Acta Mechanica 95, 1–7 (1992)

ACTA MECHANICA

The Blasius problem formulated as a free boundary value problem

R. Fazio, Messina, Italy

(Received June 24, 1991; revised September 24, 1991)

Summary. In the present paper we point out that the correct way to solve the Blasius problem by numerical means is to reformulate it as a free boundary value problem. In the new formulation the truncated boundary (instead of infinity) is the unknown free boundary and it has to be determined as part of the numerical solution. Taking into account the "partial" invariance of the mathematical model at hand with respect to a stretching group we define a non-iterative transformation method. Further, we compare the improved numerical results, obtained by the method in point, with analytical and numerical ones. Moreover, the numerical results confirm that the question of accuracy depends on the value of the free boundary. Therefore, this indicates that boundary value problems with a boundary condition at infinity should be numerically reformulated as free boundary value problems.

1 Introduction and formulation

This paper is concerned with the numerical solution of the classical problem in the boundary layer theory

$$\frac{d^3f}{d\eta^3} + \alpha f \frac{d^2f}{d\eta^2} = 0$$
$$f(0) = \frac{df}{d\eta}(0) = 0; \quad \frac{df}{d\eta}(\infty) = 1 \tag{1.1}$$

where η and $f(\eta)$ are similarity variables and $\alpha = 1/2$ or $\alpha = 1$. Blasius [1] obtained this problem in the study of a laminar boundary layer along a thin flat plate. The foundations of the boundary layer theory were established by Prandtl [2]. Oleinik [3] and Nickel [4] proved, respectively, existence and uniqueness for the original partial differential problem.

Analytic approximation to the solution of (1.1) can be obtained by series or asymptotic expansions. On the other hand we can resort to numerical methods. The first to provide a numerical solution was Topfer [5]. He discovered a transformation that allowed him to solve the boundary value problem by solving two related initial value problems. It was 50 years later that Klamkin [6] recovered that transformation as a consequence of the "partial" invariance of the problem – the boundary condition at infinity being not invariant – under a stretching group.

As generally occurs the initial interest was to obtain the qualitative behaviour of the solution of (1.1). But in recent years people became more interested in the improvement of the analytic or numerical accuracy, see Parlange, Braddock and Sander [7]. Usually, in order to obtain an idea of the accuracy involved, the value of $\frac{d^2f}{d\eta^2}(0)$ (the wall shear stress) is quoted. That value is of physical interest since it defines the skin friction around the plate.

Solving problem (1.1), as far as a numerical point of view is concerned, we have to deal with a practically unsuited condition at infinity. A typical procedure to overcome this point, see for instance Fox [8], is to apply that condition at several finite points and observe the change in the solution. More appropriately, a numerical procedure can be stated as follows: the condition $\frac{df}{d\eta}(\infty) = 1$ is replaced by $\frac{df}{d\eta}(\eta_\infty) = 1$ where η_∞ is chosen large enough in order that the second derivative is practically zero and any further integration does not change its value. Therefore, usually without mention, the numerical solution is carried out in a guess and trial framework. However, the correct problem to be solved numerically is the following one:

$$\frac{d^3 f_\varepsilon}{d\eta^3} + \alpha f_\varepsilon \frac{d^2 f_\varepsilon}{d\eta^2} = 0$$
$$f_\varepsilon(0) = \frac{df_\varepsilon}{d\eta}(0) = 0; \quad \frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = 1; \quad \frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \varepsilon \tag{1.2}$$

where the subscript ε indicates that the solution depends on its value, η_ε is unknown and $0 < \varepsilon \ll 1$ defines the value of η_ε. Thus (1.2) is a free boundary formulation of the Blasius problem. The fundamental aim of the present work is to make evident that, from a numerical point of view, (1.2) is the correct way to formulate the Blasius problem. As a secondary purpose our numerical study shows that the degree of accuracy in the numerical solution depends on the value of η_ε.

