Transformation Methods for the Blasius Problem and its Recent Variants

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Abstract—**Blasius problem is the simplest nonlinear boundary layer problem. We hope that any approach developed for this epitome can be extended to more difficult hydrodynamics problems. With this motivation we review the so called Töpfer transformation, which allows us to find a non-iterative numerical solution of the Blasius problem by solving a related initial value problem and applying a scaling transformation. The applicability of a non-iterative transformation method to the Blasius problem is a consequence of its partial invariance with respect to a scaling group. Several problems in boundary-layer theory lack this kind of invariance and cannot be solved by non-iterative transformation methods. To overcome this drawback, we can modify the problem under study by introducing a numerical parameter, and require the invariance of the modified problem with respect to an extended scaling group involving this parameter. Then we apply initial value methods to the most recent developments involving variants and extensions of the Blasius problem.**

Keywords: boundary-layer theory, scaling invariance, transformation methods, initial value methods.

1 Introduction

At the beginning of the last century L. Prandtl [17] put the foundations of boundary-layer theory providing the basis for the unification of two, at that time seemingly incompatible, sciences: namely, theoretical hydrodynamics and hydraulics. Boundary-layer theory has found its main application in calculating the skin-friction drag which acts on a body as it is moved through a fluid: for example the drag of an airplane wing, of a turbine blade, or a complete ship [18].

With the turning of this new century, as the number of applications of microelectronics devices increases, boundary-layer theory has found a renewal of interest within the study of gas and liquid flows at the micro-scale regime [6, 16].

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Blasius problem is the simplest nonlinear boundary layer problem. A recent study by Boyd pointed out how this particular problem of boundary-layer theory has arose the interest of prominent scientist, like H. Weyl, J. von Neumann, M. Van Dyke, etc., see Table 1 in [4]. The main reason for this interest is due to the hope that any approach developed for this epitome can be extended to more difficult hydrodynamics problems. Our main goal here is to show how to solve numerically the Blasius problem, and its variants and extensions, by initial value methods derived within scaling invariance theory.

2 Fluid flow on a flat plate

The model describing the steady plane flow of a fluid past a thin plate, provided the boundary layer assumptions are verified ($v \gg w$ and the existence of a very thin layer attached to the plate), is given by

$$\begin{aligned} &\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ &v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \mu\frac{\partial^2 v}{\partial z^2} \\ &v(y,0) = w(y,0) = 0 \\ &v(y,z) \rightarrow V_\infty \quad \text{as} \quad z \rightarrow \infty \ , \end{aligned} \tag{1}$$

where the governing differential equations, namely conservation of mass and momentum, are the steady-state 2D Navier-Stokes equations under the boundary layer approximations, v and w are the velocity components of the fluid in the y and z direction, V_∞ represents the main-stream velocity, see the draft in figure 1, and μ is the viscosity of the fluid. The boundary conditions at $z = 0$ are based on the assumption that neither slip nor mass transfer are permitted at the plate whereas the remaining boundary condition means that the velocity v tends to the main-stream velocity V_∞ asymptotically.

In order to study this problem it is convenient to introduce a potential (stream function) $\psi(y, z)$ defined by

$$v = \frac{\partial \psi}{\partial z}, \qquad w = -\frac{\partial \psi}{\partial y} \quad .$$

The physical motivation for introducing this function is that constant ψ lines are steam-lines. The mathematical motivation for introducing such a new variable is that the

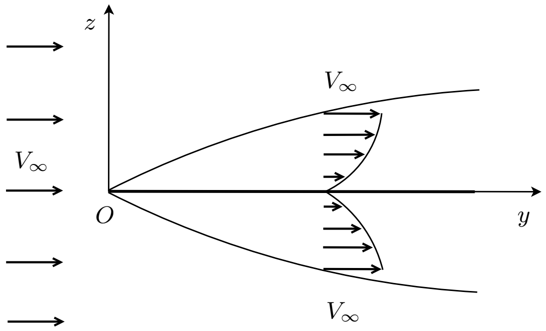

Figure 1: Boundary layer over a thin plate.

