Scaling and Spiral Equivalence from a Numerical Viewpoint

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Abstract—**The non-iterative numerical solution of nonlinear boundary value problems is a subject of great interest. This paper is concerned with the theory of non-iterative transformation methods. These methods are defined within group invariance theory. We prove the equivalence between two non-iterative transformation methods defined by the scaling group and the spiral group, respectively.**

Keywords: Scaling properties, ordinary differential equations, initial value methods.

1 Introduction

The non-iterative numerical solution of nonlinear boundary value problems (BVPs) is an outstanding subject in current research. In this context transformation methods (TMs) are founded on the group invariance theory, see Bluman and Cole [6], Dresner [8], or Bluman and Kumei [7]. These methods are initial value methods because they transform BVPs to initial value problems (IVPs).

The first application of a non-iterative TM, without any consideration of group theory, was given by Töpfer in [28] for the numerical solution of the celebrated Blasius problem in boundary layer theory. This result is quoted in several books on fluid dynamics, see, for instance, Goldstein [16, pp. 135-136]. Acrivos, Shah and Petersen [1] first and Klamkin [18] later extended Töpfer's method to a more general problem and to a class of problems, respectively. Along the lines of the work of Klamkin, for a given problem Na [20, 21] showed the relation between the invariance properties, with respect to a linear group of transformation: the scaling group, and the applicability of a non-iterative TM. Moreover, Na considered the invariance with respect to a nonlinear group of transformations: the spiral group. A review paper on this topic was written by Klamkin [19].

The invariance of one and of two or more physical parameters, if they are involved in the mathematical model, were respectively proposed by Na [22] and by Scott, Rinschler and Na [25]. A survey book written by Na [23, Chs 7-9], on the numerical solution of BVP, devoted three chapters to numerical TMs.

Here we consider two-point BVPs and we prove that two apparently different TMs, defined by the spiral group and the scaling group, respectively, are indeed equivalent. It is worth mentioning that the quoted example due to Na [20] — see Na and Tang [24], Klamkin [19], Ames [2, p. 140], Ames and Ibragimov [4], Na [23, pp. 155-158], Seshadri and Na [26, pp. 157-168] or Ames [3] — belongs to the characterized class of problems.

This paper is organized as follows. The next section is aimed at establishing the mentioned equivalence from a theoretical point of view. A preliminary note on this topic was presented at the international congress on *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics* [11]. In section 3 we verify the equivalence, by using a specific test problem, from a numerical viewpoint. The last section concerns with concluding remarks pointing out limitations and possible extensions of the proposed approach.

2 Scaling and spiral equivalence

Let us introduce the class of two-point BVPs

$$\begin{aligned} \frac{d^2u}{dx^2} &= u^{1-2\delta}\Theta\left(xu^{-\delta}, \frac{du}{dx}u^{\delta-1}\right) \qquad x \in (0,b) \\ \frac{du}{dx}(0) &= Au(0)^{1-\delta} \\ u(b)^{\zeta}\Phi\left(\frac{du}{dx}(b)u(b)^{\delta-1}\right) &= B \end{aligned} \tag{1}$$

where A and δ are arbitrary constants, $\Theta(\cdot,\cdot)$ and $\Phi(\cdot)$ are arbitrary functions of their arguments, $b > 0, \zeta \neq 0$ and $B \neq 0$. Here the governing differential equation and the boundary condition at $x = 0$ are invariant with respect to the following scaling group of transformations

$$x^* = \lambda^{\delta} x \ , \qquad u^* = \lambda u \ , \tag{2}$$

where λ is the group parameter. The non-iterative numerical solution of (1) can be obtained by the following steps:

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- we fix a value of $u^*(0)$, this defines a value of $\frac{du^*}{dx^*}(0)$ according to the boundary condition;
- next we integrate numerically, with initial data $u^*(0)$ and $\frac{du^*}{dx^*}(0)$, forwards in $[0, b^*]$ where b^* is defined by

$$b^* = b \left\{ \left[\frac{u^*(b^*)^\zeta \; \Phi\left(\frac{du^*}{dx^*}(b^*) u^*(b^*)^{\delta-1}\right)}{B} \right]^{1/\zeta} \right\}^\delta \quad ; \quad (3)$$

- finally, the following relations are defined by applying group properties

$$\lambda = \left[\frac{u^*(b^*)^\zeta \Phi\left(\frac{du^*}{dx^*}(b^*) u^*(b^*)^{\delta-1}\right)}{B} \right]^{1/\zeta}$$

$$u(0) = \lambda^{-1} u^*(0) \quad ; \quad \frac{du}{dx}(0) = \lambda^{\delta-1} \frac{du^*}{dx^*}(0)$$

$$u(b) = \lambda^{-1} u^*(b^*) \quad ; \quad \frac{du}{dx}(b) = \lambda^{\delta-1} \frac{du^*}{dx^*}(b^*) \; .$$

We note that b^* is defined implicitly, in (3), so that it can be considered as a root of an unknown function.

