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An implicit difference scheme for a moving boundary hyperbolic problem

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Abstract

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In this paper an implicit difference scheme is defined for a moving boundary hyperbolic problem, which describes a shock front propagation in a constant state. We have reformulated the problem to a fixed boundary domain where an implicit difference scheme is proposed. As is well known, the equivalent condition for the convergence of a consistent scheme is its stability. However, the only reliable methods of stability analysis are based on linear theory. Moreover, the pertinent literature provides simple examples where the linearization of a nonlinear scheme leads to incorrect stability results. On an experimental basis a discrete perturbation stability analysis was then considered. In order to investigate the convergence of the scheme we considered a particular example where an approximate similarity solution is known. In this case, we point out the numerical convergence of the scheme. Even more important is that a possible way to assess the numerical accuracy when the similarity solution does not exist is suggested.

1. Formulation and background

The aim of this paper is to define a convergent difference scheme for a moving boundary hyperbolic problem. Moving boundary problems are characterized by an unknown boundary that has to be determined as part of the solution. The considered problem describes a shock front propagation in a constant state. A preliminary note on this subject has already been presented by the authors in [7].

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Let us introduce the mathematical formulation of the problem: we would like to find $x^s(t)$, $v(x, t)$, and $e(x, t)$ {or $\sigma(x, t)$} such that

$$\rho \frac{\partial v}{\partial t} + \frac{\partial \sigma}{\partial x} = 0, \qquad \rho > 0, \quad \sigma = \mu e^{1/q},$$

$$\frac{\partial e}{\partial t} + \frac{\partial v}{\partial x} = 0, \tag{1.1a}$$

are satisfied together with the initial and boundary conditions

$$v(x, 0) = 0, \qquad e(x, 0) = e_0, \quad e_0 \geqslant 0 \quad \left\{\sigma(x, 0) = \mu(e_0)^{1/q}\right\}, \tag{1.1b}$$

$$v(0, t) = v_0 t^\delta H(t), \qquad v_0 > 0, \quad \delta > -1, \tag{1.1c}$$

$$x^s(0) = 0, \tag{1.1d}$$

$$\rho \frac{dx^s}{dt} [\![v(x^s(t), t)]\!] - [\![\sigma(x^s(t), t)]\!] = 0, \tag{1.1e}$$

$$\frac{dx^s}{dt} [\![e(x^s(t), t)]\!] - [\![v(x^s(t), t)]\!] = 0. \tag{1.1f}$$

The problem (1.1) is a mathematical model describing the shock front propagation due to a time-dependent velocity impact at the end of a thin rod (see Donato [4]). Let us denote by $u(x,t)$ the displacement at time t of the particle of the rod with reference position x (i.e., $x + u(x, t)$ the position at time t of this particle). Compressive velocity and strain fields are defined respectively by $v(x, t) = (\partial/\partial t)u(x, t)$ and by $e(x, t) = (\partial/\partial x)u(x, t)$. In this context, $\sigma(x, t)$ represents the compressive stress. Moreover, ρ, μ, q, e_0, v_0, and δ are constants, $H(t)$ is the Heaviside step function, $x^s(t)$ represents the shock front and the notation $[\![\cdot]\!]$ indicates a jump across it. The case $e_0 = 0$ is related to the physical situation of the rod initially at rest.

Here, the shock front propagation into the known state ahead of the shock—that is given by (1.1b)—is due to the time-dependent velocity impact (1.1c). Then, the Rankine–Hugoniot conditions (1.1e)–(1.1f) relate the speed of the shock and the states behind with each other. As is well known, impact problems, the point explosion problem (see Sedov [23] and Taylor [25]), and the dam breaking problem (see Grundy and Rottman [11]) are due to a shock at the initial time. We remark again that the state ahead of the shock is known. Hence, the system (1.1a) has to be hyperbolic at least for $0 \leqslant x \leqslant x^s(t)$.

Wave propagation problems governed by (1.1a), but with different boundary conditions, were studied analytically by Frydrychowicz and Singh [10] and numerically by Fazio [5].

