

The Non-Iterative Transformation Method

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Abstract

The Blasius flow is the idealized flow of a viscous fluid past an infinitesimally thick, semi-infinite flat plate. The definition of a non-iterative transformation method for the celebrated Blasius problem is due to Töpfer and dates more than a century ago. Here we define a non-iterative transformation method for Blasius equation with a moving wall, a slip flow condition or a surface gasification. The defined method allows us to deal with classes of problems in boundary layer theory that, depending on a parameter, admit multiple or no solutions. This approach is particularly convenient when the main interest is on the behaviour of the considered models with respect to the involved parameter. The obtained numerical results are found to be in good agreement with those available in literature.

Keywords: Initial value methods, non-iterative transformation method, BVPs on infinite intervals, Blasius equation, moving wall, slip flow condition, surface gasification.

1 Introduction

At the beginning of the last century Prandtl [26] put the foundations of boundary-layer theory providing the basis for the unification of two, at that time seemingly incompatible, sciences: namely, theoretical hydrodynamics and hydraulics. Boundary-layer theory has found its

main application in calculating the skin-friction drag which acts on a body as it is moved through a fluid: for example the drag of an airplane wing, of a turbine blade, or a complete ship [30]. The Blasius flow is the idealized flow of a viscous fluid past an infinitesimally thick, semi-infinite flat plate. Recently, Boyd [6] uses Blasius problem as an example were some good analysis, before the computer invention, allowed researchers of the past to solve problems, governed by partial differential equations, that might be otherwise impossible to face.

Blasius problem [4] is the simplest nonlinear boundary layer problem. A study by Boyd point out how this particular problem has arisen the interest of prominent scientist, like H. Weyl, J. von Neumann, M. Van Dyke, etc., see Table 1 in [5]. The main reason for this interest is due to the hope that any approach developed for this epitome can be extended to more difficult hydrodynamics problems.

Blasius main interest was to compute, without worrying about existence or uniqueness of its boundary value problem (BVP) solution – it was Weyl who proved in [35] that Blasius problem has one and only one solution –, the value of λ the so-called shear stress. To compute this value, Blasius used a formal series solution around $\eta = 0$ and an asymptotic expansions for large values of η , adjusting the constant λ so as to connect both expansions in a middle region. In this way, Blasius obtained the (erroneous) bounds $0.3315 < \lambda < 0.33175$.

A few years later, Töpfer [33] revised the work by Blasius and solved numerically the Blasius equation with suitable initial conditions and the classical order-four Runge-Kutta method. He then arrived, without detailing his computations, at the value $\lambda \approx 0.33206$, contradicting the bounds reported by Blasius.

Thereafter, the quest for a good approximation of λ was a main concern. This is seldom the case for the most important problems of applied mathematics: at the first study everyone would like to know if there is a method to solve a given problem, but, as soon as a problem is solved, then we would like to know how accurate is the computed solution and whether there are different methods that can provide a solution with less effort.

By using a power series, Bairstow [2] reports $\lambda \approx 0.335$, and Goldstein [19] obtains $\lambda \approx 0.332$ or, using a finite difference method, Falkner [10] finds $\lambda \approx 0.3325765$, and Howarth [20] yields $\lambda \approx 0.332057$. Fazio [12], using a free boundary formulation of the Blasius problem, finds $\lambda \approx 0.332057336215$. Boyd [5] uses Töpfer's algorithm to obtain the accurate value $\lambda \approx 0.33205733621519630$. By the Adomian's decomposition method Abbasbandy [1] finds $\lambda \approx 0.333329$, whereas a variational iteration method with Padé approximants allows Wazwaz [34] to calculate, the imprecise value, $\lambda \approx 0.3732905625$. Tajvidi et

al. [32] apply modified rational Legendre functions to get a value of $\lambda \approx 0.33209$.

At the turning of this new century, as the number of applications of microelectronics devices increases, boundary-layer theory has found a renewal of interest within the study of gas and liquid flows at the micro-scale regime, see, for instance, Gad el Hak [8] or Martin and Boyd [23].

2 Fluid flow on a flat plate

The model describing the steady plane flow of a fluid past a thin plate, provided the boundary layer assumptions are verified (the flow has a very thin layer attached to the plate and $v \gg w$), is given by

$$\begin{aligned} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\ v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= \nu \frac{\partial^2 v}{\partial z^2} \\ v(y, 0) = w(y, 0) &= 0, \quad v(y, z) \rightarrow V_\infty \quad \text{as } z \rightarrow \infty, \end{aligned} \tag{1}$$

where the governing differential equations, namely conservation of mass and momentum, are the steady-state 2D Navier-Stokes equations under the boundary layer approximations, v and w are the velocity components of the fluid in the y and z direction, V_∞ represents the main-stream velocity, see the draft in figure 1, and ν is the viscosity of the fluid. The boundary conditions at $z = 0$ are based on the assumption that neither slip nor mass transfer are permitted at the plate whereas the remaining boundary condition means that the velocity v tends to the main-stream velocity V_∞ asymptotically.

In order to study this problem, it is convenient to introduce a potential (stream function) $\psi(y, z)$ defined by

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y}. \tag{2}$$

The physical motivation for introducing this function is that constant ψ lines are stream-lines. The mathematical motivation for introducing such a new variable is that the equation of continuity is satisfied identically, and we have to deal only with the transformed momentum equation. In fact, introducing the stream function the problem can be

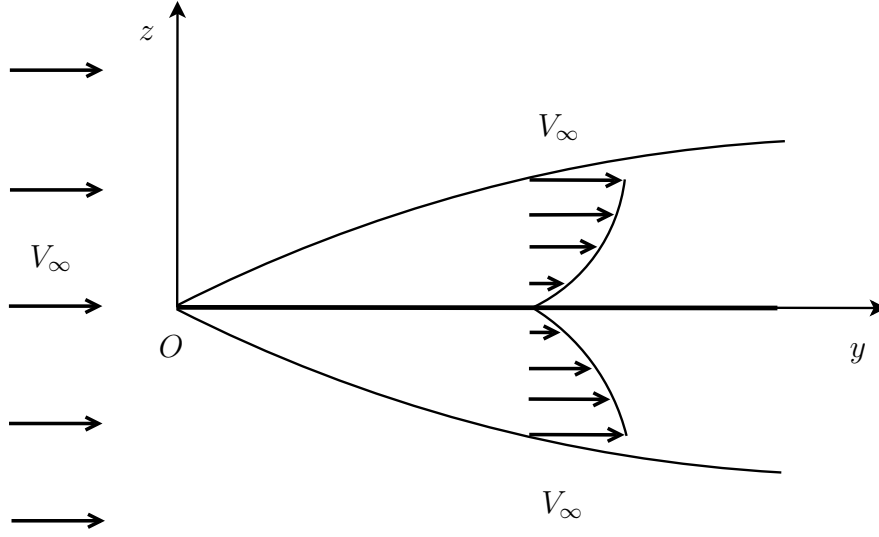


Figure 1: Sketch of boundary layer over a thin plate.

