+1/2}\mathbf{U}_{n+1} + c_{n+1/2}\mathbf{U}_n\right) = \mathbf{0} \ , \\ {}^1U_0 = \max\{E-R, 0\} \ , \quad {}^2U_0 = -R \ , \quad {}^1U_N = 0 \ , \end{gathered} \tag{4.3}$$

for $n = 0,\ 1,\ \ldots\ ,N-1$, here $\boldsymbol{\phi} = (\phi, \psi, 0)^T$, adopting Lambert's notation for the vector components [24, pp. 1-5], ${}^rU$ is the r-component of the vector $\mathbf{U}$ and

$$\begin{aligned} a_{n+1/2} &= 2\left(x_{n+3/4}-x_{n+1/4}\right) \ , \\ b_{n+1/2} &= \frac{x_{n+1/2}-x_{n+1/4}}{x_{n+3/4}-x_{n+1/4}} \ , \\ c_{n+1/2} &= \frac{x_{n+3/4}-x_{n+1/2}}{x_{n+3/4}-x_{n+1/4}} \ , \end{aligned} \tag{4.4}$$

for $n = 0, 1, \ldots, N-1$.

It is evident that (4.3) is a nonlinear system of $3 \cdot (N+1)$ equations in the $3 \cdot (N+1)$ unknowns $\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_N)^T$.

5 Richardson extrapolation

The utilization of a quasi-uniform grid allows us to improve our numerical results. The algorithm is based on Richardson extrapolation, introduced by Richardson in [28, 29]. Within Richardson extrapolation, we carry on several calculations on embedded uniform or quasi-uniform grids with total number of nodes N: e.g., for the numerical results reported in the next section we used $N = 2$, 4, 8, 16, 32, 64, 128, 256, 512 or $N = 5$, 10, 20, 40, 80, 160, 320, 640, 1280. We can identify these grids with the number of grid-points N_g where the index $g = 0$ identifies the coarsest one, 1, 2, and so on go towards the finest grid where the index has to be fixed in advance as G, so that $G = 8$ in our case. Between two adjacent grids, all nodes of the largest steps are identical to even nodes of the denser grid due to quasi-uniformity. For a scalar value U we can apply k Richardson extrapolations on the used grids

$$U_{g+1,k+1} = U_{g+1,k} + \frac{U_{g+1,k} - U_{g,k}}{q^{p_k} - 1}\,, \tag{5.1}$$

where $g \in \{0, 1, 2, \ldots, G-1\}$, $k \in \{0, 1, 2, \ldots, G-1\}$, $q = N_g / N_{g-1}$ is the grid refinement ratio, and p_k is the true order of the discretization error, see Schneider and Marchi [30] and the references quoted therein. This formula is asymptotically exact in the limit as N goes to infinity if we use uniform or quasi-uniform grids. We notice that to obtain each value of $U_{g+1,k+1}$ requires two computed solutions U in two adjacent grids, namely $g+1$ and g at the extrapolation level k. For any g, the level $k = 0$ represents the numerical solution of U without any extrapolation, which is obtained as described in section 4. The case $k = 1$ is the classical single Richardson extrapolation, which is usually used to estimate the discretization error or to improve the numerical solution accuracy. If we have computed the numerical solution on $G+1$ nested grids then we can apply equation (5.1) G times

performing G Richardson extrapolation.

Here we are interested to show how within Richardson extrapolation theory we can derive an error estimator. For any value of interest U, the numerical error E can be defined by

$$E = u - U \ , \tag{5.2}$$

where u is the exact analytical solution. Usually, we have several different sources of errors: discretization, round-off, iteration and programming errors. Discretization errors are due to our replacement of a continuous problem with a discrete one and these errors can be reduced by reducing the discretization parameters, enlarging the value of N in our case. Round-off errors are due to the utilization of floating-point arithmetic to implement the algorithms available to solve the discrete problem. This kind of error can be reduced by using higher precision arithmetic, double or, when available, fourth precision. Iteration errors are due to stopping an iteration algorithm that is converging but only as the number of iterations goes to infinity. Of course, we can reduce this kind of error by requiring more restrictive termination criteria for our iterations, the iterations of `fsolve` MATLAB routine in the present case. Programming errors are behind the scope of this work but they can be eliminated or at least reduced by adopting what is called *structured programming*. When the numerical error is caused prevalently by the discretization error and in the case of smooth enough solutions the discretization error can be decomposed into a sum of powers of the inverse of N

