Second Order Positive Schemes by means of Flux Limiters for the Advection Equation

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Abstract—**In this paper, we study first and second order positive numerical methods for the advection equation. In particular, we consider the direct discretization of the model problem and comment on its superiority to the so called method of lines. Moreover, we investigate the accuracy, stability and positivity properties of the direct discretization. The numerical results related to several test problems are reported.**

Keywords: ***advection equation, positive numerical methods, flux limiters, stability analysis, accuracy and order of convergence.***

1 Introduction

The aim of this study is to present first and second order positive numerical methods for the advection equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c) = 0 \tag{1}$$

where $c = c(\mathbf{x}, t)$ with $c \in \mathbb{R}$, $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ and $t \in \mathbb{R}^+$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$; $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^3$ is supposed to be given. This is a time-dependent partial differential equation in three spatial dimensions (3D) used in the applied sciences to describe several problems of great interest, see, for instance, the recent book by LeVeque [10]. Different methods have been developed in order to solve (1) numerically. Together with computational efficiency, another important property that should be possessed by an advection scheme is preservation of positivity, to avoid instabilities in the numerical solution. As point out by Liu and Lax [11], in the modern numerical treatment of conservation laws, the positivity is a key requirement. According to the second Godunov's barrier, second order numerical methods are not positive. We consider two different classes of schemes for linear advection equation, the first one based on direct discretization and another based on method of lines (MOL).

This study is devoted to describe the development of second order direct discretization implemented with flux limiter functions. These methods are at least second order accurate on smooth solutions and yet give well resolved, non-oscillatory discontinuities. We have here an *epitome* of modern numerical analysis: in order to be able to solve accurately a linear hyperbolic problem we have to apply nonlinear schemes involving the limiter functions. Several specific numerical tests are reported in order to show the behaviour of the considered direct discretization methods. A preliminary version of this study was presented at the 2007 WASCOM conference [3].

2 Numerical methods

In this section we consider numerical methods for solving scalar advection equation in three space dimensions

$$c_t + \sum_{s=1}^{3} (u_s c)_{x_s} = 0 \tag{2}$$

with given initial condition and appropriate boundary conditions (for instance: Dirichlet conditions at the inflow and no conditions at the outflow boundaries, or pe-

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riodic boundary conditions, etc.). Here and in the sequel the subscripts indicate partial derivatives with respect to the indicated variables, and the velocity field (u_1, u_2, u_3) may depend on the independent variables x_s. We notice that in many applications the velocity field can be taken as divergence-free, that is $\sum_{s=1}^{3}(u_s)_{x_s} = 0$.

Set a uniform Cartesian grid $\Omega_J \subseteq \Omega \subseteq \mathbb{R}^3$, where $J = (j_1, j_2, j_3)^T$ is a lattice of points in which all j_s are integers. The grid points are $x_J = (j_1\Delta x_1, j_2\Delta x_2, j_3\Delta x_3)^T$, where Δx_s are fixed step-sizes. Let C_J be an approximation to the value of the solution $c(x_J, t)$ at current time t, and $C_J^{\Delta t}$ an approximation to the value of the solution $c(x_J, t+\Delta t)$ at time $t+\Delta t$. The velocity j_s-components, for $s = 1, 2, 3$, are centered at the right, back and top face of the Cartesian cell, respectively, whereas the approximation C_J are located at its center. This is the so-called MAC (marker-and-cell) method, see Harlow and Welsh [5].

Being the governing equation linear, it would be appropriate to consider numerical schemes in the linear form

$$C_J^{\Delta t} = \sum_K \gamma_{JK} C_{J+K} \ . \tag{3}$$

Friedrichs [4] proposed to require that the coefficients in equation (3) should verify the following conditions:

(a) $\gamma_{JK} \geq 0$ for each coefficient;

(b) $\sum_K \gamma_{JK} = 1$;

(c) $\gamma_{JK} = 0$ except for a finite set of K in a neighbourhood of J;

(d) γ_{JK} depends Lipschitz continuously on $\mathbf{x}$.

Conditions (a) and (b) imply that the solution at future $C_J^{\Delta t}$ is a convex combination of the values of solution at current step C_J, and this leads to a local maximum-minimum principle. That is, the solution $C_J^{\Delta t}$ is bounded from above and below by the solution C_J locally. The condition (c) is a discrete consequence of the finite propagation speed of waves for the advection equation. Under the reported conditions, Friedrichs has shown that numerical approximations to solution that depend Lipschitz continuously on the space variables and that have positive coefficients are bounded under the discrete l^2 norm of the numerical solution, that is, they verify a bounded growth property:

$$\|\mathbf{C}^{\Delta t}\| \leq (1 + M\ \Delta)\|\mathbf{C}\| \tag{4}$$

where $\|\cdot\|$ is the discrete l^2 norm defined by

$$\|\mathbf{C}\| = \left[\sum_J (C_J, C_J)\right]^{1/2} ,$$

where $\mathbf{C}$, $\mathbf{C}^{\Delta t}$ are the numerical solution vectors at time t and $t + \Delta t$ respectively, $\Delta = \min\{\Delta t, \Delta x_1, \Delta x_2, \Delta x_3\}$, and the value of the constant M depends on the Lipschitz constant. This suggests positivity as a design principle for solving system of conservation laws in more than one space variables. The schemes (3) verifying the property (a) are called *positive* in the sense of Friedrichs.