Via the simple variable transformation

$$x = x \quad ; \quad v = ln(u) \tag{4}$$

the scaling group (2) is transformed to the following spiral group

$$x^* = \lambda^\delta x \quad ; \quad v^* = v + \mu \quad ; \quad \mu = ln(\lambda)$$

where μ is the group parameter. The problem (1), after (4), transforms to

$$\begin{aligned} \frac{d^2v}{dx^2} &= e^{-2\delta v} \Theta \left(x e^{-\delta v}, \frac{dv}{dx} e^{\delta v} \right) - \left(\frac{dv}{dx} \right)^2 \\ \frac{dv}{dx}(0) &= A e^{-\delta v(0)} \\ e^{\zeta \; v(b)} \; \Phi &\left(\frac{dv}{dx}(b) e^{\delta \; v(b)} \right) = B \quad . \end{aligned} \tag{5}$$

Therefore, $u(x) \geq 0$ for every $x \in (0, b)$.

It is interesting to note that the example due to Na [20] (where $p = -1$, see Na and Tang [24] for $p = 0$ and $p = 1$)

$$\begin{aligned} \frac{d^2T}{dr^2} + \frac{p+1}{r}\frac{dT}{dr} + q e^T &= 0 \\ \frac{dT}{dr}(0) &= 0 \\ T(1) &= 0 \end{aligned} \tag{6}$$

is a particular case of (5). To see this, let us set $v = T$, $x = r$, $\delta = -1/2, \Theta(\cdot, \cdot) = -\frac{p+1}{r}\frac{dT}{dr} e^{-T} - q + \left(\frac{dT}{dr}\right)^2 e^{-T}$, $\Phi(\cdot) = 1, A = 0, b = 1, \zeta = 1$ and $B = 1$. Now, by inverting (4), problem (6) takes the form

$$\begin{aligned} \frac{d^2u}{dx^2} + \frac{p+1}{x}\frac{du}{dx} + q u^2 - \left(\frac{du}{dx}\right)^2 u^{-1} &= 0 \\ \frac{du}{dx}(0) &= 0 \\ u(1) &= 1 \; . \end{aligned} \tag{7}$$

Therefore, along the lines of the approach outlined above a non-iterative numerical solution for (6) can be obtained by solving (7). This will be the topic of the next section.

3 An application on the non-linear heat generation

According to Na and Tang [24], the dimensionless model governing the transient distribution in the radial direction, with heat generation e^T, in plane geometry ($p = -1$), in a solid cylinder ($p = 0$) or in a sphere ($p = 1$) of radius r with $0 < r < 1$ is given by

$$\frac{\partial T}{\partial t} = \frac{1}{r^{p+1}} \frac{\partial}{\partial r} \left(r^{p+1} \frac{\partial T}{\partial r} \right) + q e^T \tag{8}$$

$$T(r, 0) = 0 \tag{9}$$

$$\frac{\partial T}{\partial r}(0, t) = 0 \tag{10}$$

$$T(1, t) = 0 \; . \tag{11}$$

where q is a dimensionless parameter, whereas (9) stands for the initial temperature of the cylinder (or the sphere), (10) represents the initial temperature radial gradient, and (11) defines the surface temperature.

Let us consider the invariance of (8) with respect to the following spiral group

$$T^* = T + \lambda \; , \qquad r^* = e^{\beta\lambda} r \; , \qquad t^* = e^{\alpha\lambda} t \; .$$

Indeed, after the application of such a group we obtain that the invariance is attained by imposing that

$$\left\{ \begin{array}{l} \alpha - 2\beta = 0 \\ 1 + 2\beta = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \beta = -\frac{1}{2} \\ \alpha = -1 \end{array} \right.$$

As a result the spiral group specializes to

$$T^* = T + \lambda \; , \qquad r^* = e^{-\frac{\lambda}{2}} r \; , \qquad t^* = e^{-\lambda} t \; . \tag{12}$$

Due to the fact that the dynamics of heat propagation in most applications is mainly performed over long time scale, it is more interesting to study the asymptotic behavior of the temperature spacial distribution. The steady state temperature space distribution, is simply obtained by setting the time derivative of temperature T to zero. As a consequence the considered model is rewritten as the following ordinary differential model

$$\frac{1}{r^{p+1}}\frac{d}{dr}\left(r^{p+1}\frac{dT}{dr}\right)+qe^{T}=0 \tag{13}$$

$$\frac{dT}{dr}(0)=0 \tag{14}$$

$$T(1)=0\ . \tag{15}$$

Obviously (13) is still invariant with regard to the finite point transformations of the spiral group (12) involving T and r. Besides, it is easy to see that the boundary condition (14) is invariant, whereas the other one (15) is not.