The similarity analysis of (1.1) where $q = \frac{1}{2}$, $\delta = 1$, $\mu/\rho = 1$, $v_0 = 1$, and $e_0 = 0$ is given by Dresner [3, pp. 77–87]. This particular problem has a physical meaning. It is related to the sudden release of a weight, under the effect of gravity, suspended from a thin rod. Henceforth, we will refer to this problem as the signalling problem. As far as representative results are concerned we can consider

$$x^s(t) = \eta^s t^{4/3}, \qquad e(0, t) = E(0)t^{2/3}, \qquad e(x^s(t), t) = E(\eta^s)t^{2/3},$$

where we found, by numerical means, $\eta^s = 0.587458$, $E(0) = 1.026560$, and $E(\eta^s) = 0.613523$. The similarity analysis is extended to the general case by Donato [4]; but only for $e_0 = 0$. As reported by Seshadri and Singh [24] rubber and some metals can be characterized by means of the constitutive law in (1.1a) where $0 < q < 1$. Let us remark here that exact solutions are crucial elements in the proper development of numerical schemes. Moreover, they are, sometimes, asymptotics for more complicated solutions after the decay of irrelevant transients induced by the initial conditions, see Barenblatt [1].

The numerical approach considered herein is useful for problems describing shock propagation in a constant or in a non-constant state. The problem (1.1) can also be considered if instead of (1.1c) we have the following boundary condition

$$e(0, t) = \left(\mu^{-1}\sigma_0\right)^q t^{\gamma q} H(t),$$

where $\sigma_0 > 0$ and $\gamma > -1$. Again this model, which describes a time-dependent stress impact, can be considered for a thin rod (see Fazio [6] and the references quoted therein). In the class mentioned previously another problem of interest is given, for instance, by the gravity-current releases studied by Grundy and Rottman [11]. Problems where contact discontinuities or development of shocks are involved need to be treated either by shock-fitting or shock-capturing methods. In the former case, the positions of the discontinuities are kept as separate variables and the solution is computed by standard methods in the other regions of interest (see Richtmyer and Morton [20, pp. 308–311, 378–383] and also Mao [17]). In the latter case some dissipative mechanism is involved in order to avoid spurious oscillations (see for instance the ENO scheme defined by Harten and Osher [13] and Harten et al. [14]).

The main difficulty in solving our moving boundary problem by a difference scheme, defined in the original physical domain, arises at the moving boundary. For the problem (1.1) we know in advance that the moving boundary is an increasing function of time and therefore it would be inadequate to use fixed spaced grids over the whole region of interest. A different way to deal with (1.1) is to introduce a transformation of variables whereby fixing the moving boundary. Here we follow the second approach. In the next section we reformulate problem (1.1) to a fixed boundary domain by a Landau-like transformation [15]. This approach can be extended in two or three space dimensions (see Gupta and Kumar [12]). Then, in the same section we propose an implicit difference scheme for (1.1) rewritten in the fixed domain. The implicit formulation is of practical interest since very large time steps can be used (see Section 4).

In Section 3 we discuss the practical application of the difference scheme introduced and point out a variant of the resolution process.

As stated before we are interested in a convergent scheme. As is well known, the main concepts to play a role in the theory of numerical approximation for evolution problems are those of consistency, stability, and convergence. For properly posed nonlinear problems the equivalence theorem states that for a consistent scheme stability and convergence are equivalent (see Rosinger [21]). Therefore, the question reduces to prove consistency and stability. Here a major difficulty arises. In fact, the most reliable methods to study the stability of a difference scheme, namely the von Neumann and the matrix methods, are applicable only to linear or linearized schemes. However, both nonlinear instability (Fornberg [9]) and nonlinear overstability (Rosinger [22]) have been assessed to show that the linearization of nonlinear schemes can lead to incorrect stability results. An alternative approach for the stability analysis is the so-called energy method (see Richtmyer and Morton [20, Chapter 6]). This approach can also be used for nonlinear problems [20, p. 142], but its applicability to a given difference scheme is not always straightforward. Therefore, in Section 4 we use a discrete perturbation stability analysis (Collatz [2, pp. 268–276] and Noye [18, pp. 124–129]) and we point out the numerical convergence ([18, pp. 119–123] and Fletcher [8, pp. 75–76]) of our scheme when it is applied to the signalling problem.

In the final section we indicate a possible way to deal with the numerical accuracy even when the problem (1.1) does not admit a similarity solution, namely for $\delta \neq 0$ and $e_0 > 0$.