rewritten as follows

$$\begin{aligned} \nu \frac{\partial^3 \psi}{\partial z^3} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial y \partial z} &= 0 \\ \frac{\partial \psi}{\partial y}(y, 0) = \frac{\partial \psi}{\partial z}(y, 0) &= 0 \\ \frac{\partial \psi}{\partial z}(y, z) &\rightarrow V_\infty \quad \text{as } z \rightarrow \infty . \end{aligned} \quad (3)$$

2.1 Blasius problem

Blasius [4] used the following similarity transformation

$$\eta = z \left(\frac{V_\infty}{\nu y} \right)^{1/2}, \quad f(\eta) = \psi(y, z) (\nu y V_\infty)^{-1/2}, \quad (4)$$

that reduces the partial differential model (3) to

$$\begin{aligned} \frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} &= 0 \\ f(0) = \frac{df}{d\eta}(0) &= 0, \quad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty, \end{aligned} \quad (5)$$

i.e., a boundary value problem (BVP) defined on a semi-infinite interval. Blasius solved this BVP by patching a power series to an asymptotic approximation at some finite value of η .

3 Töpfer transformation

In order to clarify Töpfer [33] derivation of a further transformation of variables that reduces the BVP into an initial value problem (IVP) we consider the derivation of the series expansion solution. Of course, some of the coefficients of the series can be evaluated by imposing the boundary conditions at $\eta = 0$. Moreover, we set

$$\lambda = \frac{d^2 f}{d\eta^2}(0) \quad (6)$$

where λ is a constant different from zero. So that, we look for a series solution defined as

$$f(\eta) = \frac{\lambda}{2}\eta^2 + \sum_{n=3}^{\infty} C_n \eta^n \quad (7)$$

where the coefficients λ and C_n , for $n = 3, 4, \dots$, are constants to be determined. In fact, the boundary values at the plate surface, at $\eta = 0$, require that $C_0 = C_1 = 0$, and we also have $C_2 = \lambda/2$ by the definition of λ . Now, we substitute this series expansion into the governing differential equation, whereupon we find

$$\begin{aligned} & \sum_{n=3}^{\infty} n(n-1)(n-2)C_n \eta^{n-3} \\ & + \frac{1}{2} \left(\frac{\lambda}{2}\eta^2 + \sum_{n=3}^{\infty} C_n \eta^n \right) \left[\lambda + \sum_{n=3}^{\infty} n(n-1)C_n \eta^{n-2} \right] = 0 \quad (8) \end{aligned}$$

or in expanded form

$$\begin{aligned} & [3 \ 2 \ C_3] + [4 \ 3 \ 2 \ C_4] \eta + \left[5 \ 4 \ 3 \ C_5 + \frac{1}{2} \ 2 \ \frac{\lambda \ \lambda}{2 \ 2} \right] \eta^2 \\ & + \left[6 \ 5 \ 4 \ C_6 + \frac{1}{2} \ 2 \ \frac{\lambda}{2} C_3 + \frac{1}{2} \ \frac{\lambda}{2} \ 3 \ 2 \ C_3 \right] \eta^3 \\ & + \dots = 0 . \quad (9) \end{aligned}$$

According to a standard approach, we have to require that all coefficients of the powers of η to be zero. It is an easy matter to compute the coefficients of the series expansion in terms of λ :

$$\begin{aligned} C_3 = C_4 = 0 , \quad C_5 &= -\frac{\lambda^2}{2 \ 5!} \\ C_6 = C_7 = 0 , \quad C_8 &= 11 \frac{\lambda^3}{2^2 \ 8!} \\ C_9 = C_{10} = 0 , \quad C_{11} &= -375 \frac{\lambda^4}{2^3 \ 11!} \\ & \text{and so on ...} \end{aligned}$$

The solution can be written as

$$f = \frac{\lambda\eta^2}{2} - \frac{\lambda^2\eta^5}{2 \cdot 5!} + \frac{11 \lambda^3\eta^8}{2^2 \cdot 8!} - \frac{375 \lambda^4\eta^{11}}{2^3 \cdot 11!} + \dots \quad (10)$$

where the only unknown constant is λ . In principle, λ can be determined by imposing the boundary condition at the second point, but, in this case, this cannot be done because the left boundary condition is given at infinity. However, by modifying the powers of λ we can rewrite the series expansion as

$$\lambda^{-1/3}f = \frac{(\lambda^{1/3}\eta)^2}{2} - \frac{(\lambda^{1/3}\eta)^5}{2 \cdot 5!} + \frac{11 (\lambda^{1/3}\eta)^8}{2^2 \cdot 8!} - \frac{375 (\lambda^{1/3}\eta)^{11}}{2^3 \cdot 11!} + \dots \quad (11)$$

which suggests a transformation of the form

$$f^* = \lambda^{-1/3}f, \quad \eta^* = \lambda^{1/3}\eta. \quad (12)$$

In the new variables the series expansion becomes

$$f^* = \frac{\eta^{*2}}{2} - \frac{\eta^{*5}}{2 \cdot 5!} + \frac{11 \eta^{*8}}{2^2 \cdot 8!} - \frac{375 \eta^{*11}}{2^3 \cdot 11!} + \dots \quad (13)$$

which does not depend on λ . We notice that the governing differential equation and the initial conditions at the free surface, at $\eta = 0$, are left invariant by the new variables defined above. Moreover, the first and second order derivatives transform in the following way

$$\frac{df^*}{d\eta^*} = \lambda^{-2/3} \frac{df}{d\eta}, \quad \frac{d^2 f^*}{d\eta^{*2}} = \lambda^{-1} \frac{d^2 f}{d\eta^2}. \quad (14)$$

As a consequence of the definition of λ we have

$$\frac{d^2 f^*}{d\eta^{*2}}(0) = 1, \quad (15)$$

and this explains why in these variables the series expansion does not depend on λ . Furthermore, the value of λ can be found on condition that we have an approximation for $\frac{df^*}{d\eta^*}(\infty)$, say $\frac{df^*}{d\eta^*}(\eta_\infty)$ where η_∞^* is a suitable truncated boundary. In fact, by the above relation we get

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta_\infty^*) \right]^{-3/2}. \quad (16)$$

From a numerical viewpoint, BVPs must be solved within the computational domain simultaneously (a ‘‘stationary’’ problem), whereas IVPs can be solved by a stepwise procedure (an ‘‘evolution’’ problem). Somehow, numerically, IVPs are easier than BVPs.