In the above formulation we assume, implicitly, that as $\varepsilon \to 0$ then $\eta_\varepsilon \rightsquigarrow \infty$ and $f_\varepsilon \to f$. Indeed, as shown by Rubel [9], the described procedure yields an approximation of the real solution. Moreover, Rubel provides an estimation of the error due to the truncated boundary. There the value of $f_\varepsilon(\eta_\varepsilon)$ is of interest since it defines the accuracy of $f_\varepsilon(\eta)$ in $[0, \eta_\varepsilon]$ according to Rubel's formula

$$|f(\eta) - f_\varepsilon(\eta)| \leqq \frac{\eta_\varepsilon}{f(\eta_\varepsilon)}\varepsilon; \qquad 0 \leqq \eta \leqq \eta_\varepsilon. \tag{1.3}$$

Finally the paper of Rubel defines an a priori criterion to choose the value of η_∞ (η_ε in (1.3)) in order to achieve the required accuracy. However, the specialized literature seems to neglect Rubel's work. As a consequence there is no agreement in the literature about the value of η_∞. Here we remark that the error considered in Rubel's work is due to the truncated boundary; no consideration is devoted to the error introduced by the numerical integration.

In the next Section we describe a non-iterative numerical method. The idea behind the present method is to consider the "partial" invariance of (1.2) – in the sense that the two boundary conditions at η_ε are not invariant – with respect to a stretching group, see Fazio [10]. However, this method is different from Topfer's classical one because here the free boundary η_ε has to be determined as part of the solution whereas there the far boundary η_∞ is a priori chosen. Non-iterative and iterative transformation methods for free boundary value problems have been introduced in Fazio and Evans [11] and in Fazio [12], [13].

In Section 3 we indicate how our results agree with analytical and numerical ones, also we provide an improvement in the numerical accuracy.

Finally, in the last Section, we discuss the evidences provided by the present work.

2 A non-iterative transformation method

In (1.2) the governing differential equation and the two boundary conditions at zero are invariant with respect to the action of the stretching group

$$\eta^* = \lambda^{-1}\eta; \quad f_\varepsilon^* = \lambda f_\varepsilon \tag{2.1}$$

where λ is the exponential of the group parameter. This means that solutions of (1.2) with different values of $\frac{d^2 f_\varepsilon}{d\eta^2}(0)$, η_ε, $\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon)$ and ε are transformed under (2.1) one into another. The idea expressed in the last sentence leads to a non-iterative transformation method. Let us set a value of $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0)$ and integrate numerically forwards in $[0, \eta_\varepsilon^*]$. To this end we can use any suitable initial value integrator for ordinary differential equations. Then from (2.1), taking into account the condition $\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = 1$, we have

$$\begin{aligned}
\lambda &= \left[\frac{df_\varepsilon^*}{d\eta^*}(\eta_\varepsilon^*)\right]^{1/2} \\
\frac{d^2 f_\varepsilon}{d\eta^2}(0) &= \lambda^{-3}\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0) \\
\eta_\varepsilon &= \lambda \eta_\varepsilon^* \\
f_\varepsilon(\eta_\varepsilon) &= \lambda^{-1} f_\varepsilon^*(\eta_\varepsilon^*) \\
\varepsilon &= \lambda^{-3}\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(\eta_\varepsilon^*).
\end{aligned} \tag{2.2}$$

At this stage the value of the end point η_ε^* is unknown, so we set a prefixed tolerance, say τ, and the values given by (2.2) become accepted when the condition $\varepsilon < \tau$ is achieved. Therefore the results can be compared for different values of τ, for instance, $\tau = 0.1D - 05$, $\tau = 0.1D - 08$ and so on. It is important to note that the value of τ has to be greater than the zero machine. Here it is worth noting that we have the freedom to choose the value of $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0)$, which may help in reducing the error introduced by the numerical integration if the condition $\eta_\varepsilon^* \ll \eta_\varepsilon$, or at least $\eta_\varepsilon^* < \eta_\varepsilon$, is satisfied.

In the next Section we present the numerical results calculated by the described method.