2.1 1D Advection

We start considering the 1D advection equation

$$c_t + uc_x = 0 \ , \tag{5}$$

where u is constant. A numerical approximation can be obtained by considering a direct discretization. In the case $u \geq 0$, for instance, within the finite difference approach and a five-point stencil, we can consider the α-scheme defined by

$$\begin{aligned} C_j^{\Delta t} = {} & \gamma_{-2}C_{j-2} + \gamma_{-1}C_{j-1} \\ & + \gamma_0 C_j + \gamma_1 C_{j+1} + \gamma_2 C_{j+2} \ , \end{aligned} \tag{6}$$

with coefficients γ, depending on the Courant number $\nu = u\Delta t/\Delta x$, given as follows

$$\begin{aligned} \gamma_{-2} &= -\frac{1}{2}\nu\,(1-\nu)\,\alpha \ , \\ \gamma_{-1} &= \nu\left[1 + \frac{1}{2}(1-\nu)(3\alpha - 1)\right] \ , \\ \gamma_0 &= 1 - \nu\left[1 + \frac{1}{2}(1-\nu)(3\alpha - 2)\right] \ , \\ \gamma_1 &= -\frac{1}{2}\nu(1-\nu)(1-\alpha) \ , \\ \gamma_2 &= 0 \ . \end{aligned}$$

From this scheme we recover some classical second order schemes for particular choices of α: central (Lax–Wendroff $\alpha = 0$), Fromm ($\alpha = 1/2$), and upwind (Beam–Warming for $\alpha = 1$). The α-scheme has a local truncation error given by

$$\frac{1}{6}\, u\, \Delta x^2 \left[3\alpha - 1 - 3\alpha\, \nu + \nu^2\right] c_{xxx} + O\left(\Delta t^3\right) \ . \qquad (7)$$

Moreover, the α-scheme is stable for $\nu \leq 1$ and computes the advection (exact) solution by setting $\nu = 1$. Note that, the α-scheme verifies all the conditions listed before, but the (a), i.e., the positivity condition. This is in agreement with the second Godunov's barrier: a linear positive numerical scheme is at most first order accurate.

The α-scheme can be written in conservation form

$$C_j^{\Delta t} = C_j - \frac{\Delta t}{\Delta x}\left[F_{j+1/2} - F_{j-1/2}\right]$$

where F is a numerical flux function, by setting

$$F_{j+1/2} = u\left\{C_j + \frac{1}{2}(1-\nu)\left[\alpha\left(C_j - C_{j-1}\right) + (1-\alpha)\left(C_{j+1} - C_j\right)\right]\right\} \ . \qquad (8)$$

This is important because by the Lax-Wendroff theorem we know that the shock wave velocity is computed correctly if and only if the scheme is in conservation form.

Now, we describe a different numerical approach: the method of lines or MOL. The main idea in the MOL approach is to introduce only a spatial discretization. As an example, a given five-point discretization formula defines the semi-discrete system

$$\frac{d}{dt}c_j(t) = \frac{u}{\Delta x}\left[\delta_{-2}c_{j-2}(t) + \delta_{-1}c_{j-1}(t) + \delta_0 c_j + \delta_1 c_{j+1}(t) + \delta_2 c_{j+2}(t)\right] \qquad (9)$$

where $c_j(t) \approx c(x_j, t)$ with $x_j = j\Delta x$, and the δ coefficients are given constants. The system (9) can be solved by an ODE scheme to get a numerical approximation $c_j^n \approx c(x_j, t^n)$, with $t^n = n\Delta t$. If we consider the limit as Δt goes to zero, for fixed Δx and t^n, then the approximate solution C_j, at $t = t^n$, of (6) converges to the solution $c_j(t^n)$ of (9). Therefore, for ν going to zero, and fixed Δx, the direct scheme becomes the MOL scheme with an exact time integration. In fact, as an example, when $u \geq 0$, from the α-scheme we can derive its associated MOL semi-discrete system (9) with parameters

$$\delta_{-2} = -\frac{1}{2}\alpha \ , \quad \delta_{-1} = 1 + \frac{1}{2}(3\alpha - 1) \ ,$$
$$\delta_0 = -1 - \frac{1}{2}(3\alpha - 2) \quad \delta_1 = -\frac{1}{2}(1-\alpha) \ ,$$
$$\delta_2 = 0 \ .$$

The MOL approach, that in general is less accurate than the direct discretization, is usually used to solve numerically more complex models. An example is represented by time dependent advection–diffusion–reaction (ADR) models in 3D, used in many applications. Among others we can quote the applications to the: pollutant transport in the atmosphere [16], mucilage dynamics [8], ash-fall from vulcano [12], and groundwater and surface water [15]. Significant applications solved numerically by MOL schemes can be found in the work by Verwer et al. [16]. Moreover, adaptive solver based on the MOL approach have proposed by Blom and co-workers both for 2D [1] and 3D [2] ADR problems.

2.2 Flux limiters

The considered methods satisfy the positive property, at first order accuracy, but not at second order accuracy. First order upwind methods have the advantage of keeping the solution monotonically varying in regions where the solution should be monotone, even though the accuracy is not very good. Second order accurate methods give much better accuracy on smooth solutions than the first order upwind method, but fail near discontinuities, where oscillations may appear due to their dispersive nature. On the other side, the methods implemented with flux limiters, used in order to suppress spurious oscillations, perform much better. The idea of flux limiter is to combine the best features of both methods.

