THE FALKNER-SKAN EQUATION: NUMERICAL SOLUTIONS WITHIN GROUP INVARIANCE THEORY (1)

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ABSTRACT - The iterative transformation method, defined within the framework of the group invariance theory, is applied to the numerical solution of the Falkner-Skan equation with relevant boundary conditions. In this problem a boundary condition at infinity is imposed which is not suitable for a numerical use. In order to overcome this difficulty we introduce a free boundary formulation of the problem, and we define the iterative transformation method that reduces the free boundary formulation to a sequence of initial value problems. Moreover, as far as the value of the wall shear stress is concerned we propose a numerical test of convergence. The usefulness of our approach is illustrated by considering the wall shear stress for the classical Homann and Hiemenz flows. In the Homann's case we apply the proposed numerical test of convergence, and meaningful numerical results are listed. Moreover, for both cases we compare our results with those reported in literature.

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1. Introduction and formulation

Group invariance theory (also known as Lie group theory) is widely used in numerical analysis. In this context we can quote the classical perturbation theory developed by Gröbner [26, pp. 336-343] for the numerical solution of initial value problems (IVPs), the transformation of boundary value problems (BVPs) to

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IVPs [21, Chapters 7-9], and the classification of finite difference schemes for partial differential equations according to the invariance properties of their first differential approximation [25]. The transformation of BVPs to IVPs has both analytical and numerical interest: from an analytical viewpoint, existence and uniqueness theorems can be obtained [13]; whereas from a numerical one, non-iterative methods can be defined [14-15]. A classical example is given by the Blasius problem of boundary layer theory for which existence and uniqueness were reported in [19, pp. 104-105] and a non-iterative solution was obtained in [27]. These results are a consequence of the invariance of the Blasius equation with respect to a scaling (stretching) group of transformations. Numerical methods introduced on the basis of the invariance properties have been named in several ways: «initial value method» [20], «exact shooting» [1], «method of transformation» [21], «inspectional and infinitesimal group methods» [24, Chapter 9] or «a transformation method» [7]. In the following we will use «numerical transformation methods» in order to indicate methods defined within group invariance theory.

In order to define the problem we shall deal with, let us consider the model describing the flow of a fluid past a wedge [6]

$$
\begin{aligned}
&\frac{d^3f}{d\eta^3} + f\,\frac{d^2f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0 \\
&f(0) = \frac{df}{d\eta}(0) = 0; \quad \frac{df}{d\eta}(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty
\end{aligned}
\tag{1.1}
$$

where η and f are similarity variables and β is a parameter. The governing differential equation is the classical Falkner-Skan equation. (1.1) generalizes the Blasius problem which is recovered when $\beta = 0$. For each value of the physical parameter β involved in (1.1), Weyl [28] proved that there exists a solution for which $\frac{d^2f}{d\eta^2}(\eta)$ is positive, monotone decreasing on $[0,\infty)$, and approaching zero as $\eta \rightarrow \infty$.

The uniqueness question is more complex [4] because when $\beta > 1$, besides the monotone solution characterized by Weyl, there exists a hierarchy of solutions with reversed flow. This hierarchy of solutions was studied numerically in [5]. We remark that the problem (1.1) was proposed as a caveat for numerical transformation methods [21, pp. 146-147].

Here, by following the idea introduced in [9] for the Blasius problem, we consider the free boundary formulation of (1.1) given by

$$(1.2)\qquad \begin{aligned} &\frac{d^3 f_\varepsilon}{d\eta^3} + f_\varepsilon \frac{d^2 f_\varepsilon}{d\eta^2} + \beta\left[1 - \left(\frac{df_\varepsilon}{d\eta}\right)^2\right] = 0 \\ &f_\varepsilon(0) = \frac{df_\varepsilon}{d\eta}(0) = 0; \quad \frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = 1; \quad \frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \varepsilon \end{aligned}$$

where $0 < \varepsilon \ll 1$ and η_ε is the unknown free boundary. If we fix a value of ε, then η_ε, and $f_\varepsilon(\eta)$ defined on $[0, \eta_\varepsilon]$ are sought as solution of (1.2). Moreover, it might be possible to prove the convergence of the solution of (1.2) to the solution of (1.1) as $\varepsilon \to 0$ when suitable smoothness conditions for the solution of the free BVP (1.2) hold. Work in this direction is now in progress. In sections 2 and 3 we limit ourselves to propose and to apply a practical numerical test of convergence.