3.1 Töpfer algorithm

Let us list the steps necessary to solve the Blasius problem by the Töpfer algorithm. In this way, we define a non-iterative (I)TM. We have to:

1. solve the auxiliary IVP

$$\begin{aligned} \frac{d^3 f^*}{d\eta^{*3}} + \frac{1}{2} f^* \frac{d^2 f^*}{d\eta^{*2}} &= 0 \\ f^*(0) = \frac{df^*}{d\eta^*}(0) &= 0, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1 \end{aligned} \tag{17}$$

and, in particular, get an approximation for $\frac{df^*}{d\eta^*}(\infty)$;

2. compute λ by equation (16);
3. obtain $f(\eta)$, $\frac{df}{d\eta}(\eta)$, and $\frac{d^2 f}{d\eta^2}(\eta)$ by the inverse transformation of (12).

Indeed, Töpfer solved the IVP for the Blasius equation once. At large but finite η_j^* , ordered so that $\eta_j^* < \eta_{j+1}^*$, we can compute by equation (16) the corresponding λ_j . If two subsequent values of λ_j agree within a specified accuracy, then λ is approximately equal to the common value of the λ_j , otherwise, we can march to a larger value of η and try again. Using the classical fourth order Runge-Kutta method and a grid step $\Delta\eta^* = 0.1$ Töpfer was able to determine λ with an error less than 10^{-5} . He used the two truncated boundaries $\eta_1^* = 4$ and $\eta_2^* = 6$. We notice that for $\eta^* \in [0, 4]$ we get $\lambda_1 = 0.333233336$. In figure 2 we plot the more accurate numerical solution obtained by Töpfer's algorithm defined above. We notice that this figure shows the solutions of the auxiliary IVP (17) and of the BVP (5).

4 Rubel error analysis

The boundary condition at infinity is certainly not suitable for a numerical treatment. This condition has usually been replaced by the same condition applied at a truncated boundary, see Collatz [7, pp 150-151] or Fox [18, p. 92]. In the truncated boundary formulation $f_M(\eta)$ is defined by

$$\begin{aligned} \frac{d^3 f_M}{d\eta^3} + f_M \frac{d^2 f_M}{d\eta^2} &= 0 \\ f_M(0) = \frac{df_M}{d\eta}(0) &= 0, \quad \frac{df_M}{d\eta}(M) = 1 \end{aligned} \tag{18}$$

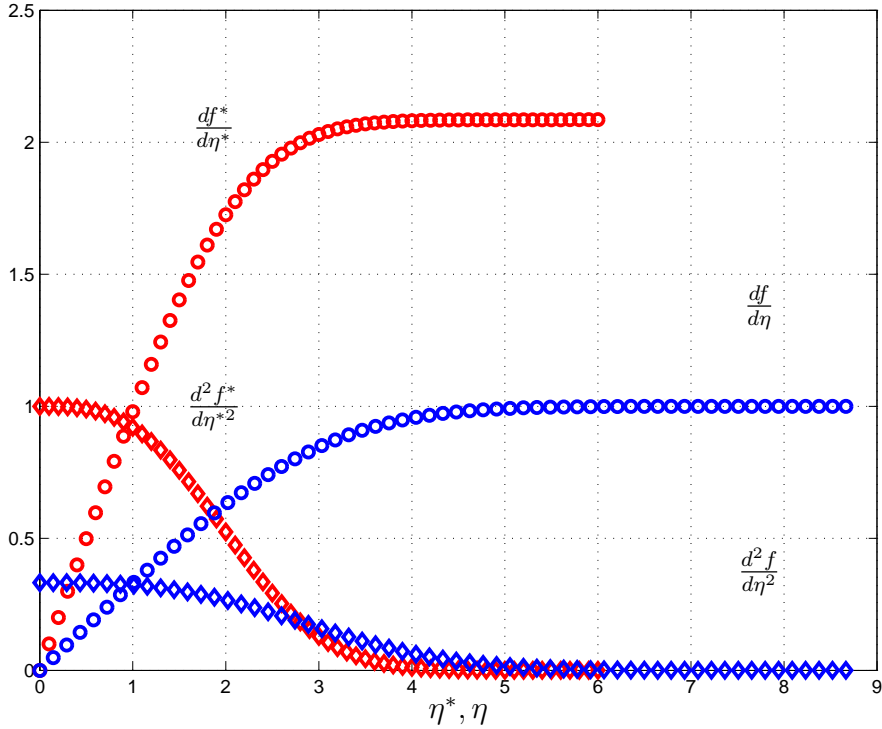


Figure 2: Blasius solution by a non-ITM for $\eta^* \in [0, 6]$ we get $\lambda_2 = 0.332057687$.

where M represents the truncated boundary. It is evident that also in (18) the governing DE and the two boundary conditions at the origin are left invariant by the scaling transformation (12).

For the error related to the truncated boundary solution $f_M(\eta)$ defined by

$$e(\eta) = |f(\eta) - f_M(\eta)|, \quad \eta \in [0, M], \quad (19)$$

the following theorem holds true.

Teorema 1 (due to Rubel [27]). *A truncated boundary formulation of the Blasius problem introduces an error which verifies the following inequality*

$$e(\eta) \leq M \frac{d^2 f_M}{d\eta^2}(M) [f_M(M)]^{-1} . \quad (20)$$

Outline of the proof. As proved by Weyl [35], it is true that

$$\frac{d^2 f}{d\eta^2}(\eta) > 0, \quad \text{for } \eta \in (0, \infty) . \quad (21)$$

By (21) and taking into account the boundary conditions in (5), we have that

$$\frac{df}{d\eta}(\eta) \text{ and } f(\eta) \text{ are increasing functions on } \eta \in (0, \infty) \quad . \quad (22)$$

As a consequence the function $\lambda^2 \frac{df}{d\eta}(\lambda M)$ is zero for $\lambda = 0$, increases with λ and tends to infinity as $\lambda \rightarrow \infty$. For some value of $\lambda \in (0, \infty)$ we must have

$$\lambda^2 \frac{df}{d\eta}(\lambda M) = 1 \quad . \quad (23)$$

This value verifies $\lambda > 1$ because λM is a finite value, $\frac{df}{d\eta}(\eta)$ is an increasing function and $\frac{df}{d\eta}(\eta) \rightarrow 1$ as $\eta \rightarrow \infty$. For this particular value of λ , due to the scaling properties, we have found that

$$f_M(\eta) = \lambda f(\lambda \eta) \quad (24)$$

because $\lambda f(\lambda \eta)$ verifies the BVP (18) that defines $f_M(\eta)$ uniquely.