equation of continuity is satisfied identically, and we have to deal only with the transformed momentum equation. In fact, introducing the stream function the problem can be rewritten as follows

$$
\begin{aligned}
&\mu\frac{\partial^3\psi}{\partial z^3} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial y\partial z} = 0 \\
&\frac{\partial\psi}{\partial y}(y,0) = \frac{\partial\psi}{\partial z}(y,0) = 0 \\
&\frac{\partial\psi}{\partial z}(y,z) \rightarrow V_\infty \quad \text{as} \quad z \rightarrow \infty \ .
\end{aligned}
\tag{2}
$$

2.1 Blasius problem

Blasius [3] introduced the following similarity transformation

$$
\eta = z\left(\frac{V_\infty}{\mu y}\right)^{1/2} \ , \qquad f(\eta) = \psi(y,z)\left(\mu y V_\infty\right)^{-1/2} \ ,
$$

that reduces the partial differential model (2) to

$$
\begin{aligned}
&\frac{d^3 f}{d\eta^3} + \tfrac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \\
&f(0) = \frac{df}{d\eta}(0) = 0 \ , \qquad \frac{df}{d\eta}(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty \ ,
\end{aligned}
\tag{3}
$$

i.e., a boundary value problem (BVP) defined on a semi-infinite interval. Blasius solved this BVP by patching a power series to an asymptotic approximation at some finite value of η.

2.2 Töpfer transformation

By considering the derivation of the series expansion solution of the Blasius problem, Töpfer [20] defined a transformation of variables that reduces the BVP into an initial value problem (IVP). However, it is much simpler to consider directly the transformation

$$
f^* = \lambda^{-1/3} f \ , \qquad \eta^* = \lambda^{1/3}\eta \ , \tag{4}
$$

and to define the non-iterative transformation method. We notice that the governing differential equation and the initial conditions at the free surface are left invariant by the new variables defined in (4). Moreover, Töpfer used the missed initial condition

$$
\frac{d^2 f^*}{d\eta^{*2}}(0) = 1 \ .
$$

The first and second order derivatives transform in the following way

$$
\frac{df^*}{d\eta^*} = \lambda^{-2/3}\frac{df}{d\eta} \ , \qquad \frac{d^2 f^*}{d\eta^{*2}} = \lambda^{-1}\frac{d^2 f}{d\eta^2} \ ,
$$

and the value of λ can be found on condition that we have an approximation for $\frac{df^*}{d\eta^*}(\infty)$. In fact, by the first of the above relations we get

$$
\lambda = \left[\frac{df^*}{d\eta^*}(\infty)\right]^{-3/2} \ . \tag{5}
$$

Let us list the steps necessary to solve the Blasius problem by the considered approach, we have to:

1. solve the IVP

$$
\begin{aligned}
&\frac{d^3 f*}{d\eta^{*3}} + \tfrac{1}{2} f^* \frac{d^2 f^*}{d\eta^{*2}} = 0 \\
&f^*(0) = \frac{df^*}{d\eta^*}(0) = 0, \qquad \frac{d^2 f^*}{d\eta^{*2}}(0) = 1
\end{aligned}
\tag{6}
$$

   and, in particular, get an approximation for $\frac{df^*}{d\eta^*}(\infty)$;

2. compute λ by equation (5);

3. obtain $f(\eta)$ by the inverse transformation of (4).

In this way we have defined an initial value method for the Blasius problem. In literature such a method is also known as a non-iterative transformation method (ITM).