In the next section we define an iterative transformation method by reducing (1.2) to a sequence of IVPs. The method is a variant of that proposed in [8] for the numerical solution of free BVPs governed by a system of two ordinary differential equations. In section 3 we list meaningful numerical results obtained with our approach for the Homann flow ($\beta = 1/2$) and the Hiemenz flow ($\beta = 1$). The value of the wall shear stress $\frac{d^2 f}{d\eta^2}(0)$ is considered for comparative purpose. The last section contains some final remarks devoted to point out the link of the present study with other works in current literature.

2. **Group invariance theory and numerical convergence**

The governing differential equation in (1.2) is invariant with respect to a simple translation in the independent variable. That is not enough in order to define a numerical transformation method. However, let us introduce the family of problems

$$(2.1)\qquad \begin{aligned} &\frac{d^3 f_\varepsilon}{d\eta^3} + f_\varepsilon \frac{d^2 f_\varepsilon}{d\eta^2} + \beta\left[h^{1/2} - \left(\frac{df_\varepsilon}{d\eta}\right)^2\right] = 0 \\ &f_\varepsilon(0) = \frac{df_\varepsilon}{d\eta}(0) = 0; \quad \frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = 1; \quad \frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \varepsilon \end{aligned}$$

where h is a parameter. The governing differential equation and the two boundary conditions at $\eta = 0$ in (2.1) are invariant with respect to the stretching group

$$\eta^* = \lambda^{-1}\,\eta; \quad f_\varepsilon^* = \lambda\, f_\varepsilon; \quad h^* = \lambda^8\, h.$$

As a consequence, a transformation method can be defined. To this end we have to assume that for a suitable range of h* the IVP

$$\frac{d^3f_\varepsilon^*}{d\eta^{*3}} + f_\varepsilon^* \frac{d^2f_\varepsilon^*}{d\eta^{*2}} + \beta\left[h^{*1/2} - \left(\frac{df_\varepsilon^*}{d\eta^*}\right)^2\right] = 0 \tag{2.2}$$

$$f_\varepsilon^*(0) = \frac{df_\varepsilon^*}{d\eta^*}(0) = 0; \quad \frac{d^2f_\varepsilon^*}{d\eta^{*2}}(0) = \text{const}$$

defined on $[0, \eta_\varepsilon^*]$, where η_ε^* is fixed by the condition

$$\left[\frac{df_\varepsilon^*}{d\eta^*}(\eta_\varepsilon^*)\right]^{-3/2} \frac{d^2f_\varepsilon^*}{d\eta^{*2}}(\eta_\varepsilon^*) = \varepsilon \Leftarrow \frac{d^2f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \varepsilon,$$

is well posed. By means of standard group analysis we obtain

$$\lambda = \left[\frac{df_\varepsilon^*}{d\eta^*}(\eta_\varepsilon^*)\right]^{1/2} \Leftarrow \frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = 1$$

$$\frac{d^2f_\varepsilon}{d\eta^2}(0) = \lambda^{-3} \frac{d^2f_\varepsilon^*}{d\eta^{*2}}(0)$$

$$\eta_\varepsilon = \lambda\, \eta_\varepsilon^*$$

$$\frac{d^2f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \lambda^{-3} \frac{d^2f_\varepsilon^*}{d\eta^{*2}}(\eta_\varepsilon^*)$$

$$h = \lambda^{-8}\, h^*.$$

The solution of the original problem is recovered when $h = 1$. In other words, in order to solve the problem (1.2), we have to find a root of the implicit function

$$\Gamma(h^*) = [\lambda(h^*)]^{-\sigma} h^* - 1$$

where λ is considered as a function of h^* because this is the only parameter to be varied. By choosing the values of h^*_0, h^*_1 and computing the corresponding values $\Gamma(h^*_0)$, $\Gamma(h^*_1)$ we can apply a root-finding method in order to define a sequence h^*_k, $k = 2, 3, \ldots,$. Hence, we have an iterative method which generalizes the non-iterative method introduced in [9]. A numerical algorithm can be easily defined by following the approach described above.