A wide variety of methods of this form has been developed. In general, a flux-limiter method can be obtained combining any low-order flux formula $F_l(C_{j-1}, C_j)$

(such as the upwind flux) and any higher-order formula $F_h(C_{j-1}, C_j)$ (such as the Lax-Wendroff one) to obtain a flux-limiter method with

$$F_{j-1/2} = F_l(C_{j-1}, C_j) + \Phi_{j-1/2}[F_h(C_{j-1}, C_j) - F_l(C_{j-1}, C_j)] \ . \quad (10)$$

If $\Phi_{j-1/2} = 0$, then this reduces to the low-order method, while when $\Phi_{j-1/2} = 1$, we obtain the higher-order one. In particular, a hint of how this can be done is seen by rewriting the Lax-Wendroff flux, obtained by (8) with $\alpha = 0$, as

$$F_{j-1/2} = u\ C_{j-1} + \frac{1}{2}u\left(1 - \frac{\Delta t}{\Delta x}u\right)\Delta C_{j-1/2} \quad (11)$$

with $\Delta C_{j-1/2} = C_j - C_{j-1}$. In this way, the flux assumes the form of the upwind one with a correction term, and then, the Lax-Wendroff method has the following form

$$C_j^{\Delta t} = C_j - u\frac{\Delta t}{\Delta x}(C_j - C_{j-1}) - \frac{1}{2}u\frac{\Delta t}{\Delta x}(\Delta x - u\Delta t)(\Delta C_{j+1/2} - \Delta C_{j-1/2}) \ . \quad (12)$$

Introducing $\delta_{j-1/2}$, a limited version of $\Delta C_{j-1/2}$, we can rewrite (11) in the following way

$$F_{j-1/2} = u\ C_{j-1} + \frac{1}{2}u\left(1 - \frac{\Delta t}{\Delta x}u\right)\delta_{j-1/2} \ . \quad (13)$$

We obtain the flux limiter methods by choosing

$$\delta_{j-1/2} = \phi(\theta_{j-1/2})\Delta C_{j-1/2} \ , \quad (14)$$

where $\phi(\theta)$ is the flux-limiter function, whose value depends on of the solution smoothness, with $\theta = \dfrac{\Delta C_{j-3/2}}{\Delta C_{j-1/2}}$. Setting $\phi(\theta) \equiv 1$ for all θ we obtain the Lax-Wendroff method, while setting $\phi(\theta) \equiv 0$ the upwind one. More generally, we might want to devise a limiter function ϕ that has values near 1 for $\theta \approx 1$, but that reduces (or perhaps increases) the slope where the data is not smooth.

A reasonably large class of flux-limiter has been studied by Sweby [13], who derived algebraic conditions on limiter function which guarantee second order accuracy and positivity. For a more recent discussion on this topic see the paper [9] by LeVeque or his book [10]. In the following, we list the functions $\phi(\theta)$ for some numerical methods.

Linear Methods:

upwind	$\phi(\theta) = 0$
Lax-Wendroff	$\phi(\theta) = 1$
Beam-Warming	$\phi(\theta) = \theta$
Fromm	$\phi(\theta) = \frac{1}{2}(1 + \theta)$

High-resolution Limiters:

superbee	$\phi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$
van Leer	$\phi(\theta) = \dfrac{\theta + \lvert\theta\rvert}{1 + \lvert\theta\rvert}$
minmod	$\phi(\theta) = minmod(1, \theta)$,

where the $minmod(\cdot, \cdot)$ function of two arguments is defined by

$$minmod(a, b) = \begin{cases} a & \text{if} \quad ab > 0 \quad \text{and} \quad |a| < |b| \\ b & \text{if} \quad ab > 0 \quad \text{and} \quad |b| < |a| \\ 0 & \text{if} \quad ab < 0 \ . \end{cases}$$

In the following we report the numerical results related to a simple test problem.

2.3 Test problem

Let us consider the problem

$$\begin{aligned} & c_t + uc_x = 0 \ , \\ & c(x, 0) = \begin{cases} 1 & \text{if } 0 \leq x \leq 0.2 \\ 0 & \text{otherwise} \end{cases} \\ & c(0, t) = c(1, t) \ , \end{aligned} \quad (15)$$

where $0 \leq x \leq 1$, $u = 1$ and $t \geq 0$. The initial datum is a unitary square wave; at the left boundary we impose periodic boundary conditions in order to have the possibility to consider long time integrations. To provide a strict comparison of the obtained numerical results, we choose not to show the computed surfaces that approximate $c(x, t)$, but to provide a frame comparison at the fixed time $t = 0.5$. Figure 1 provides such a comparison. For the upwind method we get a dumped approximation when $\nu < 1$, whereas if $\nu > 1$ we obtain