2. Fixed boundary formulation and the implicit difference scheme

The moving boundary can be fixed at $X = 1$ for all values of t greater than zero if we define X as follows

$$X = \frac{x}{x^{\mathrm{s}}(t)}. \tag{2.1}$$

This definition of X is indeterminate at $t = 0$, but since $X \in [0, 1]$ for every $t > 0$ for continuity reasons we assume that $X \in [0, 1]$ also in the limit $t = 0$. After (2.1) the moving boundary problem becomes

$$\begin{aligned} &\rho \frac{\partial v}{\partial t} - \rho \frac{X}{x^{\mathrm{s}}(t)} \frac{\mathrm{d}x^{\mathrm{s}}}{\mathrm{d}t} \frac{\partial v}{\partial X} + \frac{1}{x^{\mathrm{s}}(t)} \frac{\partial \sigma}{\partial X} = 0, \quad \sigma = \mu e^{1/q}, \\ &\frac{\partial e}{\partial t} - \frac{X}{x^{\mathrm{s}}(t)} \frac{\mathrm{d}x^{\mathrm{s}}}{\mathrm{d}t} \frac{\partial e}{\partial X} + \frac{1}{x^{\mathrm{s}}(t)} \frac{\partial v}{\partial X} = 0, \end{aligned} \tag{2.2a}$$

with initial and boundary conditions

$$v(X, 0) = 0, \qquad e(X, 0) = e_0, \tag{2.2b}$$

$$v(0, t) = v_0 t^{\delta} H(t), \qquad v_0 > 0, \quad \delta > -1, \tag{2.2c}$$

$$x^{\mathrm{s}}(0) = 0, \tag{2.2d}$$

$$[\![v(1, t)]\!] = \left(\rho^{-1} [\![\sigma(1, t)]\!] [\![e(1, t)]\!]\right)^{1/2}, \tag{2.2e}$$

$$\frac{\mathrm{d}x^{\mathrm{s}}}{\mathrm{d}t} = \left(\frac{[\![\sigma(1, t)]\!]}{\rho [\![e(1, t)]\!]} \right)^{1/2}. \tag{2.2f}$$

In (2.2e)–(2.2f) a more convenient form for the Rankine–Hugoniot conditions has been used. We shall discuss later the usefulness of (2.2e)–(2.2f).

Let us suppose that we are interested in the numerical solution of (2.2) up to a finite time T. For instance in the case of a finite rod, we may ask ourselves when will the shock front arrive at the end of the rod. We may also be concerned with, in the case of a semi-infinite rod, the time needed by the shock front in order to reach a fixed distance. In the domain of interest we introduce a network of grid points with spacings ΔX and Δt. Let $N = 1/\Delta X$ and $M = (T - \Delta t)/\Delta t$, so that $i = 0,1,2,\ldots,N$ and $j = 0,1,2,\ldots,M$. A first-order implicit difference scheme for the problem (2.2) is as follows: at each time step we solve the nonlinear system of equations

$$\begin{aligned} &v_{i,j+1} - v_{i,j} - i\frac{\left(x_{j+1}^{\mathrm{s}} - x_j^{\mathrm{s}}\right)}{x_{j+1}^{\mathrm{s}}}\left(v_{i,j+1} - v_{i-1,j+1}\right) \\ &\qquad + \frac{\mu C}{\rho x_{j+1}^{\mathrm{s}}}\left(\left(e_{i,j+1}\right)^{1/q} - \left(e_{i-1,j+1}\right)^{1/q}\right) = 0, \\ &e_{i,j+1} - e_{i,j} - i\frac{\left(x_{j+1}^{\mathrm{s}} - x_j^{\mathrm{s}}\right)}{x_{j+1}^{\mathrm{s}}}\left(e_{i,j+1} - e_{i-1,j+1}\right) \\ &\qquad + \frac{C}{x_{j+1}^{\mathrm{s}}}\left(v_{i,j+1} - v_{i-1,j+1}\right) = 0, \end{aligned} \tag{2.3a}$$

where

$$e_{i,0} = e_0, \qquad v_{i,0} = 0, \tag{2.3b}$$

$$v_{0,j+1} = \left((j+1)\Delta t\right)^{\delta}, \tag{2.3c}$$

$$x_0^{\mathrm{s}} = 0, \tag{2.3d}$$

$$v_{N,j+1} = \left(\frac{\mu}{\rho}\left(\left(e_{N,j+1}\right)^{1/q} - \left(e_0\right)^{1/q}\right)\left(e_{N,j+1} - e_0\right)\right)^{1/2}, \tag{2.3e}$$