2.3 Truncated boundary approximation

From a numerical point of view the request to evaluate $\frac{df}{d\eta}(\infty)$ cannot be fulfilled. Several strategies have been proposed in order to provide an approximation of this value. The simplest and widely used one is to introduce, instead of infinity, a suitable truncated boundary η_∞. The question on how to set a satisfactory value of η_∞ is not addressed in this work. A recent successful way to deal with such a question is to reformulate the considered problem as a free BVP [8, 9, 10]. For instance, as far as the Blasius problem is concerned, we can replace the asymptotic condition with the free boundary conditions

$$
\frac{df}{d\eta}(\eta_\epsilon) = 1, \qquad \frac{d^2 f}{d\eta^2}(\eta_\epsilon) = \epsilon \tag{7}
$$

where η_ϵ is the unknown free boundary and $0 \le \epsilon \ll 1$ is a continuation parameter, see [8] for details. For a recent survey on this topic see [12].

For the sake of simplicity we will not use the free boundary approach here, but we perform some preliminary computational tests in order to find a suitable value for the truncated boundary η_∞.

2.4 Numerical results

Figure 2 shows a sample numerical computation. We

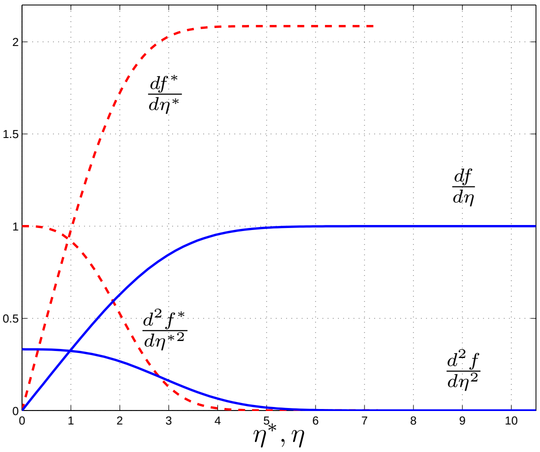

Figure 2: Blasius solution by a non-ITM.

used a variable step-size classical fourth order Runge-Kutta method, implemented in order to maintain a local error of the order of 10^{-6}. Moreover, the calculation were performed in the starred variables with a first time step equal to 0.1 and $\eta_\infty^* = 7.25$. The asymptotic value of interest was found to be

$$\frac{df^*}{d\eta^*}(\infty) \approx 2.085409 \; .$$

This value can be used in equation (5) to get

$$\frac{d^2f}{d\eta^2}(0) \approx 0.332057 \; .$$

Blasius solution was found by rescaling.

2.5 The iterative transformation method

The applicability of a non-ITM to the Blasius problem is a consequence of its partial invariance with respect to the transformation (4); the asymptotic boundary condition is not invariant. Several problems in boundary-layer theory lack this kind of invariance and cannot be solved by non-ITMs. To overcome this drawback, we can modify the problem under study by introducing a numerical parameter h, and require the invariance of the modified problem with respect to an extended scaling group involving h, see [11] for details.

An ITM can be defined as follows:

1. - the original BVP is embedded into a modified problem involving the numerical parameter h, so that it is ensured the invariance of the modified problem with respect to an extended scaling group involving h.

2. - by starting with suitable values of h_0^* and h_1^* a root-finder method is used to define a sequence h_j^*, for $j = 2, 3, \ldots,$. At each iteration the group parameter λ is obtained by solving an IVP numerically. The related sequence $\Gamma(h_j^*)$, for $j = 0, 1, 2, \ldots,$ is defined by

$$\Gamma(h^*) = h - 1 \qquad \text{with} \qquad h = h(h^*) \; , \tag{8}$$

   $\Gamma(\cdot)$ is defined implicitly by the solution of an IVP written in the starred variables and as a consequence $h = h(h^*)$.

3. - suitable termination criteria have to be used to verify whether $\Gamma(h_j^*) \to 0$ as $j \to \infty$.

4. - the solution of the original problem can be obtained by rescaling to $h = 1$.

By defining an ITM the existence and uniqueness question can be reduced to finding the number of real zeros of the transformation function $\Gamma(\cdot)$. This result can be stated as follows.