Hence, the error for $\eta \in [0, M]$ is given by

$$e(\eta) = |\lambda f(\lambda \eta) - f(\eta)| \leq |(\lambda - 1)f(\lambda \eta)| + |f(\lambda \eta) - f(\eta)| \quad . \quad (25)$$

By applying the mean value theorem of differential calculus and taking into account that $\frac{df}{d\eta}(\eta) \leq 1$ we get the relations $f(\lambda \eta) \leq \lambda \eta$ and $|f(\lambda \eta) - f(\eta)| \leq (\lambda - 1)\eta$. As a result

$$e(\eta) \leq M(\lambda^2 - 1), \quad \eta \in [0, M] \quad (26)$$

where $\lambda^2 - 1 > 0$ because $\lambda > 1$. Naturally, $\frac{df_M}{d\eta}(\eta \rightarrow \infty) = \lambda^2$, so that

$$\begin{aligned} \lambda^2 - 1 &= \frac{df_M}{d\eta}(\eta \rightarrow \infty) - \frac{df_M}{d\eta}(M) \\ &= \int_M^\infty \frac{d^2 f_M}{d\eta^2} d\eta \quad . \end{aligned}$$

To complete the proof Rubel used some manipulations, involving a first integral of the governing differential equation, to find that

$$\lambda^2 - 1 \leq \frac{d^2 f_M}{d\eta^2}(M)[f_M(M)]^{-1} \quad . \quad (27)$$

□

Remark. As a consequence of this theorem in order to control the error we can modify either the value of M or the value of $\frac{d^2 f_M}{d\eta^2}(M)$. Classically the value of M has been chosen to this end. The above Theorem shows that the error is directly proportional to M . In this context Fazio defines a free boundary formulation of the Blasius problem where the second order derivative of the solution with respect to η at the free boundary can be chosen as small as possible, see [12] for details. The free boundary can be interpreted as a truncated boundary, see Fazio [15].

5 Similarity analysis

Motivated by several problems in boundary layer theory, let us consider the class of BVPs defined by

$$\begin{aligned} \frac{d^3 f}{d\eta^3} &= \phi \left(\eta, f, \frac{df}{d\eta}, \frac{d^2 f}{d\eta^2} \right) \\ f(0) &= a \quad \frac{df}{d\eta}(0) = b + c \frac{d^2 f}{d\eta^2}(0), \quad \frac{df}{d\eta}(\eta) \rightarrow d \quad \text{as } \eta \rightarrow \infty, \end{aligned} \quad (28)$$

where a , b , c and d are given constants, with $d \neq 0$. Introducing the scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^\delta \eta, \quad (29)$$

we require the invariance of (28), but the asymptotic boundary condition so that $\delta \neq 1$, with respect to (29). The requested invariance is granted on condition that $a = b = c = 0$ and

$$\phi \left(\eta, f, \frac{df}{d\eta}, \frac{d^2 f}{d\eta^2} \right) = \eta^{1-3\delta} \Phi \left(\eta^{1/\delta} f, \eta^{(1-\delta)/\delta} \frac{df}{d\eta}, \eta^{(1-2\delta)/\delta} \frac{d^2 f}{d\eta^2} \right). \quad (30)$$

As a consequence of the above scaling invariance we can define a non-ITM.

5.1 The non-iterative algorithm

In order to define the numerical method for the characterized class of problems we have to consider the auxiliary IVP

$$\begin{aligned} \frac{d^3 f^*}{d\eta^{*3}} &= \eta^{*(1-3\delta)} \Phi \left(\eta^{*(1/\delta)} f^*, \eta^{*(1-\delta)/\delta} \frac{df^*}{d\eta^*}, \eta^{*(1-2\delta)/\delta} \frac{d^2 f^*}{d\eta^{*2}} \right) \\ f^*(0) &= \frac{df^*}{d\eta^*}(0) = 0, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = p. \end{aligned} \quad (31)$$

where p is defined by the user, we usually set $p = \pm 1$, but it is also possible to consider different values. For instance, in Fazio [12] for the Blasius problem we used $p = 1000$. We have to solve (31) on $[0, \eta_\infty^*]$, where η_∞^* is a suitable truncated boundary chosen under the condition

$$\frac{df^*}{d\eta^*}(\eta_\infty^*) \approx \frac{df^*}{d\eta^*}(\infty) . \quad (32)$$

As $d \neq 0$, we have

$$\lambda = \left[\frac{\frac{df^*}{d\eta^*}(\eta_\infty^*)}{d} \right]^{1/(1-\delta)} . \quad (33)$$

Computed the value of λ we can apply the inverse transformation of (29) to get

$$\begin{aligned} \eta &= \lambda^{-\delta} \eta^* , & f(\eta) &= \lambda^{-1} f^*(\eta^*) , \\ \frac{df}{d\eta}(\eta) &= \lambda^{\delta-1} \frac{df^*}{d\eta^*}(\eta^*) , & \frac{d^2 f}{d\eta^2}(\eta) &= \lambda^{2\delta-1} \frac{d^2 f^*}{d\eta^{*2}}(\eta^*) . \end{aligned} \quad (34)$$

In particular, we are interested to compute the missing initial condition $\frac{d^2 f}{d\eta^2}(0)$.

We are now ready to present the method of solution in the form of an algorithm.

The non-iterative algorithm.

1. Input $p, \delta, \eta_\infty^*, d$.
2. Solve (31) in $[0, \eta_\infty^*]$.
3. Compute λ by (33).
4. Rescale the numerical solution according to (34).

The above algorithm defines a non-ITM for the numerical solution of the class of problems characterized by (30) and $a = b = c = 0$.

6 The extension due to Na

We report here an important extension of the non-ITM due to Na. For this extension we have to require the invariance of physical parameters. The boundary conditions at $\eta = 0$ involve the parameters a, b and c . Those boundary conditions are left invariant by the scaling transformation on condition that the involved parameters transform as

$$a^* = \lambda a , \quad b^* = \lambda^{1-\delta} b , \quad c^* = \lambda^\delta c . \quad (35)$$

6.1 Moving wall

According to Ishak et al. [21] the moving wall boundary conditions are given by

$$f(0) = 0, \quad \frac{df}{d\eta}(0) = b, \quad \frac{df}{d\eta}(\eta) \rightarrow d \quad \text{as } \eta \rightarrow \infty, \quad (36)$$

where $d = 1 - b$ and b is a non-dimensional parameter given by the ration of the wall to the flow velocities. The boundary conditions of the Blasius problem are recovered from (36) by setting $b = 0$.

