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A moving boundary hyperbolic problem for a stress impact in a bar of rate-type material

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In this paper we present some results obtained by studying the mathematical model describing a moving boundary hyperbolic problem related to a time dependent stress impact in a bar of Maxwell-like material. Due to the impact a shock front propagates with a finite speed. Here our interest is to underline the influence of the dissipative term on the propagation of the shock front.

In the framework of the similarity analysis we are able to reduce the moving boundary hyperbolic problem to a free boundary value problem for an ordinary differential system. It is then possible, by applying two numerical transformation methods, to solve the free boundary value problem numerically. The influence of the dissipative term is evident: the free boundary (that defines the shock front propagation) is an increasing function of the dissipative coefficient.

1. Introduction and formulation

This paper presents a complete similarity analysis of a moving boundary hyperbolic problem. Some classical moving boundary hyperbolic problems can be studied by means of the similarity analysis. For instance this is the case of: blast waves due to a point explosion [1, 2], transverse waves in shock-loaded membrane [3], rainfall-runoff in sloping areas [4], or longitudinal waves propagating along a thin rod due to a velocity or a stress impact [5].

Recently the mathematical model describing the longitudinal wave propagation induced by a velocity impact in a thin rod has received a great deal of attention. Analytical investigations on it have been carried on in [6, 7] whereas numerical results have been obtained in [8, 9]. Moreover, the stress impact problem is considered in [10].

Here we shall consider a longitudinal shock front propagation determined by a stress impact at one end of a bar of rate-type material. We take into consideration Maxwell materials described by the following governing system [11–13]

$$\rho_0 \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0, \qquad \frac{\partial \sigma}{\partial t} - \Phi(\sigma) \frac{\partial v}{\partial x} = -\Psi(\sigma). \tag{1.1a}$$

In (1.1a) v is the component of velocity along the axis of the bar, σ is the tensile stress, ρ_0 and x are respectively the reference density and longitudinal coordinate at reference time $t=0$, whereas $\Phi(\sigma) = \Phi_0 \sigma^{1/p}$ and $\Psi(\sigma) = \Psi_0 \sigma^{1/q}$ (Φ_0, Ψ_0, p and q are constants) are material response functions, see [11]. Let us remark here that the governing system (1.1a) takes into account nonlinear instantaneous response and nonlinear viscoplastic response for the considered material. The system (1.1a), provided $\Phi(\sigma) > 0$, is hyperbolic.

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As far as the stress impact problem is concerned we have to consider the boundary condition

$$\sigma(0, t) = \sigma_0 t^{\delta} H(t) \tag{1.1b}$$

where σ_0 is a reference stress δ is a constant and $H(\cdot)$ is the Heaviside step function. We assume that the boundary condition (1.1b) determines the propagation of a shock front $x_s(t)$ into the bar initially at rest, i.e.,

$$v(x > x_s(t), t) = 0, \qquad \sigma(x > x_s(t), t) = 0, \qquad x_s(0) = 0. \tag{1.1c}$$

At the shock front we have the well known Rankine–Hugoniot conditions [14]:

$$\begin{aligned} &\rho_0 \frac{\mathrm{d}x_s(t)}{\mathrm{d}t} [\![v(x_s(t), t)]\!] + [\![\sigma(x_s(t), t)]\!] = 0 \\ &\frac{p}{p-1} \Phi_0^{-1} \frac{\mathrm{d}x_s(t)}{\mathrm{d}t} [\![\sigma(x_s(t), t)^{(p-1)/p}]\!] + [\![v(x_s(t), t)]\!] = 0 \end{aligned} \tag{1.1d}$$

here the notation $[\![\cdot]\!]$ indicates the jump across the shock front. The relationship between stress and strain across the shock front is therefore the same as for continuous waves.

A preliminary analysis of the model (1.1) has been worked out in [15]. A similar problem for a velocity impact in elastic materials is considered in [16–18] from a similarity viewpoint; in [19] an implicit difference scheme is introduced.

The main concern of this paper is to determine how the disspative term $(\Psi_0 \sigma^{1/q})$ influences the shock front propagation.

Within the context of similarity analysis we show that it is possible to reduce the moving boundary value problem to a free boundary value problem. Moreover, we point out that the similarity analysis can be carried on in order to solve the free boundary value problem numerically [20, 21].

2. Similarity analysis

The following stretching group

$$v^* = \mu^{\alpha} v, \qquad \sigma^* = \mu^{\beta} \sigma, \qquad x_s^*(t^*) = \mu^{\gamma} x_s(t), \qquad x^* = \mu^{\gamma} x, \qquad t^* = \mu t \tag{2.1}$$

leaves the mathematical model (1.1) invariant when $\Psi_0 = 0$ if

$$\alpha = \delta - \frac{1}{2}\frac{\delta}{p}, \qquad \beta = \delta, \qquad \gamma = 1 + \frac{1}{2}\frac{\delta}{p}$$

while for $\Psi_0 \neq 0$ we have to add the condition

$$\delta = \frac{q}{q-1}$$

Owing to the afore-mentioned invariance we can introduce the similarity variables

$$\begin{aligned} &\eta = \sigma_0^{-1/2p} \Phi_0^{-1/2} \rho_0^{1/2} x t^{-\gamma}, \qquad \eta_s = \sigma_0^{-1/2p} \Phi_0^{-1/2} \rho_0^{1/2} x_s(t) t^{-\gamma} \\ &V(\eta) = \sigma_0^{(1-2p)/2p} \Phi_0^{1/2} \rho_0^{1/2} t^{-\alpha} v(x, t), \qquad \Sigma(\eta) = \sigma_0^{-1} t^{-\delta} \sigma(x, t) \end{aligned} \tag{2.2}$$

the use of which permits to reduce the mathematical model (1.1) to the ordinary differential system in normal form