3 Numerical results

The practical application of the non-iterative transformation method of Section 2 requires the solution of an initial value problem. Any initial value solver can be used in principle, but as far as the question of accuracy is concerned, a numerical integrator with step size and error control has to be applied. Hence, we performed all the numerical integrations by using the initial value solver DIVPAG of the IMSL MATH/LIBRARY [14], with supplied Jacobian, on a RISC SYSTEM/6000 IBM computer. The tolerance we used for the error control, within the DIVPAG integrator, was $0.1D - 14$. In order to reduce the round-off error, according to the remark made in Section 2, we used $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0) = 0.1D + 04$, which implies that $\eta_\varepsilon^* < 1$. The numerical results obtained for $\alpha = 1/2$ are listed in Table 1.

Once the values of $\frac{d^2 f_\varepsilon}{d\eta^2}(0)$ and η_ε have been calculated we can integrate forwards in $[0, \eta_\varepsilon]$ in order to check the numerical values $\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon)$ and $\frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon)$. In this way we can validate the numerical results as shown in Table 1. A general discussion about the validation of numerical computations can be found in Rice [15].

Table 1. Numerical results and their validation for $\alpha = 1/2$, here $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0) = 0.1D + 04$

τ	ε	η_ε^*	η_ε	$f_\varepsilon(\eta_\varepsilon)$	$\frac{d^2 f_\varepsilon}{d\eta^2}(0)$
0.1D − 05	0.999962D − 06	0.606103	8.752700	7.031914	0.332057
0.1D − 08	0.1D − 08	0.727116	10.500242233	8.779454979	0.332057336
0.1D − 11	0.1D − 11	0.827755	11.953564643860	10.237771215862	0.332057336215

$\frac{d^2 f_\varepsilon}{d\eta^2}(0)$	η_ε	$\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon)$	$\frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon)$
0.332057	8.752700	0.999999050905	0.999974D − 06
0.332057336	10.500242233	0.999999999346	0.1D − 08
0.332057336215	11.953564643860	0.999999999999	0.1D − 11

In Table 2 the numerical results are given for $\alpha = 1$.

Table 2. Numerical results and their validation for $\alpha = 1$, again $\frac{d^2 f_\varepsilon^*}{d\eta^{*2}}(0) = 0.1D + 04$

τ	ε	η_ε^*	η_ε	$f_\varepsilon(\eta_\varepsilon)$	$\frac{d^2 f_\varepsilon}{d\eta^2}(0)$
0.1D − 11	0.1D − 11	0.660701	8.500195503235	7.283415048997	0.469599988361

$\frac{d^2 f_\varepsilon}{d\eta^2}(0)$	η_ε	$\frac{df_\varepsilon}{d\eta}(\eta_\varepsilon)$	$\frac{d^2 f_\varepsilon}{d\eta^2}(\eta_\varepsilon)$
0.469599988361	8.500195503235	1.	0.1D − 11

4 Discussion

The results obtained for the first case in Table 1, where $\tau = 0.1D - 05$, agree with Howarth's [16] classical ones, i.e., $\eta_\infty = 8.4$ and $\frac{d^2f}{d\eta^2}(0) = 0.33206$. Here we notice that the best estimates of $\frac{d^2f}{d\eta^2}(0)$ previously available, 0.332 057 34 (obtained by the Topfer method) and 0.332 057 35 (given by a revised analytical technique due to Bairstow [17]) and given in Parlange et al. [7], agree with the value listed for the case $\tau = 0.1D - 08$. However, Parlange et al. considered $\eta_\infty = 10$. Further, the case $\tau = 0.1D - 11$ gives improved values that we believe to be correct up to the twelfth decimal place. In the latter case, from (1.3), we get

$$|f(\eta) - f_\varepsilon(\eta)| \leqq 12 \cdot 10^{-13}; \qquad 0 \leqq \eta \leqq \eta_\varepsilon$$

where $\eta_\varepsilon = 11.953\,564\ldots$.