As far as the convergence question is concerned we would like to indicate here a practical numerical test. If the two problems (1.1) and (1.2) are well-posed (the latter for values of ε in a suitable interval of $\varepsilon = 0$) we would have that as $\varepsilon \to 0$ then $\eta_\varepsilon \to \infty$ and $f_\varepsilon(\eta) \to f(\eta)$ on $[0, \eta_\varepsilon]$. Let us consider a sequence of values ε_i ($i = 1, \ldots, n$) such that $0 < \varepsilon_{j+1} < \varepsilon_j$ ($j = 1, \ldots, n-1$) and solve the problem (1.2) for each value of ε in the sequence. As a first test we can verify if $\eta_{\varepsilon_{j+1}} > \eta_{\varepsilon_j}$ ($j = 1, \ldots, n-1$). Moreover, since we are solving our problem by means of an initial value method, the error related to the value of the wall shear stress is of relevant interest in determining the global error. Consequently, we can define the residuals r_j as $\left| \frac{d^2f_{\varepsilon_{j+1}}}{d\eta^2}(0) - \frac{d^2f_{\varepsilon_j}}{d\eta^2}(0) \right|$ ($j = 1, \ldots, n-1$) and verify if $r_{k+1} < r_k$ ($k = 1, \ldots, n-2$). An application of the above idea is given in the next section.

3. **Numerical results**

Let us consider first the Homann flow, corresponding to the value $\beta = 1/2$. In Table 1 a simple iteration for this case is reported.

Table 1 – Sample numerical iterations for the Homann flow.
Here $\frac{d^2f_\varepsilon^*}{d\eta^{*2}}(0) = 0.1D + 02$ and $\varepsilon \leq 10^{-3}$.

k	h^*_k	$\Gamma(h^*_k)$	η_ε^*	η_ε	$\frac{d^2f_\varepsilon}{d\eta^2}(0)$
0	565	−0.824761	129.	354.113	0.483441
1	570	0.354772D−1	1.764	3.891470	0.931440
2	569.793794	0.332237D−1	1.5855	3.490251	0.937408
3	566.753667	−0.250849D−2	1.806	3.990510	0.926975
4	566.967092	0.311817D−3	1.774	3.918604	0.927826
5	566.943495	0.699223D−5	1.777	3.925360	0.927735
6	566.942954	0.595237D−6	1.777	3.925363	0.927733
7	566.942904	0.410447D−8	1.777	3.925363	0.927733

Here and in the following the D notation indicates a double precision arithmetic. From Table 1 it is evident that the value of η_ε* is not constant in all the iterations. That may represent a difficulty in using the method because η_ε* defines the interval of integration. In the present work we started with the value $\varepsilon = 0.1$ in order to study the behaviour of $\Gamma(h^*)$. Thus, by letting the value of ε get smaller ($\varepsilon \rightarrow 0$ is of interest) we can use our information about the behaviour of $\Gamma(h^*)$ at the previous step in order to speed up the preliminary computations. Meaningful numerical results, obtained as described above, are listed in Table 2.

Table 2 – A convergence numerical test for the Homann flow.

$\varepsilon \leqslant$	h*	$\Gamma(h^*)$	η_ε^*	η_ε	$\frac{d^2 f_\varepsilon}{d\eta^2}(0)$
10^{-1}	542.670752	0.13D−8	1.009	2.216707	0.943081
10^{-2}	565.733143	0.27D−8	1.442	3.184503	0.928476
10^{-3}	566.942904	0.40D−8	1.777	3.925363	0.927733
10^{-4}	567.022195	0.39D−9	2.06	4.550585	0.927684
10^{-5}	567.027843	0.47D−8	2.302	5.085175	0.927680
10^{-6}	567.028301	0.15D−8	2.522	5.571160	0.927680

It is evident that η_ε is a decreasing function of ε. Moreover, the results for the wall shear stress reported in Table 2 clearly indicate numerical convergence. A direct validation of our results is proposed in Table 3. The results listed in Table 3 were obtained by solving the IVP (2.2) in the η and $f(\eta)$ variables (and $h^* = 1$) on $[0, \eta_\varepsilon]$ with the values of $\frac{d^2 f_\varepsilon}{d\eta^2}(0)$ and η_ε as reported in Table 2.