$$x_{j+1}^{\mathrm{s}} = x_j^{\mathrm{s}} + \Delta t\left(\frac{\mu\left(\left(e_{N,j+1}\right)^{1/q} - \left(e_0\right)^{1/q}\right)}{\rho\left(e_{N,j+1} - e_0\right)}\right)^{1/2}, \tag{2.3f}$$

for $i = 1,2,\ldots,N$. Here we approximate $v(i\Delta X,\ j\Delta t)$ by $v_{i,j}$, $e(i\Delta X,\ j\Delta t)$ by $e_{i,j}$, and $C = \Delta t/\Delta X$ is the Courant number.

The finite difference scheme (2.3) has the following truncation errors:

$$\begin{aligned} &-\frac{\Delta t}{2}\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j+1} - i\frac{\Delta X}{x_{j+1}^{\mathrm{s}}}\frac{\Delta t}{2}\frac{\Delta X}{2}\left.\frac{\mathrm{d}^2 x^{\mathrm{s}}}{\mathrm{d}t^2}\right|_{j+1}\left.\frac{\partial^2 v}{\partial X^2}\right|_{i,j+1} \\ &\qquad - \frac{\mu}{\rho x_{j+1}^{\mathrm{s}}}\frac{\Delta X}{2}\left.\frac{\partial^2\left(e^{1/q}\right)}{\partial X^2}\right|_{i,j+1}, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} &-\frac{\Delta t}{2}\left.\frac{\partial^2 e}{\partial t^2}\right|_{i,j+1} - i\frac{\Delta X}{x_{j+1}^{\mathrm{s}}}\frac{\Delta t}{2}\frac{\Delta X}{2}\left.\frac{\mathrm{d}^2 x^{\mathrm{s}}}{\mathrm{d}t^2}\right|_{j+1}\left.\frac{\partial^2 e}{\partial X^2}\right|_{i,j+1} \\ &\qquad - \frac{1}{x_{j+1}^{\mathrm{s}}}\frac{\Delta X}{2}\left.\frac{\partial^2 v}{\partial X^2}\right|_{i,j+1}, \end{aligned} \tag{2.4b}$$

$$\frac{\Delta t}{2}\left.\frac{\mathrm{d}^2 x^{\mathrm{s}}}{\mathrm{d}t^2}\right|_{j+1}. \tag{2.4c}$$

Therefore, (2.3) is consistent because the truncation errors (2.4) tend to zero as both ΔX and Δt tend to zero.

Let us remark here that, by making the appropriate adjustments, the scheme defined above will also apply to the case of a shock front propagation in a non-constant state.

3. The double iterative process and its variant

Let us now discuss the practical solution of the discrete scheme (2.3). At each time level we get from (2.3) a nonlinear system of the following general form

$$\boldsymbol{R}(z)=\boldsymbol{0}. \tag{3.1}$$

Here $z=(e_{0,j+1}, v_{1,j+1}, e_{1,j+1},\ldots, v_{N-1,j+1}, e_{N-1,j+1}, e_{N,j+1})^{\mathrm{T}}$ and $\boldsymbol{R}$ can be seen as a function of the $2N$-dimensional Euclidean space $\mathbb{R}^{2N}$ into itself. We can appreciate that the condition (2.3e) is what is needed in order to obtain a system of $2N$ equations in $2N$ unknowns. Likewise, the two conditions (2.3d) and (2.3f) can be used to define the moving boundary position. A root-finder has to be applied here because (2.3f) defines x^{s}_{j+1} implicitly. In fact, the value of $e_{N,j+1}$ being a component of the solution of (3.1) depends on the value of x^{s}_{j+1}. Therefore, we can consider x^{s}_{j+1} as a zero of the implicit function

$$f(x^{\mathrm{s}}_{j+1})=x^{\mathrm{s}}_{j+1}-x^{\mathrm{s}}_{j}-\Delta t\left(\frac{\mu\left((e_{N,j+1})^{1/q}-(e_0)^{1/q}\right)}{\rho(e_{N,j+1}-e_0)}\right)^{1/2}.$$

This means that in order to find the shock front location we use an iterative procedure at each time step.