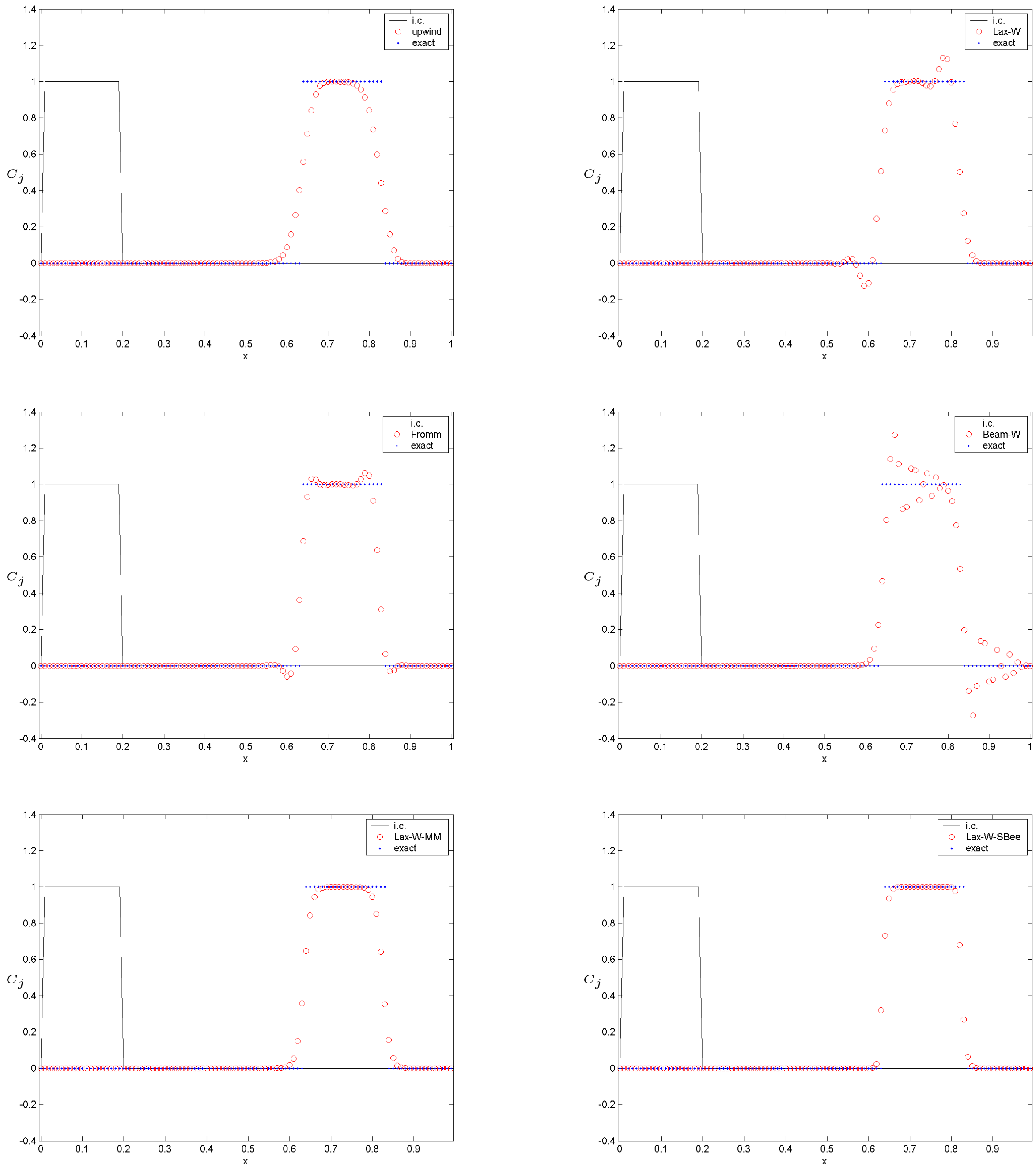


Figure 1: Square initial condition: numerical solutions at $t = 0.5$ with 100 mesh-cells and $\nu = .9$. Top-left: upwind. Top-right: Lax-Wendroff. Middle-left: Fromm. Middle-right: Beam-Warming. Bottom-left: Lax-Wendroff with MinMod limiter. Bottom-right: Lax-Wendroff with Superbee limiter.

oscillations, for instance when $\nu = 1.05$, the numerical solution has, approximatively, the range $[-1000, 1000]$ in contrast with the exact solution that verifies the conditions $0 \leq c(x,t) \leq 1$. This is the mark of numerical instability.

2.4 Multi-D peculiarities.

One of the simplest way to deal with advection problems, in 2D or 3D, is to apply a MOL approach. In this approach the 1D formula can be used as it is in any of the Cartesian directions, with or without limiting. To people coming from an ODE background, this kind of semi-discretization seems to give simple and effective schemes, which in the case of advection-diffusion-reaction models, can be easily complemented by diffusion and reaction terms. Moreover, when the conversion from PDE to ODE system is done, we have at our fingertips a good deal of numerical methods of different accuracy and stability properties. However, this kind of approach has its drawbacks. As a first step, it is necessary to write the 2D (worse in 3D) problem as a single system of ODE and the multi-D structure with initial and boundary conditions has to be rendered by a single vector. Then, we have to take into account the stability of the ODE system. Accurate solution can be computed by using higher order spatial discretization formulas with limiting. The interested reader can find full details on this topic in the research paper by Hundsdorfer et al. [6] where the positivity of the so called k-schemes and a fourth order central scheme are investigated or Hundsdorfer and Trompert [7] where the comparison of two related third order MOL and dimensional splitting (from a fully discrete one-dimensional) methods are described. However, these authors point out that the full discrete method is more efficient and reliable than the MOL method, so that in the following we consider a full discrete method (without dimensional splitting).