$$u = U_N + C_0 \left(\frac{1}{N}\right)^{p_0} + C_1 \left(\frac{1}{N}\right)^{p_1} + C_2 \left(\frac{1}{N}\right)^{p_2} + \cdots \ , \tag{5.3}$$

where C_0, C_1, C_2, $\ldots$ are coefficient that depend on u and its derivatives, but are independent on N, and p_0, p_1, p_2, $\ldots$ are the true orders of the error. The value of each p_k is usually a positive integer with $p_0 < p_1 < p_2 < \cdots$ and constitute an arithmetic progression of ratio $p_1 - p_0$. The value of p_0 is called the asymptotic order or the order of accuracy of the method or of the numerical solution U. So that, the theoretical order of accuracy of the numerical solutions $U_{g,k}$ with k

extrapolations the p_k orders verify the relation

$$p_k = p_0 + k(p_1 - p_0) \ , \tag{5.4}$$

where this equation is valid for $k \in \{0, 1, 2, \ldots, G-1\}$.

5.1 Error estimator

To show how Richardson extrapolation can be also used to get an error estimator for the computed numerical solution we use the notation introduced above. By replacing into equation (5.3) N with $2N$ and subtracting, to the obtained equation, equation (5.3) times $(1/2)^{p_0}$ we get the first extrapolation formula

$$u \approx U_{2N} + \frac{U_{2N} - U_N}{2^{p_0} - 1} \ , \tag{5.5}$$

that has a leading order of accuracy equal to p_1. Taking into account equation (5.5) we can conclude that the error estimator by a first Richardson extrapolation is given by

$$Eest = \frac{U_{2N} - U_N}{2^{p_0} - 1} \ , \tag{5.6}$$

where p_0 is the order of the numerical method used to compute the numerical solutions. Hence, it gives the real value of numerical solution error without knowledge of the exact solution. In comparison with (5.6) a safer error estimator can be defined by

$$Esafe = U_{2N} - U_N \ . \tag{5.7}$$

Of course, p_0 can be found by

$$p_0 \approx \frac{\log(|U_N - u|) - \log(|U_{2N} - u|)}{\log(2)} \ , \tag{5.8}$$

where u is again the exact solution (or, if the exact solution is unknown, a reference solution computed with a suitable large value of N), and both u and U_{2N} are evaluated at the same grid-points of U_N.

6 Numerical results

It should be mentioned that all numerical results reported in this paper were performed on an ASUS personal computer with i7 quad-core Intel processor and 16 GB of RAM memory running Windows 8.1 operating system.

The non-standard finite difference scheme, as well as the two error estimators described above, have been implemented in MATLAB. In this way, we take advantage of the available MATLAB built-in functions. In particular, for the solution of the non-linear system (4.3) we used the function `fsolve`. Among the available alternative we used the "Levenberg-Marquardt" with $TolFun = 10^{-15}$ and $TolX = 10^{-15}$ options. These values of $TolFun$ and $TolX$ define the termination criteria for `fsolve`. Usually, the `fsolve` routine performed between 5 to 11 iterations to get a numerical solution that verifies the stopping criteria.

To set a specific test problem we fixed the following values for the involved parameters

$$\sigma^2 = 0.1\ , \qquad r = 0.05\ , \qquad E = 10\ . \tag{6.1}$$

As we will see below these values provide an exact solution that remains different from zero within a large domain. For our numerical computations, we used both the two maps (3.1) with $c = 20$ and (3.2) with $c = 10$, but the results reported below are concerned with the first map because the results obtained with the second map are, indeed, very similar. In order to speed up the computations for different values of N we adopted a continuation strategy. For a small value of N, usually $N = 2$ or $N = 5$, we always used a constant initial iterate vector made with all components equal to one. Then, when refining the grid we used the accepted final iterate of the previous value of N as the first iterate for the computation with the next value of N. Figure 1 shows a reference solution. The numerical results obtained by setting $N = 128$ can be seen in the same figure. The non-uniform grid is clearly visible even if the last grid-point is not shown because it is located at infinity.