According to what said in the previous section, in order to prove the equivalence between the spiral group and the scaling one, it is enough to define a simple change of variable $W = e^T$. As a consequence the derivatives transform as follows

$$\frac{dT}{dr}=\frac{1}{W}\frac{dW}{dr}\ ,\qquad \frac{d^2T}{dr^2}=-\frac{1}{W^2}\left(\frac{dW}{dr}\right)^2+\frac{1}{W}\frac{d^2W}{dr^2}\ .$$

If we replace them in (13)-(15), then we obtain

$$\frac{d^2W}{dr^2}+\frac{p+1}{r}\frac{dW}{dr}-W^{-1}\left(\frac{dW}{dr}\right)^2+qW^2=0 \tag{16}$$

$$\frac{dW}{dr}(0)=0 \tag{17}$$

$$W(1)=1 \tag{18}$$

Insertion of the new variable W in the temperature point transformation of the spiral group (12), yields the scaling group

$$r^*=\mu^{-1/2}r\ ,\qquad W^*=\mu W\quad\text{where}\quad \mu=e^{\lambda}. \tag{19}$$

Such a BVP can be solved by defining an auxiliary IVP (written in starred variables) consisting of the governing equation (16) and the initial condition on the space derivative (17). As far as the missing initial condition is concerned, for the sake of convenience, we set $W^*(0) = 1$. Since we want to find the solution to the original problem that at $r = 1$ is $W(1) = 1$ and that these values are linked by the point transformation

$$W^*(r_0^*)=\mu W(1)\ ,\qquad r_0^*=\mu^{-1/2}, \tag{20}$$

it is enough to eliminate the parameter μ from (20) to obtain the equation

$$W^{*1/2}r^*-1=0, \tag{21}$$

whose zero(s) is just the sought value of r_0^*. We have only to be cautious, both in integrating up to a value of r^* greater than r_0^* and in finding a range for the parameter q, so that the equation (21) can have at least one root. Once the zero is found, it is immediate to deduce the value of the sought similarity parameter μ from (20) and the right value for the missing initial condition $W(0)$.

A numerical test, for several values of the parameter q, has been carried out by means of the MATLAB ODE integrator ODE45, with the accuracy and adaptivity parameters defined by default, with the help of the event locator set at picking those values of the solution where equation (21) is verified. The ODE45 solver belongs to the MATLAB ODE suite written by Samphine and Reichelt [27].

The results of such a test are reported in table 1 and the sought zeros are graphically depicted in figure 1. It was experienced that for both the cylindric case ($p = 0$) and the spherical case ($p = 1$), by choosing $q = 0$ (no heat source), we obtain the constant solution $W^* = 1$ corresponding to the initial condition imposed $W^*(0) = 1$. In this case we have only one zero. By considering values for q greater than zero we found that for the cylindric case there is the range $0 < q < 1.99$, and for the spherical case there is the range $0 < q < 3.3$, where two distinct zeros exist, whereas we have two solutions respectively for $q_1 \approx 1.99$ and for $q_2 \approx 3.3$. Furthermore, for values higher than q_1 and q_2 no solutions at all exist. This situation can be easily appreciated by looking at the figure 1. Finally, it has been shown by Na [23, pp. 165-172] that only the first solution has a physical meaning in order the hypothesis of asymptotic analysis to be valid.

Just for the sake of validation of our results, we decided to prove numerically the equivalence between a spiral group and a scaling group, by integrating the original problem (13)-(14), where the final condition (15) was replaced by the calculated missing initial condition reported in table 1, for several values of the parameter q. The obtained results are shown in figure 2.