In order to derive the higher order scheme in 2D, we start considering the first order upwind method obtained by a direct discretization. In the case $u_1 \geq 0$ and $u_2 \geq 0$, for instance, within the finite difference approach, we can consider the scheme defined by

$$C_{jh}^{\Delta t} = \gamma_{-1,0} C_{j-1h} + \gamma_{0,0} C_{jh} + \gamma_{0,-1} C_{jh-1} \qquad (16)$$

with coefficients γ, depending on Courant numbers $\nu_1 = u_1 \Delta t / \Delta x_1$ and $\nu_2 = u_2 \Delta t / \Delta x_2$, given as follows

$$\gamma_{-1,0} = \nu_1 \ , \qquad \gamma_{0,0} = 1 - \nu_1 - \nu_2 \ , \qquad \gamma_{0,-1} = \nu_2 \ .$$

In conservative form, the numerical method in 2D can be written as follows

$$C_{jh}^{\Delta t} = C_{jh} - \frac{\Delta t}{\Delta x_1} \left[F_{j+1/2h} - F_{j-1/2h} \right] - \frac{\Delta t}{\Delta x_2} \left[G_{jh+1/2} - G_{jh-1/2} \right] \qquad (17)$$

where F and G are numerical flux functions defined at the cell edges. This method is often called donor-cell upwind (DCU) method. In this method we assume that the only contribution to each flux is coming from the adjacent cell on the upwind side (the donor cell) and the numerical flux approximates the amount of concentration flowing normal to the corresponding cell edge. It is clear that the new value $C_{jh}^{\Delta t}$ is computed using only the values C_{jh}, C_{j-1h} and C_{jh-1}. This is correct only in the special cases when $u_1 = 0$ or $u_2 = 0$. In the general case, when the velocity field, at some times, may be at an angle to the grid, it is clear that also the value of C_{j-1h-1} should be involved to define $C_{jh}^{\Delta t}$. Then, in order to evaluate the contribution to fluxes by corner transport, we have to consider the transverse propagation due to mixed derivative terms in the Taylor formula:

$$c(x_1, x_2, t + \Delta t) = c(x_1, x_2, t) - u_1 \Delta t c_{x_1} - u_2 \Delta t c_{x_2} + \frac{1}{2} (\Delta t)^2 \left[u_1^2 c_{x_1 x_1} + u_1 u_2 c_{x_1 x_2} + u_2 u_1 c_{x_2 x_1} + u_2^2 c_{x_2 x_2} \right] + \ldots \qquad (18)$$

where all time derivative have replaced by spatial derivative and all derivatives are evaluated at (x_1, x_2, t^n). In this way we obtain the following method of first order accuracy in 2D, the so called corner transport upwind (CTU),

$$C_{jh}^{\Delta t} = C_{jh} - \nu_1 (C_{jh} - C_{j-1h}) - \nu_2 (C_{jh} - C_{jh-1}) + \frac{1}{2} \nu_1 \nu_2 \left[(C_{jh} - C_{jh-1}) - (C_{j-1h} - C_{j-1h-1}) + (C_{jh} - C_{j-1h}) - (C_{jh-1} - C_{j-1h-1}) \right] \ , \qquad (19)$$

which can be written also in the following form

$$C_{jh}^{\Delta t} = \gamma_{-1,-1} C_{j-1h-1} + \gamma_{-1,0} C_{j-1h} + \gamma_{0,0} C_{jh} + \gamma_{0,-1} C_{jh-1} \quad (20)$$

with coefficients γ given as follows

$$\gamma_{-1,-1} = \nu_1 \nu_2 \ , \qquad \gamma_{-1,0} = \nu_1 - \nu_1 \nu_2 \ ,$$
$$\gamma_{0,0} = 1 - \nu_1 - \nu_2 + \nu_1 \nu_2 \ , \qquad \gamma_{0,-1} = \nu_2 - \nu_1 \nu_2 \ .$$

As far as the derivation of higher order scheme is concerned, we have to consider also the higher order derivative terms in (18). By using second order central finite difference approximation, we obtain the following second order Lax–Wendroff method

$$\begin{aligned} C_{jh}^{\Delta t} = & C_{jh} - \nu_1 (C_{jh} - C_{j-1h}) - \nu_2 (C_{jh} - C_{jh-1}) + \\ & + \frac{1}{2} \nu_1 \nu_2 \Big[C_{jh} - C_{jh-1} - (C_{j-1h} - C_{j-1h-1}) \\ & + C_{jh} - C_{j-1h} - (C_{jh-1} - C_{j-1h-1}) \\ & + \frac{\nu_1}{\nu_2} (C_{j-2h} - 2C_{j-1h} + C_{jh}) \\ & + \frac{\nu_2}{\nu_1} (C_{jh-2} - 2C_{jh-1} + C_{jh}) \Big] \ , \end{aligned} \quad (21)$$

that can be written in the form (3). Note that similar formulae can be derived when the components of the velocity field have a different sign from the ones considered above.