As far as the Hiemenz flow ($\beta = 1$) is concerned, representative numerical results are listed in Table 4 for the case $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0) = 0.1D + 02$ and $\varepsilon \leqslant 10^{-6}$.

The same Table also proposes a comparison between our results and those reported in literature. Our results define the truncated boundary that is approximately required in order to achieve six decimal places of accuracy for the value of

Table 3 – Numerical validation for the results listed in Table 2.

$\frac{d^2f_\varepsilon}{d\eta^2}(0)$	η_ε	$\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon)$	$\frac{d^2f_\varepsilon}{d\eta^2}(\eta_\varepsilon)$
0.943081	2.216707	1.	0.999270D−1
0.928476	3.184503	1.	0.998599D−2
0.927733	3.925363	1.	0.998994D−3
0.927684	4.550585	1.	0.975419D−4
0.927680	5.085175	1.	0.942535D−5
0.927680	5.571160	1.	0.892296D−6

Table 4 – Comparison of the wall shear stress and truncated boundary (η_∞ or η_ε) for the Homann and the Hiemenz flows.

	this work initial value method		Nasr et al. [22] Chebyshev method		Beckett [2] finite difference method	
β	η_ε	$\frac{d^2f_\varepsilon}{d\eta^2}(0)$	η_∞	$\frac{d^2f}{d\eta^2}(0)$	η_∞	$\frac{d^2f}{d\eta^2}(0)$
0.5			3.7	0.927805	5.	0.9277
0.5	5.085175	0.927680	7.4	0.927680		
1.			3.5	1.232617	5.	1.2327
1.	5.187600	1.232588	7.	1.232588		

the wall shear stress. A reduction of the numerical accuracy or an increase of the computational cost are related to a smaller or to a greater value of the truncated boundary.

For all the experiments reported above we used a RISC System/6000 IBM computer with the DIVPAG integrator in the IMSL Math/Library [12].

Moreover, we provided a user supplied Jacobian, the value 1D-12 for the local error control and the secant method along with appropriate termination criteria.

4. Concluding remarks

The Falkner-Skan equation of boundary layer theory is of relevant interest in fluid dynamics. The classical problem for this equation is given by a two-point BVP with one of the boundary conditions specified at infinity. Such a boundary condition is not suitable for a direct numerical treatment, and some approaches have been proposed in order to overcome this difficulty.

A simple strategy, reported in [3, 10], is to impose the same boundary condition at a finite boundary (the truncated boundary). In a more complex approach a preliminary asymptotic analysis is carried on in order to define the most suitable boundary conditions to be imposed at a finite boundary. The latter approach has been developed in detail by several authors [11, 16-18]. However, as indicated in [16], an «a priori» evaluation of a convenient truncated boundary is difficult to obtain.

Here we proposed a free boundary formulation where the truncated boundary is unknown, and has to be found as part of the numerical solution. Moreover, under suitable smoothness conditions it might be possible to prove that the solution of the free boundary formulation represents an approximation of the solution of the original problem. That can also be theoretically relevant in applying the simple truncated boundary approach mentioned above. In literature such a proof is for instance available for the Blasius problem in [23], it was obtained by taking into account the invariance of the governing equation with respect to a stretching group. We have already remarked that the Falkner-Skan equation does not result to be invariant with respect to any stretching group, so that a direct generalization of the analysis worked out in [23] is not possible. However, since we solve our problem by an initial value method we propose a numerical test of convergence for the missing boundary condition at the origin (i.e., the wall shear stress). In this context the results of Table 2 are the most significant among those reported in this paper.

REFERENCES

[1] W. F. Ames, E. Adams: *Exact shooting and eigenparameter problems*, Nonlinear Anal., **1** (1976), 75-82.