Theorem 1 *Let us assume that IVPs used to define the transformation function are well posed. Then, the considered BVP has a unique solution if and only if the transformation function has a unique real zero; nonexistence (nonuniqueness) of the solution is equivalent to nonexistence of real zeros (existence of more than one real zero) of* $\Gamma(\cdot)$.

The underlying idea of the proof of this theorem is that there exists a one-to-one and onto correspondence between the set of solutions of the BVP and the set of real zeros of the transformation function, see [11]. This theorem is applied in the next section.

3 Recent developments

In this section we report on recent developments involving some extensions of the Blasius problem and the related numerical approximation. The results reported in this section were found by the **o**de113 solver, from the MATLAB ODE suite written by Samphine and Reichelt [19], with the accuracy and adaptivity parameters defined by default.

3.1 Moving surfaces

Klemp and Acrivos [15] were the first to define the similarity model of a boundary layer problem over moving surfaces. For this model the Blasius equation has to be considered along with the usual asymptotic boundary

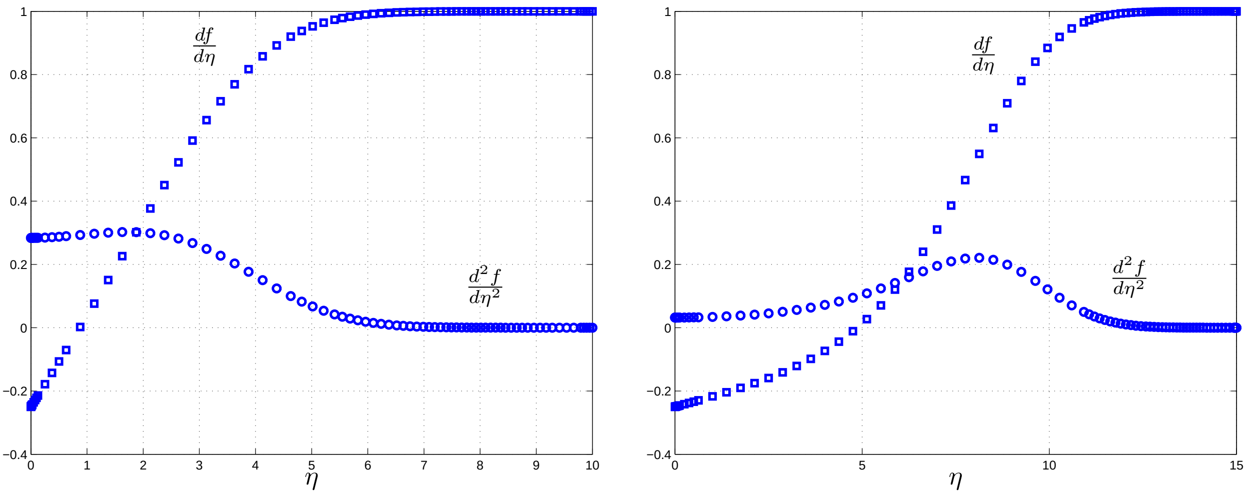

Figure 3: Blasius problem with moving plate boundary conditions. The two different solutions for $P = 0.25$.

condition at infinity, and the following non-homogeneous boundary conditions at $\eta = 0$

$$f(0) = 0 \ , \qquad \frac{df}{d\eta}(0) = -P \ , \tag{9}$$

where P is the ratio of the boundary velocity to the free stream velocity. Klemp and Acrivos studied the effect of the parameter P on the boundary layer thickness. For $P > 0$, two solutions exist only for P less than a critical value P_c, as shown numerically by Hussaini and Lakin [13]. These authors found a numerical value of P_c equal to 0.3541. Hussaini et al. [14] proved the nonuniqueness and analyticity of solutions for $P \leq P_c$, and derived the upper bound 0.46824 for P_c.