2.5 Stability analysis

As well known the von Neumann analysis can be used to determine the stability of numerical methods in the case of linear equations with constant coefficients. Moreover, any limiting approach introduce in a given numerical method a nonlinearity and therefore we have to consider only methods without limiters. In the analysis we consider a Fourier decomposition and consequently it would be sufficient to consider a single arbitrary component

$$C_{jh} = e^{i(\xi j \Delta x_1 + \eta h \Delta x_2)} \ ,$$

where i is the imaginary unit, and ξ and η are the x_1 and x_2 wave-numbers, respectively. We define

$$C_{jh}^{\Delta t} = \lambda(\Delta x_1, \Delta x_2, \Delta t) C_{jh}$$

where $\lambda(\cdot)$ can be interpreted as an amplification factor. A given method is stable if substituting these formulas in the method we find out $\lambda \leq 1$ for all ξ and η.

The DCU method (17) has an amplification factor given by

$$\lambda = 1 - \nu_1 - \nu_2 + \nu_1 e^{-i\xi\Delta x_1} + \nu_2 e^{-i\eta\Delta x_2}$$

By using the Euler identity $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$, we can verify that in this case stability on condition is given by

$$\frac{\Delta t}{\Delta x_1} |u_1| + \frac{\Delta t}{\Delta x_2} |u_2| \leq 1 \ . \quad (22)$$

For the CTU method (19) we have

$$\begin{aligned} \lambda &= 1 - \nu_1 \left(1 - e^{-i\xi\Delta x_1}\right) - \nu_2 \left(1 - e^{-i\eta\Delta x_2}\right) \\ & \quad + \frac{1}{2} \nu_1 \nu_2 \left[\left(1 - e^{-i\eta\Delta x_2}\right) - e^{-i\xi\Delta x_1} \left(1 - e^{-i\eta\Delta x_2}\right) \right. \\ & \quad \left. + e^{-i\xi\Delta x_1} - e^{-i\eta\Delta x_2} \left(1 - e^{-i\xi\Delta x_1}\right) \right] \\ &= \left[1 - \nu_1 \left(1 - e^{-i\xi\Delta x_1}\right)\right] \cdot \left[1 - \nu_2 \left(1 - e^{-i\eta\Delta x_2}\right)\right] \end{aligned}$$

so that for the stability of the method we get

$$\max \left(\frac{\Delta t}{\Delta x_1} |u_1| , \frac{\Delta t}{\Delta x_2} |u_2| \right) \leq 1 \ . \quad (23)$$

It is a simple matter to verify that the previous condition represents the stability one for the second order Lax Wendroff method (21). Note that the condition (23) is less restrictive than (22).

2.6 Test problems

Let us consider the test problem

$$\begin{aligned} & c_t + u_1 c_{x_1} + u_2 c_{x_2} = 0 \ , \\ & c(x_1, x_2, 0) = \begin{cases} 1 & \text{if } \sqrt{(x_1 + 0.2)^2 + (x_2 + 0.2)^2} \leq 0.1 \\ 0 & \text{otherwise} \end{cases} \\ & c(0, x_2, t) = c(1, x_2, t) \ , \\ & c(x_1, 0, t) = c(x_1, 1, t) \ , \end{aligned} \quad (24)$$

where $0 \leq x_1, x_2 \leq 1$, $u_1 = u_2 = 1$ and $t \geq 0$. The initial datum is a unitary cylinder with radius 0.1, centered at $(0.2, 0.2)$; at the boundaries we impose periodic boundary conditions. We report the numerical results obtained with direct discretization numerical methods of first and