Figure 2 shows two sample error estimates made by the error estimator (5.7), from top to bottom we used $N = 16$ to $N = 32$. It is easily seen that the safe

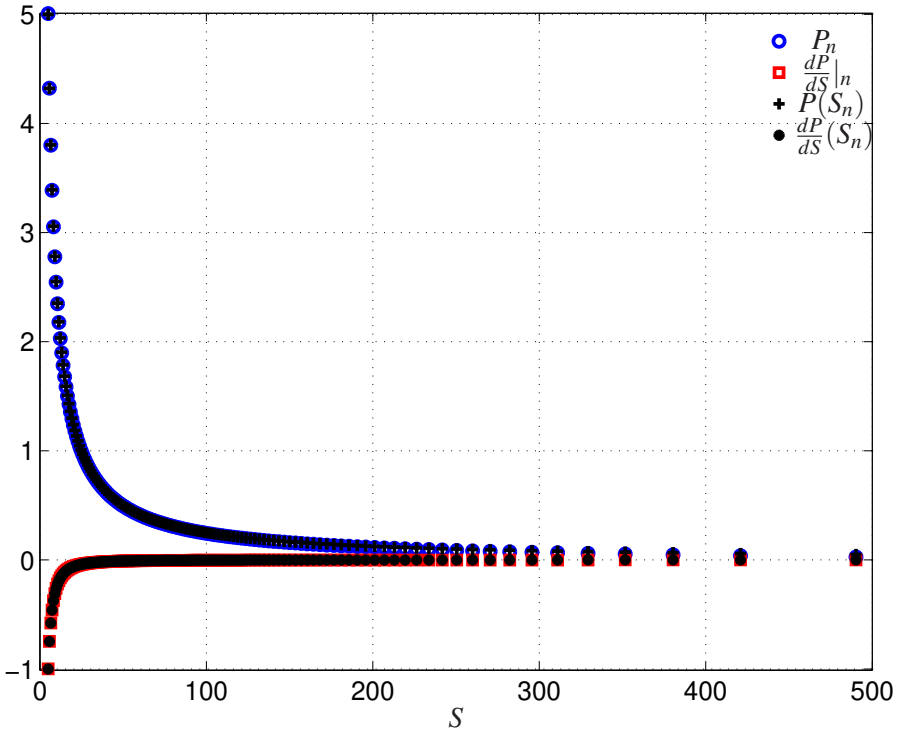

Figure 1: *Sample exact and numerical solutions for the test problem (2.1).The symbols indicate:* $+$ *the exact solution,* $\bullet$ *its first derivative,* $\circ$ *the numerical solution, and* $\square$ *its numerical first derivative.*

estimator defined by equation (5.7) provides upper bounds for the true error.

In table 1 we report, for different values of N, several extrapolations for the free boundary value R of our test problem (2.1). These values where computed according to the extrapolation formula

$$R_{2N,k+1} = R_{2N,k} + \frac{R_{2N,k} - R_{N,k}}{2^{k+1} - 1}, \quad \text{for } k = 0, 1, 2, \ldots . \tag{6.2}$$

In this table, since the values of N can be seen on the first column, we omitted the first subscript for the notation defined in equation (5.1) and used in equation (6.2) for the extrapolated values of R. We notice that the first reported values are far from the correct value of R. However, it is noteworthy that these incorrect values are used by Richardson's extrapolation formula to compute the accurate value of $R_3 = 5.000153$ in the last column and the row for $N = 512$.