With regard to table 1, it has to be remarked the comparison between our values and the ones obtained by Na and Tang in their paper [24] where they found the solution to the same problem in (13)-(15) by means of a spiral group. For small values of q, we can observe a difference of about $\pm 10^{-4}$ both for cylindric case and for spherical case.It is likely that this difference can be due to the fact that the present values have been obtained by means of adaptive step integrator, whereas the others results, nearly for sure, by using a constant step routine. Unfortunately for values of the parameter q greater than 0.9 (not reported in table 1), the values of the missing initial condition provided by Na and Tang are clearly incorrect or misreported. Correctness of our results is also confirmed by the numerical validation depicted in figure 2.

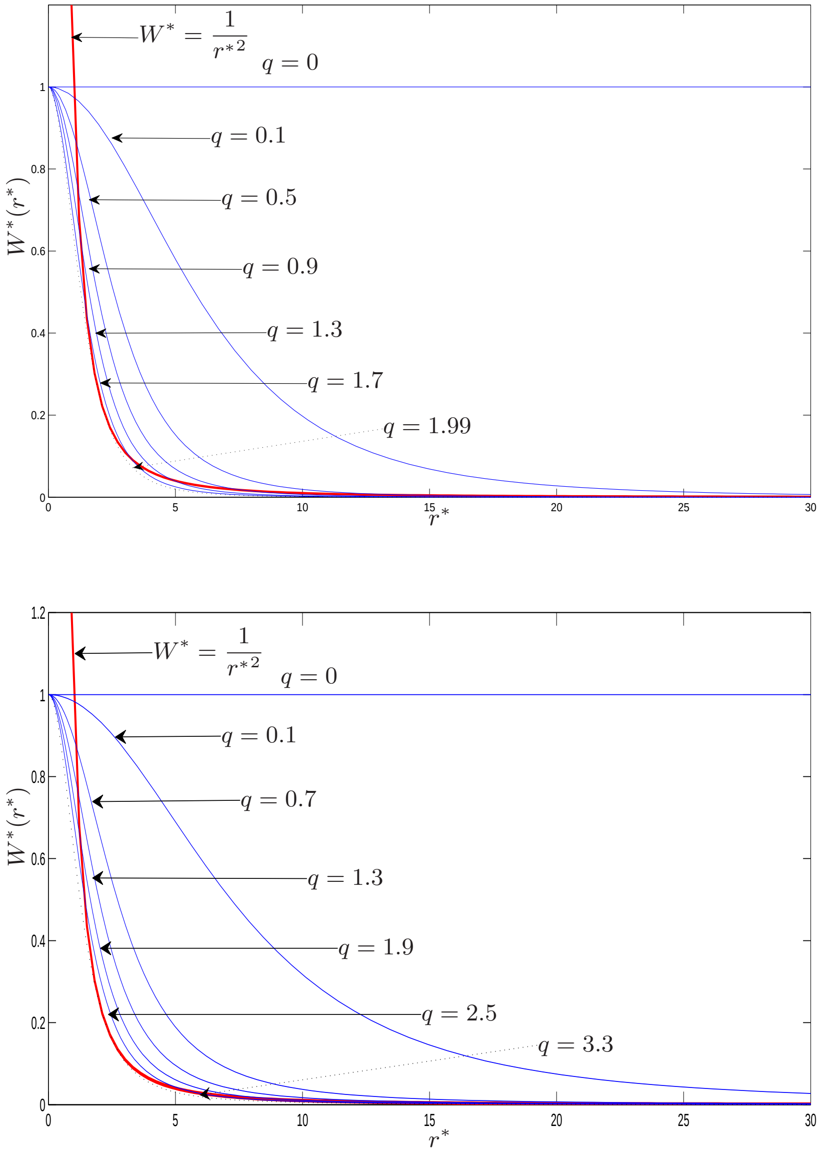

Figure 1: Numerical results for the problem (16-18): top $p = 0$ and bottom $p = 1$ (dotted lines correspond to the upper values of q for the existence of solutions).

	cylinder			sphere	
q	$ln(W(0))$	T(0)(Na-Tang)	q	$ln(W(0))$	T(0)(Na-Tang)
0.1	0.025577704422179	0.0252	0.1	0.016864300837286	0.0168
0.5	0.138768257096320	0.1380	0.7	0.127512507202391	0.1270
0.9	0.276610074783917		1.3	0.259102236854185	
1.3	0.456703715190792		1.9	0.424694208534499	
1.7	0.730434831089254	2.0	2.5	0.651628572111586	
1.9	0.983251982794000		3.1	1.056357138615421	

Table 1: Numerical results. Our numerical results are validated in figure 2.

