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A nonlinear hyperbolic free boundary value problem

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Summary. The present paper is concerned with the application of a non-iterative transformation method to the numerical solution of a nonlinear hyperbolic free boundary value problem.

Making use of the similarity analysis approach to the hyperbolic model describing time dependent velocity impact to nonlinear inhomogeneous thin rods we recover a free boundary value problem.

Since exact solutions are known only in some particular cases, we consider application of numerical methods of integration.

Usually iterative numerical methods of solution are known to be applicable to free boundary value problems. However, we can prove that the ordinary differential equation related to the model in point is invariant with respect to a stretching group of transformations. This is the hint to apply group properties and to develop an ad hoc non-iterative transformation method.

1 Introduction

In the following we discuss the application of a non-iterative transformation method [1] to [2] to the numerical solution of a free boundary value problem deduced by applying similarity transformations to a quasilinear hyperbolic model.

The hyperbolic problem is that of nonlinear inhomogeneous thin rods subjected to a time dependent velocity impact [3]—[5].

The present work outlines a possible way to complete the early investigations [3]—[5] on similarity analysis approach to that model.

In Section 2 we reformulate the problem in point and we underline the reduction to a free boundary value problem.

The existence of an "associated group" [6]—[7], that allows us to consider application of the non-iterative transformation method, is pointed out in Section 3. Moreover in the same Section we explain application of the non-iterative transformation method.

Then application of our numerical method to a homogeneous "superelastic" rod [6] is reported in the last Section together with some general remarks.

Furthermore the obtained numerical results are directly validated by integrating the ordinary differential problem as an initial one.

2 Mathematical formulation

Let us consider the quasilinear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{E_0}{\varrho_0} \frac{\partial}{\partial x} \left[x^n \left(-\frac{\partial u}{\partial x} \right)^{1/q} \right] = 0 \tag{2.1}$$

with boundary and initial conditions

$$
\begin{array}{lll}
\dfrac{\partial u}{\partial t}(0, t) = V_0 t^\delta; & u(x \geqq x_w, t) = 0; & t > 0 \\[2ex]
u(x, 0) = 0; & \dfrac{\partial u}{\partial t}(x, 0) = 0; & x \geqq 0
\end{array}
\tag{2.2}
$$

where E_0, ϱ_0, $q \neq 0$, V_0 and δ are constants, while u is the displacement of a point, x and t are respectively Lagrangian space and time coordinates.

The partial differential equation (2.1) arises in the context of uniaxial theory for small deformation [8] of thin inhomogeneous rods where a nonlinear stress-strain relationship has been assumed [3].

The boundary and initial conditions (2.2) are related to a time dependent velocity impact problem [3].

In the mathematical model are present some parameters describing: nonlinearity ($q \neq 1$ and $q > 0$); inhomogeneity ($n \neq 0$); and time dependent velocity impact ($\delta \neq 0$).

Since the differential problem is nonlinear exact solutions have been obtained only for very few particular situations, namely the linear case and the homogeneous case with constant velocity impact ($\delta = 0$) [5].

According to [5] we introduce a similarity transformation as follows

$$
u(x, t) = V_0 t^{\delta+1} F(\eta); \qquad \eta = K x^p t^{-m}
\tag{2.3}
$$

where

$$
K = \left(\frac{\varrho_0 q}{E_0}\right)^{q/(1+q)} \left(\frac{1}{V_0}\right)^{(1-q)/(1+q)}
$$

$$
p = 1 - nq/(1+q); \qquad m = 1 + \delta(1-q)/(1+q)
$$

and we have to require that

$$
q > 0; \qquad p > 0 \qquad \text{and} \qquad m > 0.
$$

For (2.1) the wave propagation velocity c is given by

$$
c^2 = \frac{E_0}{\varrho_0}\frac{x^n}{q}\left(-\frac{\partial u}{\partial x}\right)^{(1-q)/q}.
\tag{2.4}
$$

In the similarity representation the wave front has the functional form

$$
x_w(t) = (\eta_w/K)^{1/p}\, t^{m/p}
\tag{2.5}
$$

and it is important to note that in this expression the only unknown is the value of η_w. Then the constant value of η_w characterizes the wave front propagation.