6.1.1 The non-ITM

The applicability of a non-ITM to the Blasius problem is a consequence of its invariance with respect to a scaling transformation; note that the asymptotic boundary condition is not invariant. In order to apply the non-ITM we consider b as a parameter involved in the scaling invariance, i.e., we define the extended scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^{-1} \eta, \quad b^* = \lambda^2 b. \quad (37)$$

By setting a value of P^* , we can integrate the Blasius equation (5) in the star variables with initial conditions

$$f^*(0) = 0, \quad \frac{df^*}{d\eta^*}(0) = b^*, \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = \pm 1, \quad (38)$$

in order to compute an approximation $\frac{df^*}{d\eta^*}(\eta_\infty)$ for $\frac{df^*}{d\eta^*}(\infty)$ and the corresponding value of λ according to the equation

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta_\infty) + b^* \right]^{1/2}. \quad (39)$$

Once the value of λ is computed by equation (39), then we can find the missed initial condition by the equation

$$\frac{d^2 f}{d\eta^2}(0) = \lambda^{-3} \frac{d^2 f^*}{d\eta^{*2}}(0), \quad (40)$$

For the application of the method defined above, we remark that the plus (for $b < 0.5$) or minus (when $b > 0.5$) sign must be used for the second derivative in (38). Moreover, the computation of a value at infinity is unsuitable from a numerical viewpoint and therefore we use a truncated boundary η_∞^* instead of infinity. In figure 3 we plot $\frac{d^2 f}{d\eta^2}(0)$ versus b . From this figure we realize that our problem has an unique solution for $b \geq 0$, while two solutions exist for $b < 0$.

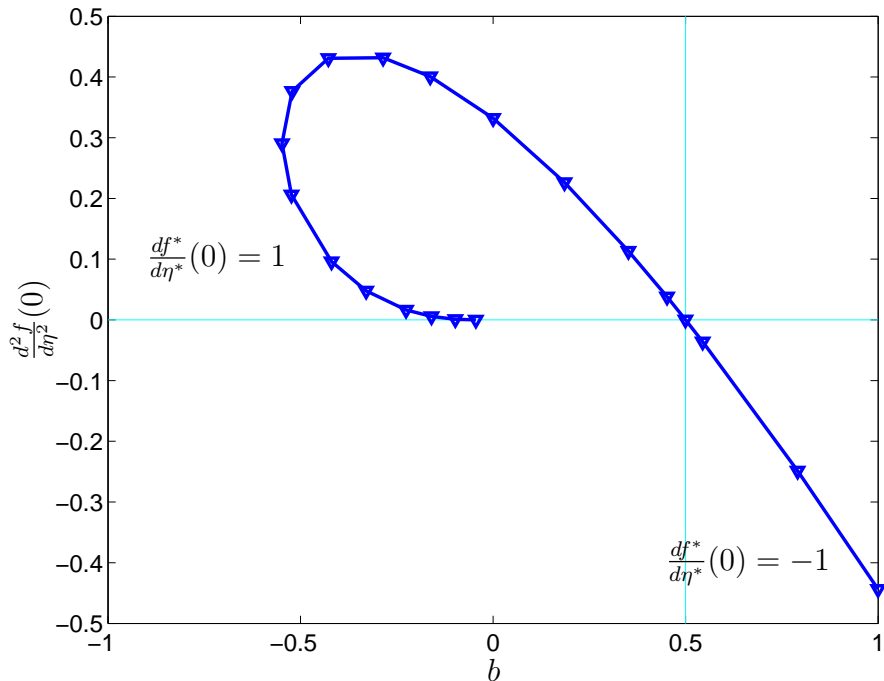


Figure 3: Blasius problem with moving wall boundary conditions. Non-ITM: missing initial condition versus b .

The values of the skin friction coefficient are positive for $b < 1/2$ and they become negative when $b > 1/2$. From a physical viewpoint, this means that a drag force is exerted by the flow on the plate in the first case, while in the second the force is of opposite type. Figure 3 also shows that the solutions exist until a critical, negative, value of the parameter, say b_c , is reached. The boundary layer separate from the surface at $b = b_c$, and, therefore, for smaller values of b the Navier-Stoker equations has to be solved because the hypotheses of boundary layer theory felt down. We have separation for a positive value of the skin friction coefficient and not at the point where this coefficient vanish as in the classical boundary layer theory. The zero value of the skin friction coefficient when $b = 1/2$ corresponds to equal velocity of the plate and the free stream and does not mark separation.

From the data in table 1 we get $b_c \approx -0.548210$. This value is in good agreement with the value $b_c = -0.5483$ computed by Ishak et al. [21] using an iterative method: the second order Keller's Box finite difference method. As mentioned before, the case $b = 0$ is the Blasius problem (5). In this case our non-ITM becomes the original method defined by Töpfer [33]. The obtained skin friction coefficient is in good agreement with the values available in literature, see for

$\frac{d^2 f^*}{d\eta^{*2}}(0)$	b^*	$\frac{df^*}{d\eta^*}(\infty)$	$\frac{d^2 f}{d\eta^2}(0)$	b
1	-500.	$1.55 \cdot 10^4$	$5.46 \cdot 10^{-7}$	-0.033393
	-100.	$2.34 \cdot 10^3$	$9.42 \cdot 10^{-6}$	-0.044591
	-5.	36.325698	0.005704	-0.159613
	-1.	2.917762	0.376537	-0.521441
	0.	2.085393	0.332061	0.
	1.	2.440648	0.156689	0.290643
	5.	5.771518	0.028287	0.464187
	100.	$1.00 \cdot 10^2$	$3.53 \cdot 10^{-4}$	0.499557
	500.	$5.00 \cdot 10^2$	$3.16 \cdot 10^{-5}$	0.499960
	-1	100.	99.822681	$-3.54 \cdot 10^{-4}$
10.		9.433763	-0.011673	0.514568
5.		4.182424	-0.035939	0.544519
2.		0.528464	-0.248722	0.790994
1.719		$-4.73 \cdot 10^{-5}$	-0.443715	1.000027

Table 1: Moving wall boundary condition: non-ITM numerical results.

instance Fazio [12] or Boyd [5]. The values reported in the last line are related to the Sakiadis problem [28, 29] and were found by a few trial and miss attempts. The obtained skin friction coefficient is in good agreement with the values reported in literature, e.g.: Sakiadis [28], Ishak et al. [21], Cortell [3] and Fazio [17] with an iterative TM .

Figure 4 shows the solution of the Sakiadis problem, describing the behaviour of a boundary layer flow due to a moving flat surface immersed in an otherwise quiescent fluid, corresponding to $b = 1$. Actually, this is a case of practical interest if we are considering the plate as an idealization of an airplane wing.