4 Concluding remarks

The previous sections indicate how two apparently different non-iterative TMs are equivalent under a simple variable transformation. The underlying idea of the transformation in point was to solve the transformed problem instead of the original one, see Fazio [9, 10] for other ap-

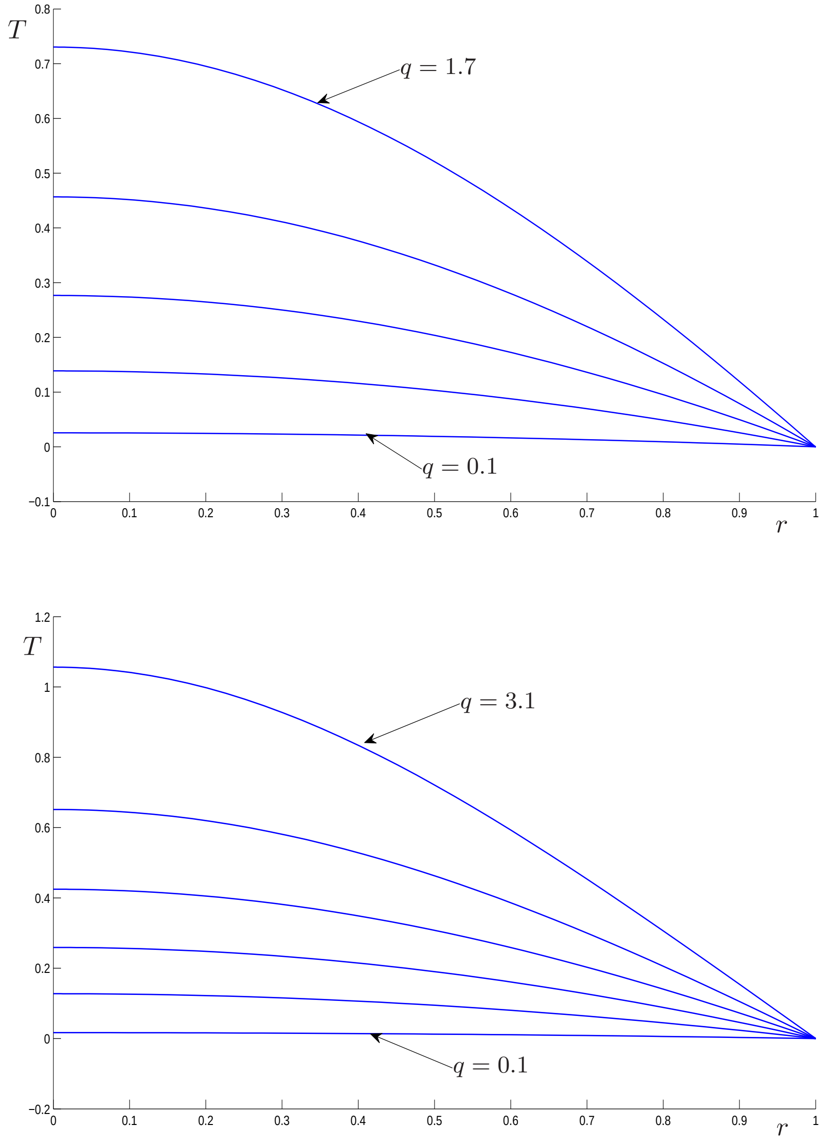

Figure 2: Top for $p = 0$, bottom for $p = 1$: solutions to the problem (13)-(15) obtained through the missing initial condition reported in table 1 for each considered q.

plications of this idea.

In closing, we can outline some further implications coming out from this work. First of all, non-iterative TMs are *ad hoc* methods. Their applicability depends upon the invariance properties of the governing differential equation and the given boundary conditions. Consequently, it is easy to realize that non-iterative TMs can not be extended to every BVP. The introduction of a variable transformation linking two different invariant groups is a possible way to extend the applicability of non-iterative TMs. However, it is a simple matter to show a differential equation not admitting any group of transformations, see for instance Hill [17, pp. 81-82] who reported a classical example due to Bianchi [5, pp. 470-475]. Consequently, it is easy to realize that non-iterative TMs cannot be extended to every BVPs. On the other hand, a further extension of the present method is possible if one or more physical parameters are involved in the mathematical model, see Na [21] and Scott, Rinschler and Na [25], cf. also Na [23, Chapters 8 and 9]. Moreover, BVPs governed by the most general second order differential equation in normal form can be solved iteratively by extending a scaling group via the introduction of a numerical parameter h so as to recover the original problem as $h \to 1$, see Fazio [12, 13, 14]. The extension of this iterative TM to moving boundary problems governed by parabolic equations have been considered in [15].

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