The results listed in Table 2 can be compared to those of Nasr, Hassanien and El-Hawary [18] where $\eta_\infty = 6.9$ and $\frac{d^2f}{d\eta^2}(0) = 0.469\,6000$ obtained by a Chebyshev expansion or to those of Quartapelle and Rebay [19] in which $\eta_\infty = 7$ and $\frac{d^2f}{d\eta^2}(0) = 0.469\,599\,99$ and calculated by a numerical method that makes use of the integral condition $\int_0^\infty \frac{d^2f}{d\eta^2}(\eta)\, d\eta = 1$. Once again we believe that the value of the wall shear stress given in Table 2 is correct up to the twelfth decimal place. Again from (1.3) we obtain

$$|f(\eta) - f_\varepsilon(\eta)| \leqq \frac{9}{7} \cdot 10^{-12}; \qquad 0 \leqq \eta \leqq \eta_\varepsilon$$

where $\eta_\varepsilon = 8.500\,195\ldots$.

The Blasius problem has been considered and solved in this paper. We now consider further applications of our non-iterative transformation method. The method is applicable to a quasi-steady form of the Blasius problem, see Duck [20], where the condition at infinity is

$$\frac{df}{d\eta}(\infty) = 1 + \mu \cos(t)$$

with μ of $O(1)$. t represents the time. Moreover, the introduction of the new independent variable ξ, given in (4.1), proposed by Schultz-Grunow, see Schlichting [21], allows us to reduce several problems in the boundary layer theory, involving self-similar solution, to the Blasius problem:

$$\xi = \frac{1}{2A} \ln(1 + 2A\eta). \tag{4.1}$$

The transformation (4.1) can be used for flows along longitudinally curved walls with blunt or sharp leading edges by choosing $A = R\Delta/2$ as the curvature parameter. Here Δ represents the boundary-layer thickness and R the Reynolds number.

Formally, the non-iterative transformation method introduced so far can be used in order to solve any problem in the class

$$\frac{d^3f_\varepsilon}{d\eta^3} = f_\varepsilon^{\,1-3\delta}\Phi\left(f_\varepsilon^{-\delta}\eta, f_\varepsilon^{\delta-1}\frac{df_\varepsilon}{d\eta}, f_\varepsilon^{2\delta-1}\frac{d^2f_\varepsilon}{d\eta^2}\right)$$

$$f_\varepsilon(0) = \frac{df_\varepsilon}{d\eta}(0) = 0; \qquad \frac{df_\varepsilon}{d\eta}(\eta_\varepsilon) = C; \qquad \frac{d^2f_\varepsilon}{d\eta^2}(\eta_\varepsilon) = \varepsilon \tag{4.2}$$

where δ is a constant different from zero, $\Phi(\cdot, \cdot, \cdot)$ is an arbitrary function of its arguments, C is a given constant, and $\varepsilon \ll 1$ as before. Here we have to consider the partial invariance of (4.2) with respect to

$$\eta^* = \lambda^\delta\eta; \quad f_\varepsilon^* = \lambda f_\varepsilon. \tag{4.3}$$

From (4.3), we have

$$\lambda = \left[\frac{df_\varepsilon^*}{d\eta^*}(\eta_\varepsilon^*)\right]^{1/(1-\delta)}$$

$$\frac{d^2f_\varepsilon}{d\eta^2}(0) = \lambda^{2\delta-1}\frac{d^2f_\varepsilon^*}{d\eta^{*2}}(0)$$

$$\eta_\varepsilon = \lambda^{-\delta}\eta_\varepsilon^* \tag{4.4}$$

$$f_\varepsilon(\eta_\varepsilon) = \lambda^{-1}f_\varepsilon^*(\eta_\varepsilon^*)$$

$$\varepsilon = \lambda^{2\delta-1}\frac{d^2f_\varepsilon^*}{d\eta^{*2}}(\eta_\varepsilon^*)$$

instead of (2.2).

Of course, by setting $\Phi = -\alpha f_\varepsilon^{2\delta-1}\dfrac{d^2f_\varepsilon}{d\eta^2}$, $C = 1$ and $\delta = -1$ we recover from (4.2) the problem (1.2), moreover (4.3) and (4.4) reduce to (2.1) and (2.2) respectively.