6.2 Slip flow condition

We consider now the case of a rarefied flow where the no-slip condition at the wall, considered in the previous section, must be replaced by a slip-flow condition, see for instance Gad-el-Hak [8]. For an isothermal wall, the slip condition can be defined as

$$v(y, 0) = \frac{2 - \sigma}{\sigma} \ell \frac{\partial v}{\partial z}(y, 0) , \quad (41)$$

where ℓ is the mean free path, and σ is the tangential momentum accommodation coefficient. Within a similarity transformation this

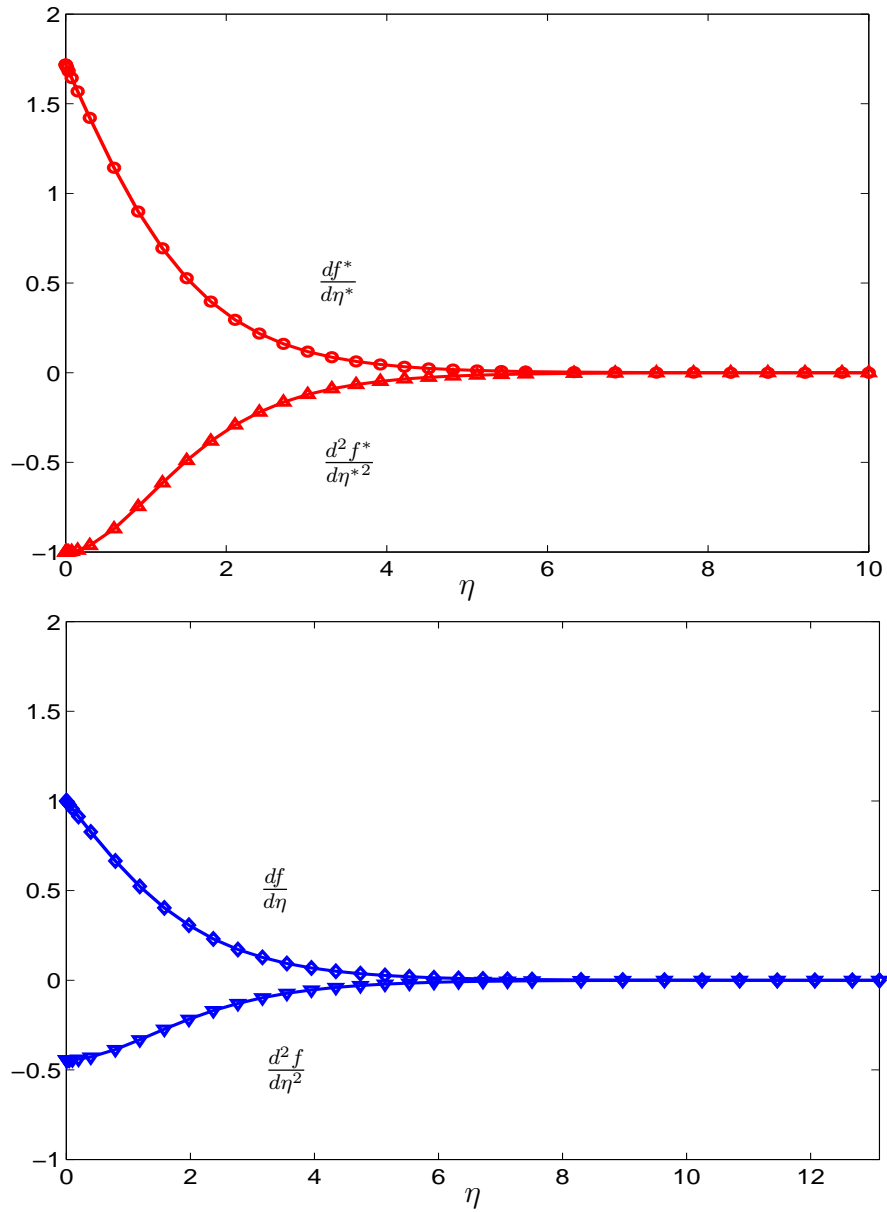


Figure 4: Numerical results of the non-ITM. Top frame: solution of the IVP; bottom frame: solution of the Sakiadis problem found after rescaling.

slip boundary condition becomes

$$\frac{df}{d\eta}(0) = c \frac{d^2 f}{d\eta^2}(0) , \quad (42)$$

where c is a non-dimensional parameter, that takes into account the behaviour at the surface, defined by

$$c = \frac{2 - \sigma}{\sigma} Kn Re y^{1/2} , \quad (43)$$

where Kn and Re are the Knudsen and Reynolds numbers based on y .

For the Blasius problem with slip condition we implemented the extended non-ITM.

6.2.1 The non-ITM

In order to apply the non-ITM we consider c as a parameter involved in the scaling invariance, i.e., we defined the extended scaling group

$$f^* = \lambda f , \quad \eta^* = \lambda^{-1} \eta , \quad c^* = \lambda^{-1} c . \quad (44)$$

Henceforth, λ is defined, once again, by equation (16).

Sample numerical results are reported on table 2. As mentioned

c^*	$\frac{df^*}{d\eta^*}(\infty)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	c
0.	2.085393	0.	0.332061	0.
0.1	2.090453	0.047836	0.330856	0.144584
0.5	2.191907	0.228112	0.308153	0.740255
1.	2.440648	0.409727	0.262266	1.562257
5.	5.771518	0.866323	0.072122	12.011992
10.	10.554805	0.947436	0.029162	32.488159
15.	15.455238	0.970545	0.016458	3.815517
20.	20.394883	0.980638	0.010857	90.321389
25.	25.353618	0.986053	0.007833	125.880941

Table 2: Slip boundary condition: non-iterative numerical results.

before, the case $c = 0$ is the classical Blasius problem (5). It is possible to compare the results listed in the last two columns of table 2 with similar results, obtained via a shooting method, shown in figure 1 of the proceedings report by Martin and Boyd [23]. It is clear that our non-ITM would be faster and easier to implement than any iterative algorithm.

As far as the non-ITM is concerned, we set a value of c^* and get the numerical solution of the problem for a different value of c . As an example, figure 5 shows a sample numerical integration for $c = 1.562257$ obtained by fixing $c^* = 1$. Note that the solution of the Blasius problem with slip boundary condition was computed by rescaling. If we need the solution for a specific value of c , then we can