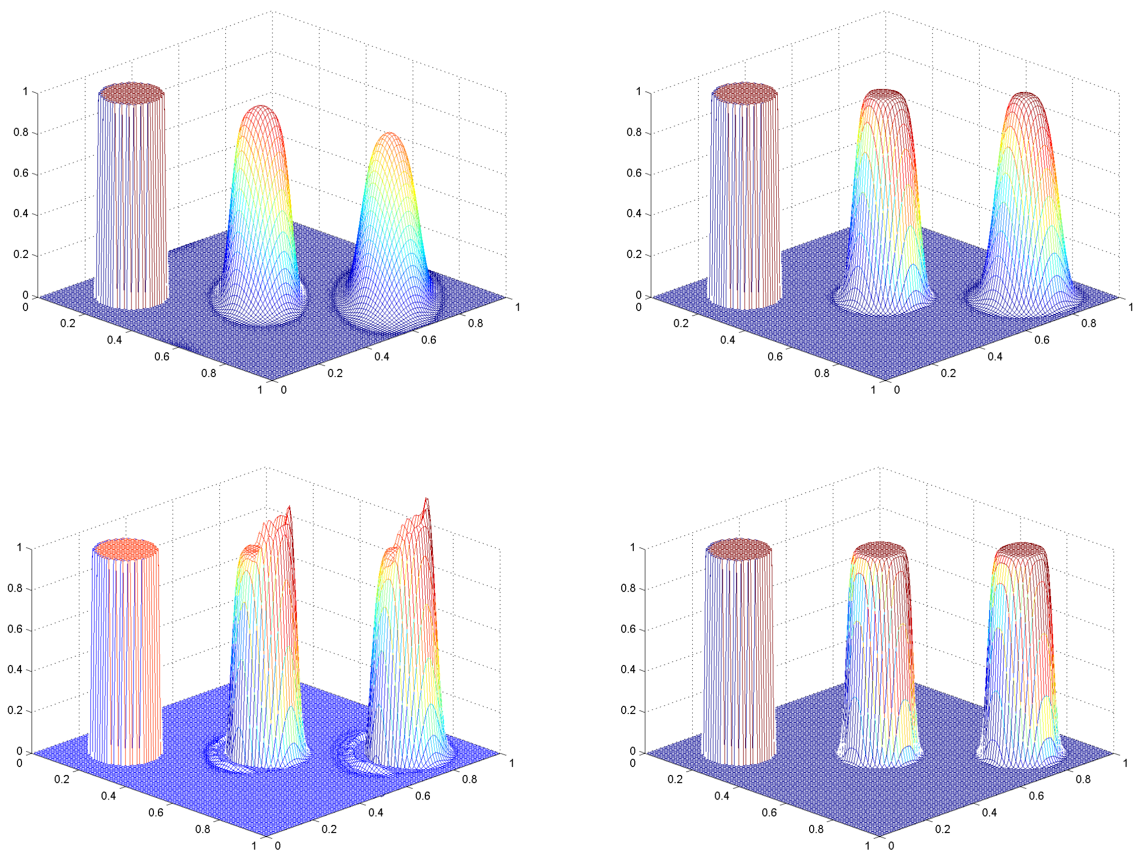


Figure 2: Advection equation with 100×100 spatial grid, $\nu_1 = \nu_2 = 0.95$ and final time $t_{\max} = 0.6$. Top left: first order DTU method; top right: first order CTU method; bottom Left: second order Lax-Wendroff method without limiter; bottom right: second order Lax-Wendroff method with van Leer limiter.

second order accuracy, with and without flux limiter in Figure 2. By considering the different behaviour of the methods and limiters used, we have found that the best results are obtained by the second order method with van Leer limiter.

The second order Lax-Wendroff method, with van Leer limiter, has been used for solving the following test problem

$$\begin{aligned} & c_t + (u_1\, c)_{x_1} + (u_2\, c)_{x_2} = 0 \\ & c(x_1, x_2, 0) = \tanh\left(-\frac{1}{2} x_2\right) \end{aligned} \tag{25}$$

with $-4 \le x_1, x_2 \le 4$ and $t \ge 0$ and where the velocity field is given by

$$\begin{aligned} & u_1(x_1,x_2) = -\omega(r)\, x_2 \;, \quad u_2(x_1,x_2) = \omega(r)\, x_1 \;, \\ & r = \sqrt{x_1^2 + x_2^2} \;, \\ & \omega(r) = \frac{V(r)}{r\, V_{\max}} \;, \quad V(r) = \frac{\tanh(r)}{\cosh^2(r)} \;. \end{aligned} \tag{26}$$

Here $V(r)$ represents the tangential velocity around the center of the domain, and $V_{\max}$ is its maximum value. Figure 3 shows the velocity on a 25×25 grid, with this number of cells it is possible to grasp the structure of the velocity field. Far from the center the velocity is smaller compared to its vicinity, and at the center the velocity is zero. This problem provides a simple model describing the mixing of cold and hot air, due to the rotational ve-

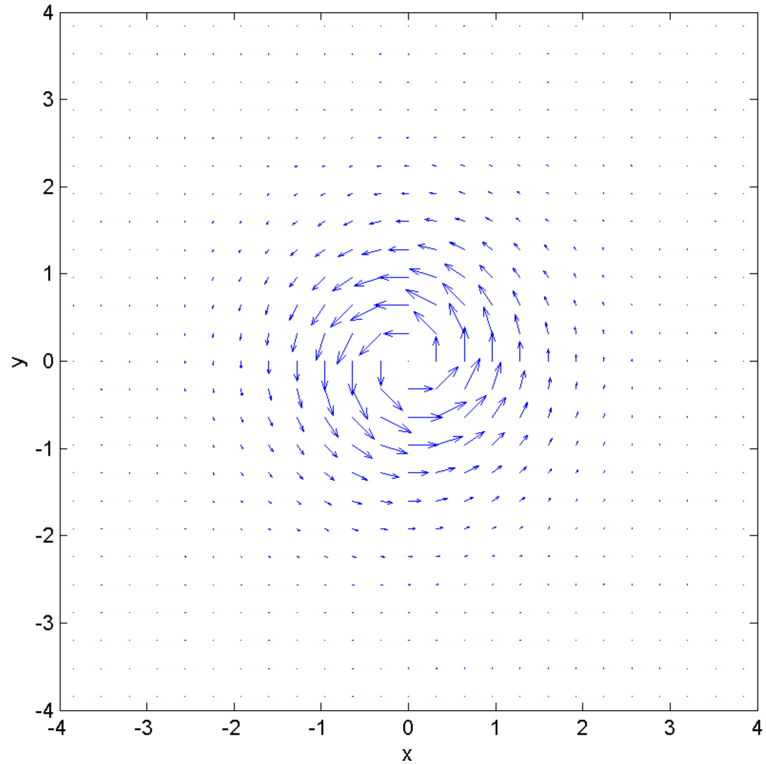


Figure 3: Velocity field for problem (25)-(26) with $V_{\max} = 0.385$.

locity field, which is similar to the cyclonic air motion at low pressure systems observed on the weather maps.