Here we suggest to apply the simple secant method because it does not need the computation of the first derivative of $f(\cdot)$ and its order of convergence is close to that of Newton's method. However, at the first time step $j=0$ we have to use two guessed values in order to start the secant method. As a result the residual for the function $f(\cdot)$ is of interest in order to have a trial for the guessed values. On the other hand as we proceed further in time, i.e. for $j=1, 2,\ldots, M$, the shock front location x^{s}_j and the value given by

$$x^{\mathrm{s}}_j+\Delta t\left(\frac{\mu\left((e_{N,j})^{1/q}-(e_0)^{1/q}\right)}{\rho(e_{N,j}-e_0)}\right)^{1/2} \tag{3.2}$$

can be used as starting values.

The nonlinear system (3.1) can be solved by the Newton method

$$J(z^k)\Delta z=-\boldsymbol{R}(z^k), \tag{3.3}$$

where J is the Jacobian matrix of $\boldsymbol{R}$ with respect to z and $\Delta z=z^{k+1}-z^k$. Provided the initial guess z^0 is close enough to the solution of (3.1) the Newton method converges quadratically (see Ortega and Rheinboldt [19, pp. 310–313]). Now, we have for z^0 good hints since we can use $(\Delta t)^\delta \mathbf{1}$ for $j = 0$—here $\mathbf{1}$ represents the vector in $\mathbb{R}^{2N}$ with all components equal to unity—and the solution at the previous time step for all $j = 1,2,\ldots, M$. (3.3) is a linear system with a pentadiagonal matrix. This allows us to apply a direct method (see Mann [16, pp. 593–596]).

It is now evident that we have to deal with a double iterative process. For the convergence of the shock front location we require the absolute value of the difference between two subsequent iterations to be less than a prefixed tolerance, say TOL. The accepted value may then be used in order to check the residual of the equation. In a similar way, if the maximum absolute value of the components of the difference of two subsequent iterations is less than, say, TOLL we accept the convergence of the Newton method. Again, the accepted values may be used to compute the residual in the root mean square (rms) norm for the system (3.1). The two residuals introduced so far, along with the numerical values of TOL and TOLL, are then indicators of the convergence of the double iterative process.

A quicker way to solve the problem at hand is to apply the iteration to approximate the shock front only at the first time step. This means that the moving boundary position given by (3.2) is accepted without iterations after the first time step.

4. Numerical results

As a practical application of our difference scheme we considered its application to the signalling problem. The same numerical value was used for TOL and TOLL throughout the computations, that is 1E–06. Some numerical experiments were carried out, by a discrete perturbation stability analysis, in order to assess the stability of our scheme. In the classical theoretical setting, the analysis has to be restricted to errors introduced in the initial and boundary conditions. Moreover, the analysis can be extended to consider also the effect of round-off errors. In this case, a discrete error is introduced at arbitrary grid-points and its effect on the computation of the solution given by the finite difference scheme is examined. This includes for instance, the effect of errors of the same magnitude with the same or alternating sign. The stability of the scheme is equivalent to the decay (or at least non-growth) of any discrete error as the computation proceeds. It is evident that an exhaustive application of the above methodology is quite tedious. Moreover, since our scheme was implicit and nonlinear we applied this analysis on an experimental basis. At each experiment we solved the signalling problem twice: once without and then with the discrete error. Table 1 lists two sample cases related to the value $v_{0,1} = \Delta t + p$ at the fixed boundary $t = 0$, where p is an error for the correct boundary value. As indicated before this means that at the first time step the first guess for the Newton method was given by $z^0 = (\Delta t + p)\mathbf{1}$. In Table 1 we limit ourselves to verify the effect of the introduced error only on x_j^s, $j = 1,2,\ldots, M$. We have to check also the effect on $v_{i,j}$ and $e_{i,j}$ for $i = 1,2,\ldots, N-1$, $j = 1,2,\ldots, M$. Let us remark that in Table 1 we have used a value of p of the same order with respect to $v_{0,1}$. As far as the computation related to the first column of Table 1 is concerned the absolute values of the residual for the shock front equation and the absolute values of the residual in the root mean square (rms) norm for the nonlinear system (3.1) were smaller than 2E–07 and 6E–09 respectively. More numerical