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\eta}&=\frac{\alpha\gamma\eta V-\delta\Sigma-\Psi_0\sigma_0^{(1-q)/q}\Sigma^{1/q}}{\gamma^2\eta^2-\Sigma^{1/p}}\\ \frac{\mathrm{d}\Sigma}{\mathrm{d}\eta}&=\frac{\gamma\eta(\delta\Sigma+\Psi_0\sigma_0^{(1-q)/q}\Sigma^{1/q})-\alpha V\Sigma^{1/p}}{\gamma^2\eta^2-\Sigma^{1/p}}\end{aligned} \tag{2.3a}$$

along with the boundary conditions

$$\Sigma(0)=1, \qquad \Sigma(\eta_s)=\left(\frac{p-1}{p}\frac{1}{\gamma^2\eta_s^2}\right)^{-p}, \qquad V(\eta_s)=-\frac{\Sigma(\eta_s)}{\gamma\eta_s}. \tag{2.3b, c, d}$$

Here (2.3c) and (2.3d) are obtained by the Rankine–Hugoniot conditions. Since η_s is unknown, (2.3) defines a free boundary value problem.

The main result is the following: if $\Psi_0=0$ or $q=1$ then (2.3a), (2.3c) and (2.3d) result invariant with respect to the stretching group

$$V^*=\omega^{2p-1}V, \qquad \Sigma^*=\omega^{2p}\Sigma, \qquad \eta^*=\omega\eta \tag{2.4}$$

which does not exist when $\Psi_0\neq 0$ and $q\neq 1$. We conclude that in order to solve numerically the free boundary value problem (2.3) we can use for the non-dissipative case the non-iterative transformation method [20], while investigation of the dissipative case requires the use of the iterative transformation method [21].

3. Numerical methods

In the following we give the general outlines of the numerical solution for the free boundary value problem (2.3). First we consider the non-dissipative case $\Psi_0=0$. From (2.4) we easily find that

$$\omega=(\Sigma^*(0))^{1/2p}, \qquad \eta_s=\omega^{-1}\eta_s^*, \qquad V(0)=\omega^{(1-2p)}V^*(0). \tag{3.1}$$

Therefore, it is possible to obtain η_s by a numerical integration backwards in $[0, \eta_s^*]$, where η_s^* can be chosen at our convenience. Next, let us consider the dissipative case $\Psi_0\neq 0$. In order to apply the iterative transformation method we have to introduce a numerical parameter h. If we extend the stretching group (2.4) by

$$h^*=\omega h \tag{3.2}$$

then the differential system (2.3a) has to be modified as follows

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\eta}&=\frac{\alpha\gamma\eta V-\delta\Sigma-\Psi_0\sigma_0^{(1-q)/q}h^{2p(1-1/q)}\Sigma^{1/q}}{\gamma^2\eta^2-\Sigma^{1/p}}\\ \frac{\mathrm{d}\Sigma}{\mathrm{d}\eta}&=\frac{\gamma\eta(\delta\Sigma+\Psi_0\sigma_0^{(1-q)/q}h^{2p(1-1/q)}\Sigma^{1/q})-\alpha V\Sigma^{1/p}}{\gamma^2\eta^2-\Sigma^{1/p}}\end{aligned} \tag{3.3}$$

Since (3.2) extends the stretching group (2.4), along with (3.1) we have to use

$$h=\omega^{-1}h^{*}. \tag{3.4}$$

Here we note that (2.3a) is recovered from (3.3) by setting $h=1$. Thus we will find the correct value of η_s if we get $h=1$ from (3.4). Then, fixed a value of η_s^*, we define h^* iteratively by a root finder. Our purpose is to find a root of the transformation function

$$\Gamma(h^*)=[\omega(h^*)]^{-1}h^*-1. \tag{3.5}$$

We remark that $\omega(h^*)$ is an unknown function of h^*. Hence, the values of $\Gamma(h^*)$ have to be computed by solving a related initial value problem. At every iteration step we have to integrate (3.3) backwards in $[0, \eta_s^*]$. Once again η_s^* can be chosen at our convenience.