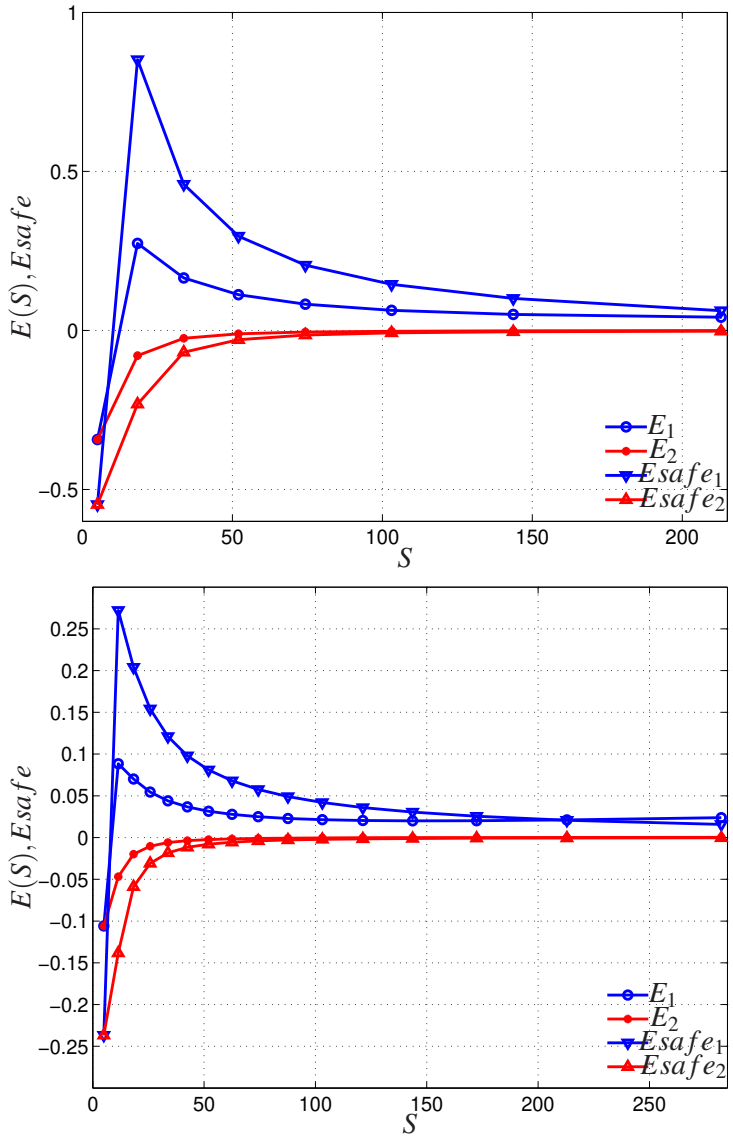

Figure 2: *Safe error estimates provided by equation (5.7) and true error by equation (5.2) for the test problem (2.1). Here* E_1 *is the solution true error,* E_2 *true error for the solution first derivative,* $Esafe_1$ *the safe error estimate for the solution, and* $Esafe_2$ *the safe error estimate for the solution first derivative.*

Now we can apply the formula (5.8) to the computed value of R_0 in table 1. The computed values of p_0 are given by 0.650819676, 1.009888863, 1.376800758, 1.693756030, 1.893889652, 1.998098336, 2.107064507, and 2.485357150. There-

N	R_0	R_1	R_2	R_3
2	2.1860735			
4	3.2077586	3.548320		
8	4.1100007	4.410748	4.533952	
16	4.6572864	4.839715	4.900996	4.925466
32	4.8940600	4.972985	4.992024	4.998093
64	4.9714936	4.997305	5.000779	5.001363
128	4.9928640	4.999987	5.000370	5.000343
256	4.9983436	5.000170	5.000196	5.000184
512	4.9997042	5.000158	5.000156	5.000153

Table 1: Richardson's extrapolations for the free boundary value of the test problem (2.1) with parameters fixed in (6.1). Note that for this problem the exact value is $R = 5$.

fore, our claim that the finite difference scheme is second order is correct.

7 Conclusions

In this paper, we have presented the results obtained by a MATLAB version of a non-standard finite difference scheme for the numerical solution of the so-called perpetual American put option model of financial markets. This model can be derived from the celebrated Black-Scholes model letting the time go to infinity. Even for non-linear Black-Scholes models, there is no known formula for the price of an American put with a finite exercise time; indeed there are closed formulae for prices of linear models or infinite exercise time American put, American call option and European put and call options. A variety of numerical methods and approximations for the American put option price have been developed over the years. An overview of the various methods can be found for example in Barone-

Adesi [4]. The problem considered here is a free boundary problem defined on a semi-infinite interval so that it is a non-linear problem complicated by a boundary condition at infinity. By using non-uniform maps, we have shown how it is possible to apply the boundary condition at infinity exactly in contrast with the definition of a truncated boundary that introduces an error related to the replacement of infinity by a finite value, see for instance Nielsen et al. [27].

As future work, it would be of interest to apply our non-standard difference scheme to non-linear perpetual America option models. Moreover, it would be relevant to extend our numerical scheme to Black-Scholes models that are governed by partial differential equations defined on infinite domains. Of course, we can apply the Landau's transform to the original moving boundary problem to get a problem defined on a fixed domain. The semi-discretization in time of the transformed problem with standard methods like the first order Euler or high order Runge-Kutta type will result in a sequence of problems in the class (2.4).

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