[2] P. M. BECKETT: *Finite difference solution of boundary-layer type equations*, Internat. J. Computer Math., **14** (1983), 183-190.

[3] L. COLLATZ: *The Numerical Treatment of Differential Equations*, 1960, Springer-Verlag, Berlin, 3rd ed.

[4] W. A. COPPEL: *On a differential equation of boundary-layer theory*, Philos. Trans. Roy. Soc. London Ser. A, **253** (1960), 101-136.

[5] A. H. CRAVEN, L. A. PELIETIER: *Reverse flow solutions of the Falkner-Skan equation for $\lambda > 1$*, Mathematika, **19** (1972), 135-138.

[6] V. M. FALKNER, S. W. SKAN: *Some approximate solutions of the boundary layer equations*, Philos. Mag., **12** (1931), 865-896.

[7] R. FAZIO: *A noniterative transformation method applied to two-point boundary value problems*, Appl. Math. Comput., **39** (1990), 79-87.

[8] R. FAZIO: *The iterative transformation method: free or moving boundary value problems*, in «Computational and Applied Mathematics, Vol. I Algorithms and Theory», IMACS 13th World Congress, Dublin, 1991, (C. Brezinski & U. Kulish eds.), North-Holland, Amsterdam, 1992, 147-152.

[9] R. FAZIO: *The Blasius problem formulated as a free boundary value problem*, Acta Mech., **95** (1992), 1-7.

[10] L. FOX: *Numerical solution of two-point boundary value problems in ordinary differential equations*, 1957, Clarendon Press, Oxford.

[11] F. R. DE HOOG, R. WEISS: *An approximation theory for boundary value problems on infinite intervals*, Compunting, **24** (1980), 227-239.

[12] IMSL, Math/Library User's Manual, Fortran subroutines for mathematical applications, Version 1.1, 1989.

[13] H. B. KELLER: *Numerical methods for two-point boundary value problems*, 1968, Waltham, Blaisdell.

[14] M. S. KLAMKIN: *On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations*, SIAM Rev., **4** (1962), 43-47.

[15] M. S. KLAMKIN: *Transformation of boundary value problems into initial value problems*, J. Math. Anal. Appl., **32** (1970), 308-330.

[16] M. LENTINI, H. B. KELLER: *Boundary value problems on semi-infinite intervals and their numerical solutions*, SIAM J. Numer. Anal., **17** (1980), 577-604.

[17] P. A. MARKOWICH: *A theory for the approximation of solution of boundary value problems on infinite intervals*, SIAM J. Math. Anal., **13** (1982), 484-513.

[18] P. A. MARKOWICH: *Analysis of boundary value problems on infinite intervals*, SIAM, J. Math. Anal., **14** (1983), 11-37.

[19] R. E. MEYER: *Introduction to mathematical fluid dynamics*, 1971, Wiley, New York.

[20] T. Y. NA: *An initial value method for the solution of a class of nonlinear equations in fluid mechanics*, J. Basic Eng. Trans. ASME, **92** (1970), 91-99.

[21] T. Y. NA: *Computational methods in engineering boundary value problems*, 1979, Academic Press, New York.

[22] H. NASR, I. A. HASSANIEN, H. M. EL-HAWARY: *Chebyshev solution of laminar boundary layer flow*, Internat. J. Computer Math., **33** (1990), 127-132.

[23] L. A. RUBEL: *An estimation of the error due to the truncated boundary in the numerical solution of the Blasius equation*, Quart. Appl. Math., **13** (1955), 203-206.

[24] R. SESHADRI, T. Y. NA: *Group invariance in engineering boundary value problems*, 1985, Springer-Verlag, New York.

[25] JU. I. SOKIN: *The method of differential approximation*, 1983, Springer-Verlag, New-York.

[26] H. J. STETTER: *Analysis of discretization methods for ordinary differential equations*, 1973, Springer Tracts in Natural Philosophy, Vol. **23**, Berlin.

[27] K. TÖPFER: *Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. Phys., **60** (1912), 397-398.

[28] H. WEYL: *On the differential equation of the simplest boundary-layer problems*, Ann. Math., **43** (1942), 381-407.