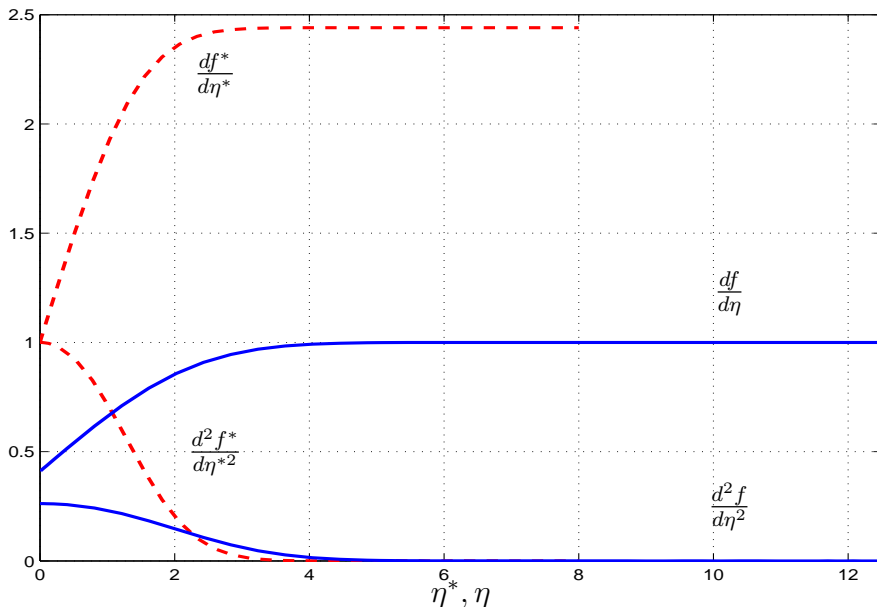


Figure 5: Blasius problem with slip condition. Numerical solution by a non-ITM with $c^* = 1$ and $c = 1.562257$.

apply interpolation techniques to the results of table 2.

In figure 6 we plot $\frac{df}{d\eta}(0)$ and $\frac{d^2f}{d\eta^2}(0)$ versus c . It is easily seen that as c goes to infinity then $\frac{df}{d\eta}(0)$ goes to one while $\frac{d^2f}{d\eta^2}(0)$ tends to zero.

7 Surface gasification boundary condition

In this section we consider a problem that does not belong to the framework used so far. This is the surface gasification flow that, with and without subsequent gas-phase flame-sheet burning, was first formulated and analysed by Emmons [9]. For such a flow we have to

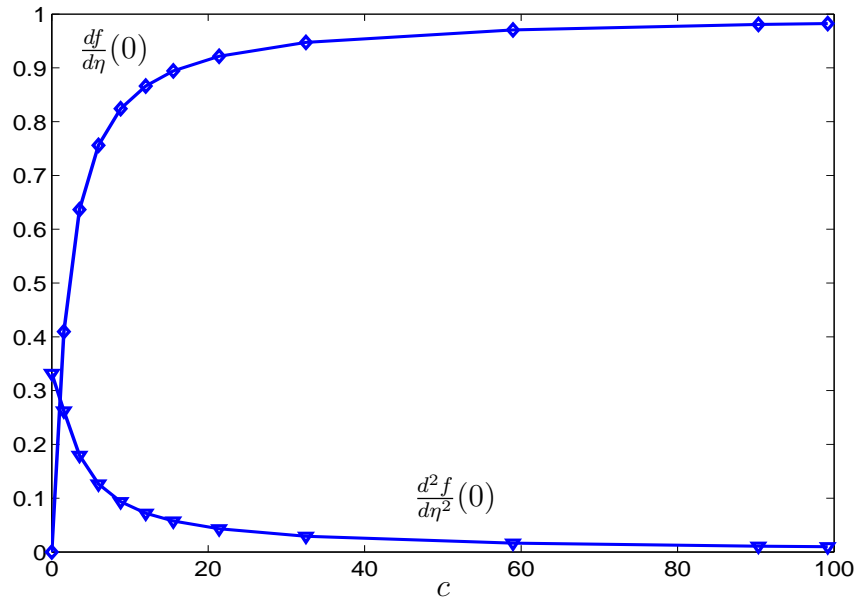


Figure 6: Blasius problem with slip boundary condition. Non-ITM: behaviour of $\frac{df}{d\eta}(0)$ and $\frac{d^2f}{d\eta^2}(0)$ with respect to c .

consider the variant of the celebrated Blasius problem

$$\frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} = 0 \tag{45}$$

$$f(0) = -s \frac{d^2 f}{d\eta^2}(0), \quad \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty,$$

where s is the classical Spalding heat transfer number [31]. This transfer number for slow vaporization belongs to the interval $[0, 0.1]$ and varies from $s = O(1)$ to $s \approx 20$ for strong burning. This problem has been studied recently by Lu and Law [22]. These authors define an iterative method that has been shown to produce more accurate numerical results than the classical approximate solutions.

7.1 The non-ITM

In the present case we consider the extended scaling group

$$f^* = \lambda f, \quad \eta^* = \lambda^{-1} \eta, \quad s^* = \lambda^{-2} s. \tag{46}$$

Let us notice that the governing differential equation and the two boundary conditions at $\eta = 0$ in (45) are left invariant under (46) and,

on the contrary, the asymptotic boundary condition is not invariant. By setting a value of s^* , we can integrate the Blasius governing differential equation in (45) in the star variables on $[0, \eta_\infty^*]$ with initial conditions

$$f^*(0) = -s^* , \quad \frac{df^*}{d\eta^*}(0) = 0 , \quad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1 , \quad (47)$$

in order to compute $\frac{df^*}{d\eta^*}(\eta_\infty^*) \approx \frac{df^*}{d\eta^*}(\infty)$. Here η_∞^* is a suitable truncated boundary. The value of λ can be found by

$$\lambda = \left[\frac{df^*}{d\eta^*}(\eta_\infty^*) \right]^{1/2} . \quad (48)$$

After using (48) to get the value of λ , we can apply the scaling invariance to obtain the missing initial conditions

$$f(0) = \lambda^{-2} s^* , \quad \frac{d^2 f}{d\eta^2}(0) = \lambda^{-3} . \quad (49)$$

For the reader convenience, in table 3 we list sample numerical results. The case $s^* = s = 0$ is, again, the Blasius problem (45).

s^*	$\frac{df^*}{d\eta^*}(\infty)$	$-f(0)$	$\frac{d^2 f}{d\eta^2}(0)$	s
0.	1.655301	0.	0.469553	0.
0.25	2.025902	0.175643	0.346795	0.506476
0.5	2.485809	0.317129	0.255152	1.242904
0.75	3.048481	0.429556	0.187877	2.286361
1.	3.726397	0.518031	0.139016	3.726397
1.25	4.528469	0.587401	0.103770	5.660586
1.5	5.469166	0.641403	0.078184	8.203749
1.75	6.548781	0.683845	0.059670	11.460366
2.	7.779561	0.717055	0.046086	15.559122

Table 3: Surface gasification boundary condition: non-ITM results.

In this case our non-ITM reduces to the original method defined by Töpfer [33]. The obtained skin friction coefficient is in good agreement with the values available in literature, see for instance the value 0.46599988361 computed by Fazio [12]. On the other hand, our value is different from the value 0.490 obtained by a 3 – 2 iteration solution of Lu and Law [22]. For the numerical results reported here, depending on the behaviour of the numerical solution, we have used $\eta_\infty^* = 5$ or $\eta_\infty^* = 10$.