More recently, a modified Blasius equation, taking into account the effect of P on the boundary layer thickness, has been introduced by Allan [1]. Moreover, Allan and Syam [2], using an homotopy analysis method, define an implicit relation between the wall shear stress and the moving wall parameters. The study of these relation shows that two solutions exist when $P \leq P_c \approx 0.354\ldots$, one solution exists for $P = P_c$ and no solution exists for $P > P_c$.

We have used the ITM in order to investigate the existence and uniqueness question for the Blasius model on a moving plate. For the modified problem we defined the boundary condition

$$\frac{df}{d\eta}(0) = -h \ P$$

and used the extended scaling group

$$f^* = \lambda f \ , \qquad \eta^* = \lambda^{-1}\eta \ , \qquad h^* = \lambda^2 h \ . \tag{10}$$

so that λ is defined by

$$\lambda = \left[\frac{df^*}{d\eta^*}(\infty)\right]^{1/2} \ . \tag{11}$$

Here we report on sample numerical results. First let us consider the case $P = 0.25$. Sample numerical results are reported in Table 1. In this case $\Gamma(\cdot)$ has two differ-

h^*	$\Gamma(h^*)$	h^*	$\Gamma(h^*)$
1.	-0.528765	10.	-0.008382
3.	0.198514	3.	0.198514
2.454091	0.045354	9.716410	0.016266
2.339221	0.008091	9.903562	$-1.52 \cdot 10^{-4}$
2.319037	0.001371	9.901831	$-2.96 \cdot 10^{-6}$
2.315626	$2.21 \cdot 10^{-4}$	9.901798	$-5.79 \cdot 10^{-8}$
2.315076	$3.87 \cdot 10^{-5}$	.	.
2.314979	$6.78 \cdot 10^{-6}$	.	.
2.314963	$1.19 \cdot 10^{-6}$	.	.
2.314960	$2.08 \cdot 10^{-7}$	.	.
$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$
-0.25	0.283928	-0.25	0.032094

Table 1: Fluid flow on a moving plate: numerical results by the ITM.

ent zeros. Figure 3 shows the two corresponding solution found by the ITM. It is evident, from the two frames of this figure, that the truncated boundary approach has to be supplemented by some preliminary numerical experiments and this is more relevant in the case of non uniqueness of solution. In fact, if we set $\eta_\infty = 10$, then we miss the solution shown in the right frame of figure 3. For the ITM we used the convergence criterion $|\Gamma(\cdot)| < 10^{-6}$. On the contrary, if we set $P = 0.5$, then we find that $\Gamma(\cdot)$ has always the same sign, so that no solution is available for such a case.

3.2 Slip flow condition

We consider now the case of a rarefied flow where the no-slip condition at the wall, considered in the previous section, must be replaced by a slip-flow condition, see for instance Gad-el-Hak [6]. For an isothermal wall, the slip condition can be defined as

$$v(y,0) = \frac{2-\sigma}{\sigma}\ell\frac{\partial v}{\partial z}(y,0) \ ,$$

where ℓ is the mean free path, and σ is the tangential momentum accommodation coefficient. Within a similarity transformation this slip boundary condition becomes

$$\frac{df}{d\eta}(0) = P\,\frac{d^2 f}{d\eta^2}(0) \ ,$$

where P is a non-dimensional parameter, that takes into account the behaviour at the surface, defined by

$$P = \frac{2-\sigma}{\sigma} Kn\ Re\ y^{1/2} \ ,$$

where Kn and Re are the Knudsen and Reynolds numbers based on y.

For the Blasius problem with slip condition we implemented both the non-ITM and the ITM. In order to apply the non-ITM we had to require that P is a parameter involved in the scaling invariance, i.e., we defined the extended scaling group

$$f^* = \lambda f \ , \qquad \eta^* = \lambda^{-1}\eta \ , \qquad P^* = \lambda^{-1}P \ . \tag{12}$$

As far as the application of the ITM is concerned, we used a modified problem with the boundary condition

$$\frac{df}{d\eta}(0) = h\ P\,\frac{d^2 f}{d\eta^2}(0) \ ,$$

and the extended scaling group

$$f^* = \lambda f \ , \qquad \eta^* = \lambda^{-1}\eta \ , \qquad h^* = \lambda^{-1}h \ . \tag{13}$$

Henceforth, in both cases λ is defined, once again, by equation (11).