The numerical results listed in Table 1 show that the accuracy inherent in the numerical solution depends on the value of ε. This means that if we are trying to solve problem (1.1) by a guess and trial procedure the numerical accuracy depends on the value of the far boundary η_∞. Therefore, as a main conclusion we can say that the present study indicates that all problems with a boundary condition at infinity should be formulated as free boundary value problems before any effective attempt to obtain a numerical solution is made.

Acknowledgement

This work was supported by the C.N.R. contract N. 90.04202. MZ78.

References

[1] Blasius, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Z. Math. Phys. **56**, 1–37 (1908) (Engl. transl. in NACA Tech. Memo. 1256).
[2] Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Proc. Third Intern. Math. Congr. Heidelberg, 484–494 (1904) (Engl. transl. in NACA Tech. Memo. 452).
[3] Oleinik, O. A.: On a system of equations in boundary layer theory. Zhurn. Vychislit. Math. Math Fiz. **3**, 489–507 (1963) (Engl. transl. in: USSR Comput. Math. Math. Phys., 3).
[4] Nickel, K.: Einige Eigenschaften von Lösungen der Prandtlschen Grenzschicht-Differentialgleichungen. Arch. Rat. Mech. Anal. **2**, 1–31 (1958).
[5] Topfer, K.: Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Z. Math. Phys. **60**, 397–398 (1912).
[6] Klamkin, M. S.: On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations. SIAM Rev. **4**, 204–210 (1962).
[7] Parlange, J. Y., Braddock, R. D., Sander, D.: Analytical approximations to the solution of the Blasius equation. Acta Mech. **38**, 119–125 (1981).
[8] Fox, L.: Numerical solution of two-point boundary value problems in ordinary differential equations. Oxford: At the Clarendon Press 1957.
[9] Rubel, L. A.: An estimation of the error due to the truncated boundary in the numerical solution of the Blasius equation. Quart. Appl. Math. **13**, 203–206 (1955).
[10] Fazio, R.: Metodi di transformazione per l'integrazione numerica di problemi al contorno libero o mobile. Ph. D. Thesis., University of Messina 1990.
[11] Fazio, R., Evans, D. J.: Similarity and numerical analysis for free boundary value problems. Int. J. Comput. Math. **31**, 215–220 (1990) and **39**, 249 (1991).
[12] Fazio, R.: Normal variables transformation method applied to free boundary value problems. Int. J. Comput. Math. **37**, 189–199 (1990).
[13] Fazio, R.: The iterative transformation method: free or moving boundary value problems. In: Proceedings of the 13th IMACS World Congress on Computation and Applied Mathematics, Dublin (Ireland) 22–26 July 1991 (Vichnevetsky, R., Miller, J. J. H., eds.) pp. 242–243. Vol. 1.
[14] IMSL, MATH/LIBRARY User's Manual, Fortran subroutines for mathematical applications, Version 1.1, 1989.
[15] Rice, J. R.: Numerical methods, software and applications. New York: McGraw-Hill 1983.
[16] Howarth, L.: On the solution of the laminar boundary layer equations. Proc. Roy. Soc. London **A 164**, 547–579 (1938).
[17] Bairstow, L.: Skin friction. J. Roy. Aero. Soc. **19**, 3–23 (1925).
[18] Nasr, H., Hassanien I. A., El-Hawary, H. M.: Chebyshev solution of laminar boundary layer flow. Int. J. Comput. Math. **33**, 127–132 (1990).
[19] Quartapelle, L., Rebay, S.: Numerical solution of two-point boundary value problems. J. Comput. Phys. **86**, 314–354 (1990).
[20] Duck, P. W.: A numerical method for treating time-period boundary layers. J. Fluid Mech. **204**, 549–561 (1989).
[21] Schlichting, H.: Boundary-layer theory. New York: McGraw-Hill 1979.

Author's address: Dr. R. Fazio, Department of Mathematics, University of Messina, Contrada Papardo, Salita Sperone 31, I-98166 Sant'Agata, Messina, Italy