Figure 7 shows a sample numerical integration for $s^* = 1$ that is transformed under (46) to $s \approx 3.726397$. We notice that the solution of the Blasius problem with surface gasification boundary condition is computed by rescaling. Moreover, by rescaling we get $\eta_\infty^* < \eta_\infty$.

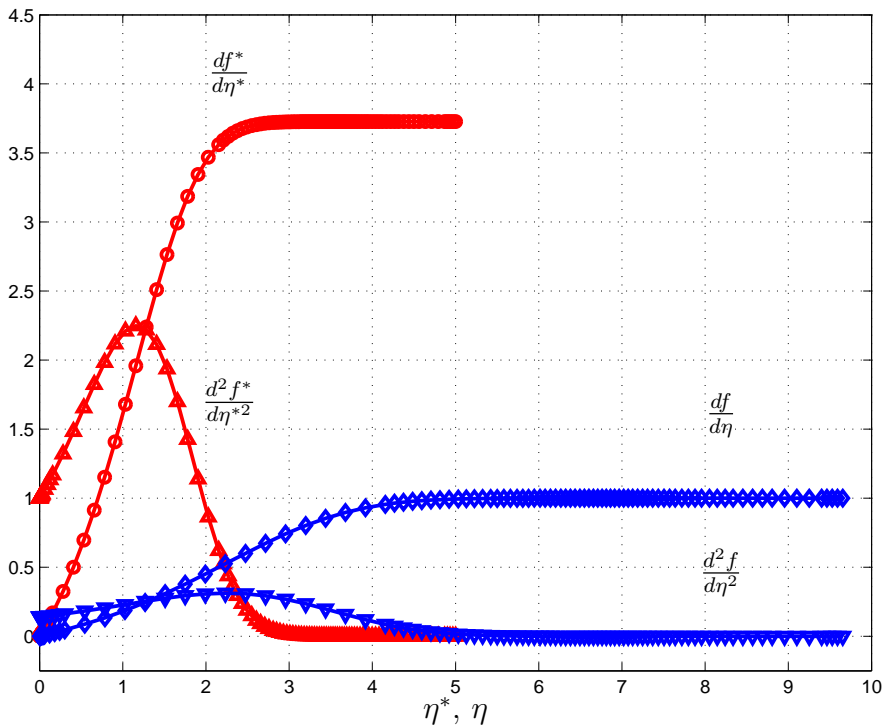


Figure 7: Surface gasification boundary conditions with $s^* = 1$. Numerical solution by the non-ITM.

In figure 8 we plot $f(0)$ and $\frac{d^2 f}{d\eta^2}(0)$ versus s . We know, from the literature, that as s goes to infinity then $f(0)$ goes to -0.876 . Moreover, as it is easily seen, as s goes to infinity then $\frac{d^2 f}{d\eta^2}(0)$ goes to zero.

8 Concluding remarks

The main contribution of this paper is the extension of the non-ITM proposed by Töpfer [33] for the numerical solution of the celebrated Blasius problem [4] to classes of problems depending on a parameter. By requiring the invariance of the involved parameter we are able to solve the given BVP non-iteratively but for a different value of the parameter. This kind of extension was considered first by Na [24],

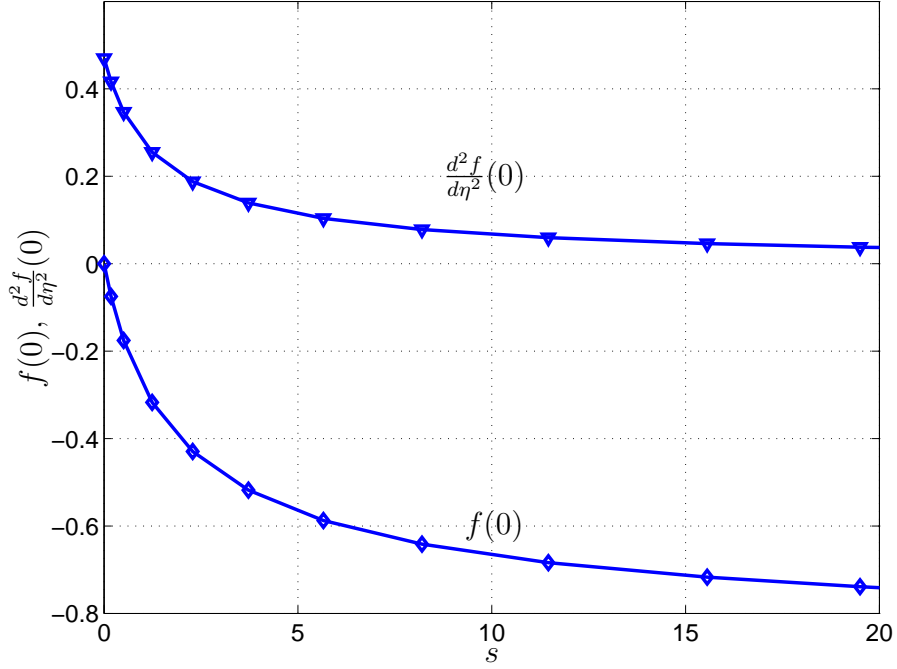


Figure 8: Blasius problem with surface gasification boundary conditions. Non-ITM: behaviour of $f(0)$ and $\frac{d^2 f}{d\eta^2}(0)$ with respect to s .

see also NA [25, Chapters 8-9]. Here we defined a non-ITM for Blasius equation with moving wall, surface gasification or slip boundary conditions.

Finally, the reader should be advised that non-ITM cannot be applied to all problems of boundary layer theory. As an example, let us consider the Falkner-Skan model [11]

$$\frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} + P \left[1 - \left(\frac{df}{d\eta} \right)^2 \right] = 0, \quad (50)$$

$$f(0) = \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty,$$

where f and η are similarity variables and P is a parameter related to the functional form of the fluid mainstream velocity. If we test the invariance of the governing differential equation in (50) under the extended scaling group

$$\eta^* = \lambda^{\alpha_1} \eta, \quad f^* = \lambda^{\alpha_2} f, \quad P^* = \lambda^{\alpha_3} P, \quad (51)$$

where λ is, again, the group parameter and α_j , for $j = 1, 2, 3$, are

constant to be determined, then we get three invariant conditions

$$\alpha_2 - 3\alpha_1 = 2(\alpha_2 - \alpha_1) = \alpha_3 = \alpha_3 + 2(\alpha_2 - \alpha_1) . \quad (52)$$

Now, it is a simple matter to show that the linear system defined by (52) has the unique solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. However, as a last word on this topic we can mention an iterative extension of our transformation method that has been developed in [13, 14] and successfully applied to the Falkner-Skan model [13, 16].

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