Taking into account the similarity representation (2.3), from (2.1), (2.2) we obtain

$$
\left[p^{(1+q)/q}\left(-\frac{dF}{d\eta}\right)^{(1-q)/q} - m^2\eta^2\right]\frac{d^2F}{d\eta^2} - p^{1/q} n \frac{q^2}{1+q}\,\eta^{-1}\left(-\frac{dF}{d\eta}\right)^{1/q}
$$

$$
-m(m - 2\delta - 1)\,\eta\,\frac{dF}{d\eta} - \delta(\delta+1)\,F = 0
\tag{2.6}
$$

along with the boundary conditions

$$F(0) = 1/(1 + \delta) \tag{2.7.1}$$

$$F(\eta_w) = 0 \tag{2.7.2}$$

$$\frac{dF}{d\eta}(\eta_w) = -(m^2 \eta_w^2 p^{-(1+q)q})^{q/(1-q)}. \tag{2.7.3}$$

The boundary condition (2.7.3) is called the "similarity-characteristic relationship", it comes out by differentiating (2.5) with respect to t and by comparing the result with the wave propagation velocity (2.4).

Actually, in the case of nonlinear problems, the similarity-characteristic relationship allows us to consider the ordinary differential problem (2.6), (2.7) as a free boundary value one. The conditions (2.7.2) and (2.7.3) define the free boundary η_w. It is worth noticing that the value of $\eta = \eta_w$ is a singular point for the second order ordinary differential equation (2.6).

3 The associated group and a non-iterative numerical method

As a main result we can prove that the ordinary differential equation (2.6) and the conditions (2.7.2) and (2.7.3) admit a stretching group of transformation.

One possible way to express this associated group is the following

$$\eta^* = \omega^{(1-q)/(1+q)}\eta; \qquad F^* = \omega F. \tag{3.1}$$

The existence of this group allows us to apply the non-iterative numerical method formulated in [1]—[2].

Let us discuss here an ad hoc application of that method to the free boundary value problem (2.6), (2.7).

From (3.1) we have

$$\omega = F^*(0)/F(0); \qquad \eta_w = \omega^{(q-1)/(1+q)}\eta_w{}^* \tag{3.2}$$

and also

$$\frac{dF}{d\eta}(0) = \omega^{-2q/(1+q)} \frac{dF^*}{d\eta^*}(0). \tag{3.3}$$

So that a first numerical integration inwards in $[0, \eta_w{}^*]$, with any guess of the free boundary, allows us to find a value of $F^*(0)$ and then, by applying (3.2), (3.3), to evaluate η_w and $\frac{dF}{d\eta}(0)$.

The above treatment differs from the general theory given in [1]—[2] because here we are interested also in the value of $\frac{dF}{d\eta}(0)$.

In fact we want to use this value and the condition (2.7.1) to integrate numerically (2.6) forward in $[0, \eta_w]$. In this way we can avoid, at the present step, the singular point of (2.6) in $\eta = \eta_w$. Furthermore within the second numerical integration we achieve a numerical solution of the problem (2.6), (2.7).

Thus, so doing, we validate the numerical results since, in $\eta = \eta_w$, we can check the two conditions (2.7.2) and (2.7.3).

4 Numerical results and concluding remarks

For illustrative purpose, we treat numerically the homogeneous case arising by setting $n = 0$ in the considered model.

Moreover, the further choice $q = 1/2$ is related to a thin rod called "superelastic" [6]. As noted in [3] materials such as rubbers and some metals have the constitutive law characterized by $q < 1$.

Numerical results related to these assumptions are given in Table 1.

Table 2 reports a direct validation of the numerical results listed in Tab. 1 obtained by integrating (2.6) as an initial value problem in $[0, \eta_w]$.

We believe that it might be of interest to give a qualitative representation of the numerical solutions, and we do that in Fig. 1.

Next, some remarks and conclusions drawn from the study can be given.