Table 1
$x^s(t)$, discrete perturbation stability analysis; $C = 40$

t	ΔX			
	0.005	0.005 $p = 0.1$	0.00125	0.00125 $p = 0.1$
0.05			0.012461	0.017971
0.1			0.029225	0.039218
0.2	0.079121	0.090570	0.071071	0.085723
0.4	0.185556	0.204961	0.176294	0.190126
0.6	0.310991	0.334694	0.301131	0.311674
0.8	0.451206	0.476552	0.440681	0.448981
1.0	0.603632	0.628965	0.592328	0.599228
1.2	0.766543	0.790952	0.754389	0.760366
1.4	0.938704	0.961764	0.925674	0.931001
1.6	1.119184	1.140759	1.105282	1.110126
1.8	1.307259	1.327369	1.292505	1.296975
2.0	1.502345	1.521086	1.486771	1.490937

experiments were performed in order to test the stability of the scheme with respect to discrete errors. All the results clearly indicated the stability of the scheme (2.3) for the problem under consideration.

In Appendix A Tables A.1, A.2, and A.3 report the comparison between the numerical results and the approximate similarity solution. The same value of the Courant number, namely $C = 40$, was used consistently for four values of the space step. The numerical convergence has been made evident by computing the errors in the rms norm. For the reader's convenience the errors in the rms norm related to Tables A.1, A.2, and A.3 have been plotted against the space step, on a log by log scale, in Fig. 1. From Fig. 1 it is easily seen that the order of convergence is

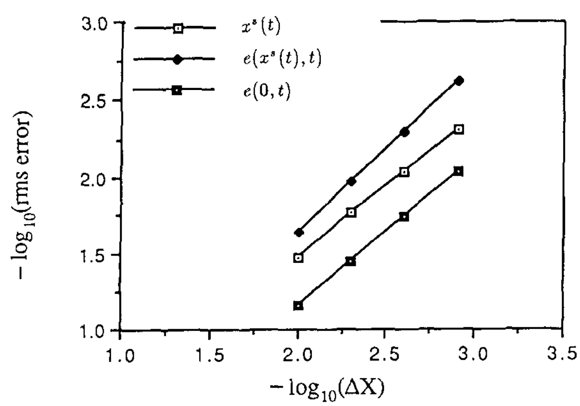

Fig. 1. Numerical convergence.

$O(\Delta X)$. Further evidence from Fig. 1 is the following: the numerical accuracy of 1%, i.e. $-\log_{10}$ (rms error) = 2, has been obtained by the results found with $\Delta t = 0.05$. Higher-order schemes are worth considering only if a security problem is concerned, say for instance in the design of nuclear reactors.

The numerical results for $x^s(t)$, again for $C = 40$, obtained by the variant of the double iterative process proposed in the previous section are listed in Table A.4. By comparison of the errors in the rms norms it is easily seen that the former approach (iteration at all time steps) is more accurate than the latter one (iteration only at the first time step). However, the errors in the rms norm listed in Table A.4 indicate numerical convergence as well.

5. Conclusions

In this section we would like to point out the main evidence from the present work.

Here we defined an implicit difference scheme for a shock front propagation in a constant state. The implicit formulation is of practical interest since very large time steps can be used, significantly reducing the computer time needed to solve a specific problem. In particular, the time step can be increased, in comparison with standard explicit schemes, by factors of thirty or more (see Section 4). The signalling problem of Section 1 may be considered as a benchmark problem since we know an approximate similarity solution for it. This particular problem was considered in order to test our difference scheme. The stability of the scheme was investigated by a discrete stability analysis. Moreover, we verified the convergence of the numerical solution to the approximate similarity solution.

As a first remark, we would like to underline that the fixed boundary formulation allows us to follow the shock front propagation accurately. In this context, the present work suggests that implicit difference schemes are more suitable than explicit ones for problem (2.2) and similar type problems. In fact, (2.2a) and the condition (2.2d) prevent us to consider an explicit difference scheme.

As a further remark the proposed approach can also be applied to problems with a shock front propagation in a non-constant state.