Sample numerical results are reported in Tables 2 and 3. Figure 4 shows a sample numerical integration for $P = 1.562257$. Note that the solution of the Blasius problem with slip flow condition was computed by rescaling.

For the ITM, we always used $h_0^* = 0.1$ and $h_1^* = 1$ but for the case $P = 50$ where, in order to speed up the convergence, we set $h_1^* = 0.5$. By setting $|\Gamma(\cdot)| < 10^{-6}$, as a convergence criterion, the Regula Falsi method converged within 8 iterations in all cases.

P^*	$\frac{df^*}{d\eta^*}(\infty)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	P
0.	2.085393	0.	0.332061	0.
0.1	2.090453	0.047836	0.330856	0.144584
0.5	2.191907	0.228112	0.308153	0.740255
1.	2.440648	0.409727	0.262266	1.562257
5.	5.771518	0.866323	0.072122	12.011992
10.	10.554805	0.947436	0.029162	32.488159
20.	20.394883	0.980638	0.010857	90.321389
25.	25.353618	0.986053	0.007833	125.880941

Table 2: Non-iterative numerical results.

$\frac{df^*}{d\eta^*}(\infty)$	$\frac{df}{d\eta}(0)$	$\frac{d^2f}{d\eta^2}(0)$	P
2.085393	0.	0.332061	0.
2.087710	0.033151	0.331509	0.1
2.262516	0.293841	0.293841	1.
3.644351	0.718686	0.143737	5.
5.203210	0.842545	0.084255	10.
13.894469	0.965399	0.019308	50.

Table 3: Slip boundary condition: numerical results by the ITM.

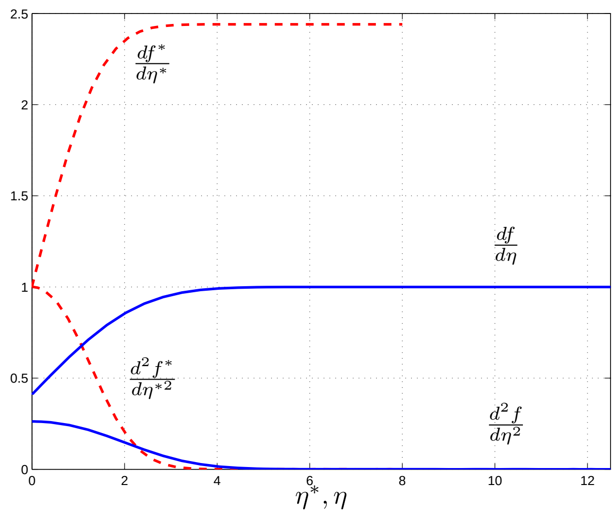

Figure 4: Blasius problem with slip condition. Numerical solution by a non-ITM with $P^* = 1$ and $P = 1.562257$.

4 Future Work

The ideas outlined in this paper can be applied to other problems of boundary-layer theory. As an example let us consider the Falkner-Skan equation with relevant boundary conditions

$$\begin{aligned}
&\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0 \\
&f(0) = \frac{df}{d\eta}(0) = 0 \ , \qquad \frac{df}{d\eta}(\infty) = 1 \ ,
\end{aligned} \tag{14}$$

where f and η are appropriate similarity variables and β is a parameter. This problem describes the flow of a fluid past a wedge, see [7]. The application of the ITM to (14) has been reported in [9] but only in the simple case where $0 \leq \beta \leq 1$. It is well known that the case $\beta > 1$ is more interesting, because the Falkner-Skan model loses the uniqueness property and a hierarchy of solution with reversed flow exists [5].

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