4. Numerical results and discussion

In the present analysis our interest is to point out the role played by the dissipative term, see Section 1. Let us denote with ϑ the dissipative coefficient

$$\vartheta = \Psi_0\sigma_0^{(1-q)/q}.$$

First, by assuming $\vartheta=0$ we obtained the numerical results listed in Table 1 by setting either $\eta_s^*=1$ or $\eta_s^*=0.5$

Table 1
Results related to a nonlinear model ($p=3$) in the non-dissipative case $\vartheta=0$

δ	$V(0)$	η_s
−0.5	−1.286499	1.07735
0.5	−1.211072	0.66995
1.0	−1.206158	0.571939
1.5	−1.203906	0.500607
2.0	−1.202708	0.44592

The value of $p=3$ used here is related to a constitutive law between strain and stress where the strain is proportional to the power two over three of the stress. As reported in [6] a relationship of the type mentioned before with an exponent greater than zero but smaller than one is used to describe rubbers and some metals.

Here we notice how it is possible to validate the numerical results by a further integration backwards in $[0, \eta_s]$. For instance, in the case $\delta=2$ we obtain $V(0)=-1.202709$ and $\Sigma(0)=1.000001$.

Then we considered $\vartheta\neq 0$. As well known every iterative numerical procedure needs an appropriate criterion of convergence. In order to accept a value of η_s^k we require

$$|\Gamma(h_k^*)|<\mathrm{Tol}, \qquad |\eta_s^k-\eta_s^{k-1}|<\mathrm{Tol}$$

where Tol is a prescribed tolerance. The Table 2 shows the iterations obtained by means of the secant method. There we used $\mathrm{Tol}=1\times 10^{-6}$.

For the reader's convenience Fig. 1 represents the self-similarity solution for the problem at hand. The choice $p=3$ and $q=2$, results in the viscoplastic strain rate proportional to the stress to the power one over

Table 2

Results obtained for $p=3$, $\delta=2$, $q=2$, $\vartheta=-1$ and $\eta_s^*=1$

k	h_k^*	$\Gamma(h_k^*)$	$V(0)$	η_s
0	2.000000	-0.15111		
1	4.000000	0.431889		
2	2.518389	0.29×10^{-1}	-1.441716	0.408756
3	2.410143	-0.64×10^{-2}	-1.419068	0.412262
4	2.429466	0.73×10^{-4}	-1.423061	0.411643
5	2.429248	0.18×10^{-6}	-1.423015	0.41165
6	2.429248	-0.50×10^{-11}	-1.423015	0.41165

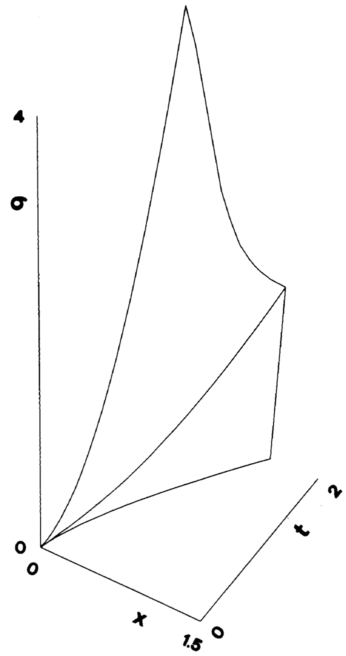

Fig. 1. Shock front propagation for $p=3$, $q=2$, $\delta=2$, i.e. $\sigma(0, t)\propto t^2$ and $x_s(t)\propto t^{4/3}$, and $\vartheta=-1$.

six; this agrees with some physical experiments showing that the viscoplastic strain rate increases with increasing stress, see [22].

As discussed before a direct validation in the case $\vartheta=-1$ led to $V(0)=-1.423014$ and $\Sigma(0)=0.999999$. Table 3 lists further numerical results; those not shown before were obtained iteratively as discussed above.

In order to discuss the influence of the dissipative term upon the shock front propagation it is convenient to rewrite here the functional form of the moving boundary

$$x_s(t)=\sigma_0^{-1/2p}\Phi_0^{-1/2}\rho_0^{1/2}\eta_s t^{\gamma}$$

where $\gamma=1+\frac{1}{2}(\delta/p)$; $\delta=q/(q-1)$ if $\vartheta\neq0$. Therefore, for a given material the behaviour of the shock front depends only on η_s. The influence of the dissipative term is evident: it turns out that η_s is an increasing function of ϑ. Since $\mathrm{d}x_s(t)/\mathrm{d}t$ represents the shock speed, attenuation effects on shock front propagation will occur for negative values of ϑ, while they do not occur for positive values of ϑ.

Table 3
Results connected with $p=3$, $\delta=2$, $q=2$ and $\eta_s^*=1$.

ϑ	$V(0)$	η_s
−1.0	−1.423015	0.41165
−0.5	−1.316325	0.42825
0.0	−1.202708	0.44592
0.5	−1.08136	0.464515
1.0	−0.951518	0.483819

All computations were performed on a RISC SYSTEM/6000 IBM computer with the DIVPAG integrator in the IMSL MATH/LIBRARY [23]. The DIVPAG allows us to apply step size and local error control. An user supplied Jacobian and a value of $1 \cdot 10^{-12}$ for the error control were used within the DIVPAG.

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