As far as the boundary conditions are concerned, we used the zero extrapolation ones instead of the available exact values. In Figure 4, we plot a top view of the numerical solution, at final time $t_{\max} = 4$ on a 160×160 spatial grid, for $V_{\max} = 0.385$. For this test we fixed a Courant number equal to 0.9. Two sample animations of the numerical solution, obtained on a 80×80 spatial grid, are available on the web page http://mat520.unime.it/fazio/Mixing.html

Problem (25)-(26) was used by Tamamidis and Assanis [14] and by Hundsdorfer and Trompert [7] to evaluate the performance of different numerical methods. In fact, for this problem it is possible to carry on convergence tests since its exact solution is known:

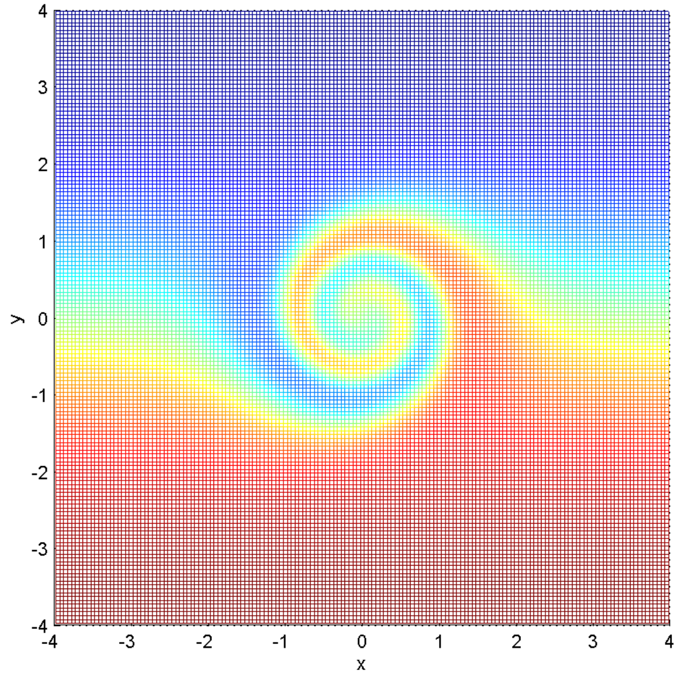
$$c(x_1, x_2, t) = \tanh\left(\frac{1}{2}x_1 \sin(\omega(r)t) - \frac{1}{2}x_2 \cos(\omega(r)t)\right) . \quad (27)$$

To study the convergence of the method, we calculated the numerical solution on successively refined spatial grids, with $\Delta x_1 = \Delta x_2$, using 40×40, 80×80, 160×160 grid-cells and a variable time step in order to ensure a Courant number equal to 0.9. We define the root-mean-squared (RMS) error in two-dimensional space

$$E_{\text{RMS}} = \sqrt{\frac{1}{JH}\sum_j \sum_h E_{jh}^2}$$

where $E_{jh} = |c(x_{1j}, x_{2h}, t_{\max}) - C_{jh}|$ is the relative error with $x_{1j} = -4 + (j + 0.5)\Delta x_1$ for $j = 0, 1, 2, \ldots, J$ and $x_{2h} = -4 + (h + 0.5)\Delta x_2$ for $h = 0, 1, 2, \ldots, H$.

The corresponding RMS errors are shown in table 1,



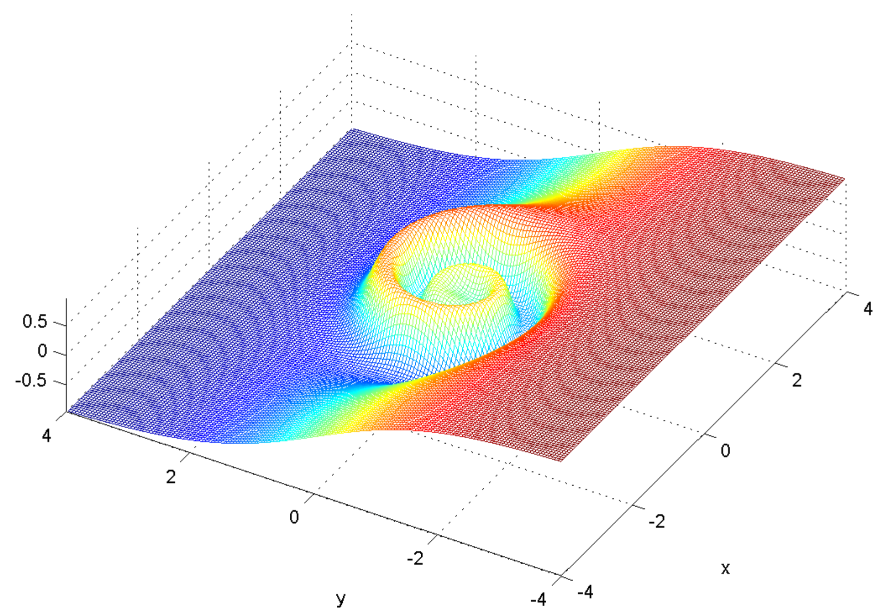


Figure 4: Sample numerical solution for problem (25)-(26). Top: top view; bottom: side view.

where we also report the convergence order. Note the influence of the refined grid on the reduction of the RMS error, that decreases for decreasing values of $\Delta x_i, (i = 1, 2)$, with fixed Courant number.

Table 1: RMS error and order results.

Grid refinement	Δx_1	E_{RMS}	Order
40×40	0.2	0.0299	
			1.8490
80×80	0.1	0.0083	
			2.2876
160×160	0.05	0.0017	

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