First we consider a homogeneous rod ($n = 0$). In the linear case ($q = 1$) integrating (2.6) we obtain the exact solution [3]

$$F(\eta) = \frac{1}{1+\delta}(1-\eta)^{\delta+1}$$

that satisfies the boundary condition (2.7.1).

From the boundary condition (2.7.2) we can deduce

$$\eta_w = 1 \qquad \text{for all value of } \delta .$$

Differentiating the exact solution we get

$$\frac{dF}{d\eta}(\eta_w) = 0 \qquad \text{for all value of } \delta \neq 0 .$$

Tab. 1. Numerical results related to the homogeneous case ($n = 0$) of a thin "superelastic" rod ($q = 1/2$)

δ	$\eta_w{}^*$	η_w	$\frac{dF}{d\eta}(0)$
0.1	0.5	0.884536	−1.091861
0.1	1.0	0.884511	−1.091767
0.5	0.5	0.658305	−1.204177
0.5	1.0	0.658505	−1.204054
1.0	0.5	0.510552	−1.247305
1.0	1.0	0.510583	−1.246965

Tab. 2. Numerical validations obtained directly from the integration of (2.6) as an initial value problem with the values of η_w and $\frac{dF}{d\eta}(0)$ given by the arithmetic mean of those in Tab. 1. # indicates values obtained by the condition (2.7.3).

δ	$F(\eta_w)$	$\frac{dF}{d\eta}(\eta_w)$	$\frac{dF}{d\eta}(\eta_w)\#$
0.1	0.001309	−0.835373	−0.835410
0.5	0.002147	−0.590006	−0.590038
1.0	0.002029	−0.463529	−0.463430

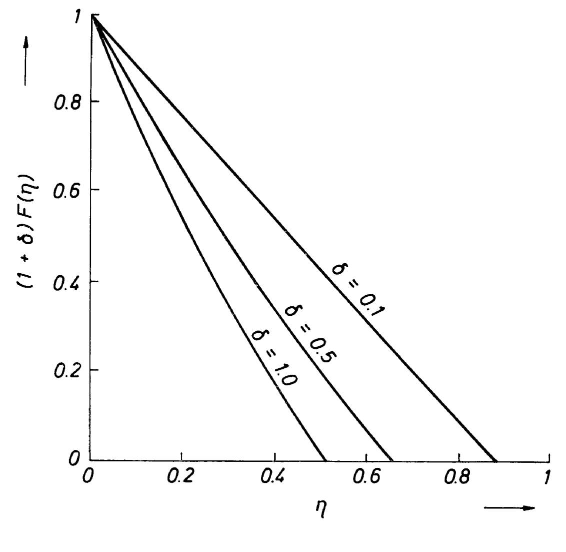

Fig. 1. Qualitative behaviour of the numerical solutions

It is worthwhile to remark that, in the nonlinear case, from the condition (2.7.3) we have

$$\frac{dF}{d\eta}(\eta_w) < 0 \qquad \text{for all value of } q \neq 1$$

The numerical results listed in Tab. 2 and shown in Fig. 1 are in agreement with this observation.

Since the model problem at hand is nonlinear we used the DIVPAG integrator, in the IMSL MATH/LIBRARY, with step size and error control [9].

As remarked in the previous Sections the differential equation (2.6) has a singular point in η_w (for every choice of η_w). To overcome the singular point in η_w*, in the first numerical integration, we applied the so called "discrete perturbation stability analysis" concept by introducing a discrete perturbation [10]—[11], say ε, in the datum given by the boundary condition (2.7.3). Usually we used $\varepsilon = -1D - 12$. As it is well known for nonlinear problems analytical stability depends upon the initial data, so it could be possible that in the case of $n \neq 0$ and for different values of q or δ, before obtaining acceptable results, we will have to try a set of discrete values of the guess η_w*.

In the present paper we deal with a problem related to compressive velocity impact. In [6] there was considered a physical situation from which arises a tensile velocity impact. That case is related to a large weight suspended from the end of a thin wire ($\delta = 1$). Certainly the present numerical method of solution can be applied also to that context.

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