The numerical results listed in Appendix A allow us to make a last remark on their meaning. Let us forget here the approximate similarity solution and the related errors in the rms norm. We can ask ourselves: is there any way to assess the accuracy inherent in the numerical solution? In order to answer this question the following point is of great importance. The numerical results of Table A.1 and Table A.4—except at the first time step, obviously—are monotonically increasing and decreasing with respect to the space step size respectively. Since those of Table A.1 are always greater than those of Table A.4 they approximate from above and below the exact shock front location. As a consequence *the common decimals in the corresponding entries of Tables A.1 and A.4 are correct*. In other words we have here a numerical inclusion for the shock front position. To conclude we have enough evidence to suggest that the variant of the double iteration process, namely iterative only at the first time step, is worth considering also when a similarity solution for the problem (1.1) does not exist, that is for $\delta \neq 0$ and $e_0 > 0$.

Acknowledgments

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Appendix A

Table A.1
$x^s(t)$, iteration for all time steps

t	ΔX				similarity
	0.01	0.005	0.0025	0.00125	
0.05				0.012461	0.010821
0.1			0.031399	0.029225	0.027267
0.2		0.079121	0.073642	0.071071	0.068709
0.4	0.199372	0.185556	0.179079	0.176294	0.173137
0.6		0.310991	0.304239	0.301131	0.297289
0.8	0.467528	0.451206	0.444204	0.440681	0.436278
1.0		0.603632	0.596281	0.592328	0.587458
1.2	0.783526	0.766543	0.758755	0.754389	0.749121
1.4		0.938704	0.930420	0.925674	0.920056
1.6	1.136747	1.119184	1.110377	1.105282	1.099352
1.8		1.307259	1.297919	1.292505	1.286294
2.0	1.520726	1.502345	1.492478	1.486771	1.480301
rms error	0.034296	0.017044	0.009050	0.004831	

Table A.2
$e(0, t)$, iteration for all time steps

t	ΔX				similarity
	0.01	0.005	0.0025	0.00125	
0.05				0.129203	0.139326
0.1			0.205097	0.215489	0.221166
0.2		0.325572	0.342065	0.348187	0.351079
0.4	0.516813	0.542985	0.552703	0.555478	0.557303
0.6		0.720715	0.726864	0.728746	0.730273
0.8	0.861909	0.877334	0.881750	0.883299	0.884663
1.0		1.020410	1.023926	1.025308	1.026560
1.2	1.144004	1.153784	1.156790	1.158067	1.159236
1.4		1.279707	1.282399	1.283599	1.284704
1.6	1.392591	1.399640	1.402123	1.403262	1.404315
1.8		1.514596	1.516931	1.518020	1.519029
2.0	1.619688	1.625321	1.627543	1.628590	1.629563
rms error	0.022909	0.010736	0.005032	0.002365	

Table A.3
$e(x^s(t), t)$, iteration for all time steps

t	ΔX				similarity
	0.01	0.005	0.0025	0.00125	
0.05				0.062107	0.083268
0.1			0.098589	0.112425	0.132180
0.2		0.156512	0.178443	0.195138	0.209822
0.4	0.248434	0.283213	0.309718	0.323301	0.333072
0.6		0.393346	0.418101	0.428324	0.436447
0.8	0.449423	0.491508	0.513137	0.521355	0.528718
1.0		0.580841	0.599540	0.606629	0.613523
1.2	0.624090	0.663502	0.679835	0.686266	0.692817
1.4		0.740981	0.755506	0.761525	0.767803
1.6	0.779786	0.814331	0.827497	0.833239	0.839288
1.8		0.884305	0.896456	0.901991	0.907847
2.0	0.921498	0.951460	0.962846	0.968220	0.973901
rms error	0.069946	0.035916	0.018081	0.009140	

Table A.4
$x^s(t)$, iteration only at the first time step

t	ΔX				similarity
	0.01	0.005	0.0025	0.00125	
0.05				0.012461	0.010821
0.1			0.031399	0.024921	0.027267
0.2		0.079121	0.062798	0.064472	0.068709
0.4	0.199372	0.158241	0.162454	0.167693	0.173137
0.6		0.274595	0.284711	0.291216	0.297289
0.8	0.398745	0.409337	0.422543	0.429719	0.436278
1.0		0.557740	0.572853	0.580481	0.587458
1.2	0.691885	0.717375	0.733785	0.741769	0.749121
1.4		0.886706	0.904072	0.912363	0.920056
1.6	1.031351	1.064658	1.082774	1.091344	1.099352
1.8		1.250427	1.269162	1.277992	1.286294
2.0	1.405237	1.443378	1.462649	1.471724	1.480301
rms error	0.055914	0.029031	0.014168	0.006840	

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